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ALGEBRAIC SURFACES WITH NONHYPERELLIPTIC LINEAR PENCIL OF GENUS 4 AND IRREGULARITY ONE

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We construct algebraic surfaces with nonhyperelliptic linear pencil of genus 4 and of rank 3 whose slope is equal to 4 and with irregularity one. Furthermore, we consider the converse. Namely, we obtain the structure of the surfaces with the above properties.

1. Introduction

Throughout the paper, all the varieties are defined over the field $\mathbb{C}$ of complex numbers.

The set of smooth projective curves of genus 4 is separated into 3 types. Namely, Eisenbud-Harris general case(cf. [1]), Eisenbud-Harris special nonhyperelliptic case (cf. [6]), and hyperelliptic case. Let C be a smooth projective curve of genus 4. If C is Eisencud-Harris general (E-H general, for short), C has two base point free pencil of degree 3. In this case, C is obtained as an irreducible divisor of $\mathbb{P}^1 \times \mathbb{P}^1$ that is linearly equivalent to triple of a diagonal divisor. If C is Eisenbud-Harris special nonhyperelliptic (E-H special, for short), then C has only one base point free pencil of degree 3. In this case, C is obtained as a divisor of the Hirzebruch surface of degree 2 that is linearly equivalent to the triple of the tautological divisor. If C is hyperelliptic, then C is obtained as a bisection of the Hirzebruch surface of degree 3.

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Let $f : S \to B$ be a surjective morphism from a smooth projective surface S onto a smooth projective curve B such that the genus of a general fiber is 4. Denote by $K_{S/B}$ the relative canonical divisor, and by $\omega_{S/B} := \mathcal{O}_S(K_{S/B})$ the relative dualizing sheaf. Then the direct image $f_*\omega_{S/B}$ is a locally free sheaf of rank 4. If we assume f is not isotrivial, then it is well-known that $K_{S/B}^2 > 0$ and $\Delta(f) := \deg f_*\omega_{S/B} > 0$, and the slope $\lambda(f) := K_{S/B}^2/\Delta(f)$ is defined. If we assume further that f is relatively minimal, the following inequalities hold (cf, [5], [8], [9].)

(1) If a general fiber is E-H general, then $\lambda(f) \geq 7/2$.

(2) If a general fiber is E-H special, then $\lambda(f) \geq 24/7$.

Furthermore, if $q(S) > b$, then $\lambda(f) \geq 4$ holds. (cf. [13].)

In this note, we investigate a smooth projective surface with the following properties:

(i) There is a surjective morphism $f : S \to \mathbb{P}^1$ whose general fiber is E-H special nonhyperelliptic curve of genus 4.
(ii) f is relatively minimal and not isotrivial.
(iii) $q(S) = 1$ and $\lambda(f) = 4$.

This paper is organaized as follows:

In §2, we set some notations and consider the basic results. A curve obtained as a double cover over an elliptic curve branched over six points is a nonhyperelliptic curve of genus 4. (cf. [13].) We give the condition for the curve to be E-H special from the viewpoint of the branch locus.

In §3, we construct surfaces with the above properties. In the last of this section, we prove that our examples are isomorphic to a fiber product of some rational ruled surface and some elliptic ruled surface over $\mathbb{P}^1 \times \mathbb{P}^1$.

In §4, we consider the converse of §3 and classify the surfaces with the above properties. In order to prove some statements, we use the slope equality of nonhperelliptic E-H special fibration of genus 4. (cf. [6].)

2. Preliminaries

Notations 2.1. For a variety X, denote by $e(X)$ the topological Euler number of X. If X is smooth, denote by K_X the canonical divisor of X, and put $\omega_X := \mathcal{O}_X(K_X)$. For two divisors D_1 and D_2 over X, $D_1 \sim D_2$ means linearly equivalence. Denote by $p_g(X) := \dim H^0(\mathcal{O}_X(K_X))$ the geometric genus of X, and by $q(X) := \dim H^1(\mathcal{O}_X)$ the irregularity of X. Moreover, if $\dim X = 2$, denote by $\chi(\mathcal{O}_X) := 1 - q(X) + p_g(X)$ the Euler-Poincaré characteristic of X. For a divisor D, denote by $\mathrm{Bs}|D|$ the base locus of $|D|$.

For a Hirzebruch surface $\Sigma_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$, let $\mu : \Sigma_d \to \mathbb{P}^1$ be the ruling, Δ_0 the tautological divisor. Namely, $\Delta_0^2 = d$. If Γ is a fiber of μ, then denote by Δ_∞ the section of μ with $\Delta_\infty \sim \Delta_0 - d\Gamma$.

Definition 2.2. Let C be a reduced curve on a smooth surface S. A singularity $p \in C$ is said to be *negligible* if one of the following (i) and (ii) holds:

(i) p is a double point of C.

(ii) p is a triple point and after a blow-up of S at p, the multiplicity of C at the inverse image of p is less than 3.

Remark 2.3. Let $\mathfrak{B}$ be a reduced curve on a smooth projective surface Y. Assume there is an invertible sheaf L with $L^{\otimes 2} \cong \mathcal{O}_Y(\mathfrak{B})$. We can construct a double cover $\mathfrak{h} : \widehat{Y} \to Y$ in the total space of L branched along $\mathfrak{B}$. Let $p' \in \widehat{Y}$ be an inverse image of a point $p \in \mathfrak{B}$ by $\mathfrak{h}$. Then p' is a rational double point of $\widehat{Y}$ if and only if p is a negligible singularity of $\mathfrak{B}$.

Lemma 2.4. *Let $C \subset \mathbb{P}^2$ be a smooth cubic curve, and $Q_0 \subset \mathbb{P}^2$ a conic. Assume that $Q_0|_C$ is a reduced divisor of C of degree* 6*. If $h_0 : \widetilde{C} \to C$ is a double cover branched along $Q_0|_C$, then we have the following:*

(1) *If Q_0 is a smooth conic, then $\widetilde{C}$ is an E-H general curve of genus* 4.

(2) *If Q_0 is a union of two lines, then $\widetilde{C}$ is an E-H special curve of genus* 4.

PROOF Consider the double cover over $\mathbb{P}^2$ branched along Q_0.

The following lemma is trivial.

Lemma 2.5. *Let C be an elliptic curve, and δ a divisor on C of degree* 6*. Then δ is of type* (2) *of Lemma* 2.4 *if and only if δ is a pull back of some divisor of degree* 2 *by some triple cover $C \to \mathbb{P}^1$.*

Remark 2.6. Note that in the Hiezebruch surface of degree 2, the restriction of the tautological divisor to an E-H special curve of genus 4 gives a canonical divisor of the curve. Namely, the restriction of the rational map defined by the complete linear system of the tautological divisor to the curve is nothing but the canonical map of the curve. If the curve is embedded into $\mathbb{P}^3$ together with the Hirzebruch surface Σ_2, then the image of Σ_2 is a quadric cone with a vertex, and the image of the curve does not go through the vertex.

3. Construction

Fix a non-negative integer d, and consider the Hirzebruch surface Σ_d. Let a be a non-negative integer. Then we have $\mathrm{Bs}|2\Delta_0 + a\Gamma| = \emptyset$ which leads us to the fact that there exist irreducible and nonsingular members in $|2\Delta_0 + a\Gamma|$. Let $\mathfrak{D}_1 \in |2\Delta_0 + a\Gamma|$ be such a member. Since the restriction map $H^0(\mathcal{O}_{\Sigma_d}(2\Delta_0 + a\Gamma)) \to H^0(\mathcal{O}_{\mathfrak{D}_1}(2\Delta_0 + a\Gamma))$ is surjective, there exists an irreducible and nonsingular member $\mathfrak{D}_2 \in |2\Delta_0 + a\Gamma|$ such that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ intersect at $4a + 4d$ points transversally. A general member $\mathfrak{D}_3$ of the pencil generated by $\mathfrak{D}_1$ and $\mathfrak{D}_2$ is irreducible and nonsingular and intersects with $\mathfrak{D}_1$ and $\mathfrak{D}_2$ transversally. Let $\{p_j\}_{j=1,\cdots,4a+4d}$ be the set of the intersection points of $\mathfrak{D}_1$, $\mathfrak{D}_2$ and $\mathfrak{D}_3$, and $\varepsilon : \widetilde{\Sigma} \to \Sigma_d$ the blow-up at the set. Put $\mathcal{E}_i := \varepsilon^{-1}(p_j)$ for $j = 1, \cdots, 4a + 4d$.

Let $\widetilde{\mathfrak{D}}_i$ be the proper transform of $\mathfrak{D}_i$ by ε, and put $\widetilde{\mathfrak{D}} := \sum_{i=1}^{3} \widetilde{\mathfrak{D}}_i$. Then we have $\widetilde{\mathfrak{D}} \sim \varepsilon^*(6\Delta_0 + 3a\Gamma) - 3\sum_{i=1}^{4a+4d} \mathcal{E}_i$, and we obtain a cyclic triple cover $h : S \to \widetilde{\Sigma}$ branched along $\widetilde{\mathfrak{D}}$. Note that $\widetilde{\mathfrak{D}}$ is a union of three disjoint smooth curves $\widetilde{\mathfrak{D}}_1$, $\widetilde{\mathfrak{D}}_2$ and $\widetilde{\mathfrak{D}}_3$, and hence, S is smooth. Furthermore, $\widetilde{\Sigma}$ has a base point free linear pencil $|\widetilde{\mathfrak{D}}_i|$. Let $\beta : \widetilde{\Sigma} \to \mathbb{P}^1$ be the pencil. Then h can be considered as follows: If we put $q_i := \beta(\widetilde{\mathfrak{D}}_i)$ for $i = 1, 2, 3$, and if we let $\gamma : C \to \mathbb{P}^1$ be a cyclic triple cover branched at $\{q_1, q_2, q_3\}$, then we have $S = \widetilde{\Sigma} \times_{\mathbb{P}^1} C$, and h coincides with a natural projection $S \to \widetilde{\Sigma}$. Since C is an elliptic curve, S has an elliptic pencil, and we obtain $q(S) > 0$.

Lemma 3.1. *In the above notations, we have the following:*

$$
\begin{aligned}
&K_S^2 = 12(a + d - 2) \\
&e(S) = 24(a + d) - 12 \\
&p_g(S) = 3(a + d - 1) \\
&q(S) = 1
\end{aligned}
$$

PROOF Since $K_S \sim h^*(\varepsilon^*(2\Delta_0 + (2a + d - 2)\Gamma) - \sum_{j=1}^{4a+4d} \mathcal{E}_j)$ by the adjunction formula, we obtain the formula for K_S^2 by the complicated calculation. We have

$$
\begin{aligned}
e(S) &= 3(e(\Sigma_d) - 3e(\widetilde{B}_i)) + 3e(\widetilde{B}_i) \\
&= 3(8 + 6g(\widetilde{B}_i) - 6) - 6g(\widetilde{B}_i) + 6 \\
&= 24(a + d) - 12.
\end{aligned}
$$

By Noether's formula, we have $\chi(\mathcal{O}_S) = 3(a + d - 1)$. Note that the natural morphism $\alpha : S \to C$ is a hyperelliptic fibration of genus $a + d - 1$, and $\lambda(\alpha) = 4(a + d - 2)/(a + d - 1)$, which is the minimum of the slope of fibrations of genus $a + d - 1$ by [13]. Hence, we obtain the formula for the irregularity of S and the geometric genus of S. (See [13].)

Remark 3.2. (1) Put $f := \mu \circ \varepsilon \circ h : S \to \mathbb{P}^1$. Then f is an E-H special non-hyperelliptic fibration of genus 4 with $\lambda(f) = 4.$.

(2) The value $\chi(\mathcal{O}_S)$ is obtained by another way as follows:

If we put $\delta := \varepsilon^*(2\Delta_0 + a\Gamma) - \sum_i \mathcal{E}_i$, then we have

$$h_*\mathcal{O}_S \cong \mathcal{O}_{\widetilde{\Sigma}} \oplus \mathcal{O}_{\widetilde{\Sigma}}(-\delta) \oplus \mathcal{O}_{\widetilde{\Sigma}}(-2\delta),$$

and hence, by using Leray's spectral sequence and the Riemann-Roch theorem, we obtain

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{\widetilde{\Sigma}}) + \chi(\mathcal{O}_{\widetilde{\Sigma}}(-\delta)) + \chi(\mathcal{O}_{\widetilde{\Sigma}}(-2\delta)) = 3a + 3d - 3.$$

Similarly, we have

$$\begin{aligned} p_g(S) &= \dim H^0(\mathcal{O}_{\widetilde{\Sigma}}(K_{\widetilde{\Sigma}})) + \dim H^0(\mathcal{O}_{\widetilde{\Sigma}}(\delta + K_{\widetilde{\Sigma}})) + \dim H^0(\mathcal{O}_{\widetilde{\Sigma}}(2\delta + K_{\widetilde{\Sigma}})) \\ &= a + d - 1 + \dim H^0(\mathcal{O}_{\widetilde{\Sigma}}(\varepsilon^*(2\Delta_0 + (2a+d-2)\Gamma) - \textstyle\sum_j \mathcal{E}_j). \end{aligned}$$

Hence, by the result of Lemma 3.1, we obtain

$$\dim H^0\Big(\mathcal{O}_{\widetilde{\Sigma}}(\varepsilon^*(2\Delta_0 + (2a+d-2)\Gamma) - \sum_j \mathcal{E}_j\Big) = 2a + 2d - 2.$$

On the other hand, we have

$$\dim H^0(\mathcal{O}_{\Sigma_d}(2\Delta_0 + (2a+d-2)\Gamma)) = 6a + 6d - 3,$$

that is less than $\dim H^0(\mathcal{O}_{\widetilde{\Sigma}}(\varepsilon^*(2\Delta_0 + (2a+d-2)\Gamma) - \sum_j \mathcal{E}_j) + 4a + 4d$. This implies that a member of $|2\Delta_0 + (2a+d-2)\Gamma|$ going through any $4a+4d-1$ points of $\{p_j\}_j$ goes through the remaining one point, which is similar to the Cayley-Bacharach theorem. (cf. e.g., [7].)

(3) If $p_{j_1}, p_{j_2} \in \{p_j\}_{j=1,\cdots,4a+4d}$ are contained in the same fiber Γ_0 of the ruling $\Sigma_d \to \mathbb{P}^1$, then the proper transform $\widetilde{\Gamma}_0$ is a (-2)-curve, and dominated by 3 disjoint (-2)-curves by h. Let $\{\widetilde{\Gamma}_k\}$ be a set of (-2)-curves of $\widetilde{\Sigma}$ and $\{C_l\}$ be a set of (-2)-curves of S obtained by the same way. Let $\widehat{\nu} : S \to \widehat{S}$ and $\nu : \widetilde{\Sigma} \to \widehat{\Sigma}$ be the contractions of these (-2)-curves $\{\widetilde{\Gamma}_k\}$ and $\{C_l\}$. We have the following commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\widehat{\nu}"] \arrow[d, "h"'] & \widehat{S} \arrow[d, "\widehat{h}"] \\ \widetilde{\Sigma} \arrow[r, "\nu"'] & \widehat{\Sigma}. \end{tikzcd}$$

(4) Consider the $\mathbb{P}^1$-bundle $\widetilde{h} : X := \mathbb{P}(\mathcal{O}_{\widetilde{\Sigma}} \oplus \mathcal{O}_{\widetilde{\Sigma}}(\mathfrak{D}_i)) \to \widetilde{\Sigma}$. Let H be the tautological divisor of X. Note that S can be considered as a member of $|3H|$, which implies that any irreducible and nonsingular member $S_0 \in |3H|$ has the

same invariants as S. Let $h_{S_0}: S_0 \to \widetilde{\Sigma}$ be the restriction of $\widetilde{h}$ to S_0. Then h_{S_0} is a triple cover and $f_0 := \mu \circ \varepsilon \circ h_{S_0}: S_0 \to \mathbb{P}^1$ is an E-H special nonhyperelliptic fibration of genus 4. Furthermore, $h_{S_0}^* \mathfrak{D}_i$ gives a hyperelliptic fibraton $\alpha_0 : S_0 \to C_0$ of genus $a+d-1$ over an elliptic curve C_0, and we have the following commutative diagram:

$$\begin{array}{ccc} S_0 & \xrightarrow{g_0} & C_0 \\ {\scriptstyle h_{S_0}}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \widetilde{\Sigma} & \xrightarrow[\beta]{} & \mathbb{P}^1, \end{array}$$

where $\gamma : C_0 \to \mathbb{P}^1$ is a triple cover.

In the following arguments, we use the same notations S, C, f, α and h as the case of cyclic covering, instead of S_0, C_0, f_0, α_0 and h_{S_0}, respectively.

Consider the natural morphism $((\mu \circ \varepsilon) \times \beta) : \widetilde{\Sigma} \to \mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\mathcal{F}_0$ a fiber of $\mu \circ \varepsilon$ and by $\widetilde{\mathcal{F}_0}$ a fiber of β. We have $\mathcal{F}_0 \widetilde{\mathcal{F}_0} = 2$ and hence, $\deg((\mu \circ \varepsilon) \times \beta) = 2$. Let $\tau : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be a natural projection with $f = \tau \circ ((\mu \circ \varepsilon) \times \beta) \circ h$ and $\widehat{\tau} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ the other natural projection. Moreover, let Γ_0 be a fiber of τ and $\Delta_{0,0}$ a fiber of $\widehat{\tau}$. Then the branch locus $\mathfrak{B}_0$ of $((\mu \circ \varepsilon) \times \beta)$ satisfies $\mathfrak{B} \sim 2\Delta_{0,0} + 2(a+d)\Gamma_0$, and has at most ordinary nodes as its singularities. If Γ_0' is a fiber of τ containing the singularity of $\mathfrak{B}_0$, then the fiber of $\tau \circ (\mu \circ \varepsilon) \times \beta$ dominating Γ_0' is a union of two (-1)-curves and a (-2)-curve. Namely, it is the case where two points of $\{p_j\}$ are contained in the same fiber of μ.

Next, consider the natural morphism $(f \times \alpha) : S \to \mathbb{P}^1 \times C$. This is also a double cover. Since

$$\lambda(\alpha) = \frac{K_S^2}{\chi(\mathcal{O}_S)} = \frac{4((a+d-1)-1)}{a+d-1},$$

and since this is the lower bound of the slope for the fibration of genus $a+d-1$, α does not have the degenerate fiber with the positive H-index in the sence of [13]. Let $\iota : \mathbb{P}^1 \times C \to \mathbb{P}^1$ and $\widetilde{\iota} : \mathbb{P}^1 \times C \to C$ be the natural projections, and denote by $\widehat{\Delta}$ a fiber of ι. Then the branch locus $\mathfrak{B}$ of $(f \times \alpha)$ satisfies $\mathfrak{B} \sim 2(a+d)\widehat{\Delta} + \widetilde{\iota}^* m_0$ for some $m_0 \in \mathrm{Div}(C)$ with $\deg m_0 = 6$, and has at most the negligible singularities.

Consider the natural morphism $(\mathrm{id}_{\mathbb{P}^1} \times \gamma) : \mathbb{P}^1 \times C \to \mathbb{P}^1 \times \mathbb{P}^1$. It is clear that any fiber of f is mapped onto the same fiber of τ by $((\mu \circ \varepsilon) \times \beta) \circ h$ and $(\mathrm{id}_{\mathbb{P}^1} \times \gamma) \circ (f \times \alpha)$. Furthermore, if two fibers $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ of α are mapped onto the same fiber of $\widehat{\tau}$ by $((\mu \circ \varepsilon) \times \beta) \circ h$, then they are mapped onto the same fiber by $(\mathrm{id}_{\mathbb{P}^1} \times \gamma) \circ (f \times \alpha)$.

By considering above, we obtain the following:

Theorem 3.3. *Let the notations be as above. Then S is the minimal resolution of the fiber product* $\widehat{\Sigma} \times_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathbb{P}^1 \times C)$

PROOF Any point of S that is not contained in any (-2)-curve is mapped to the same point by $((\mu \circ \varepsilon) \times \beta) \circ h$ and $(\mathrm{id}_{\mathbb{P}^1} \times \gamma) \circ (f \times \alpha)$. Therefore, we have the natural morphism $S \to \widehat{\Sigma} \times_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathbb{P}^1 \times C)$. It is clear that this morphism contracts only (-2)-curves of Remark 3.2 (3).

Remark 3.4. From Theorem 3.3, we have $\mathfrak{B} = (f \times \alpha)^* \mathfrak{B}_0$. If $\widehat{\Delta}$ is generic, then the divisor $\mathfrak{B}|_{\widehat{\Delta}}$ is of type (2) of Lemma 2.4 by Lemma 2.5. Namely, we obtain another evidence for the fact that a general fiber of $f : S \to \mathbb{P}^1$ is an EH-special curve of genus 4.

Let Γ_0 be a fiber of τ such that $\mathfrak{B}_0$ contact at a point $p_0 \in \Gamma_0$, and put $\widehat{\Delta} := ((\mu \circ \varepsilon) \times \beta)^* \Gamma_0$. $\mathfrak{B}$ contacts with $\widehat{\Delta}$ at 3 points of $((\mu \circ \varepsilon) \times \beta)^{-1}(p_0)$. Hence, the fiber of f dominating Γ_0 is a union of 2 elliptic curves intersect at 3 points transversally, and a fiber with H-index $3/7$. (cf. [6].)

4. Classification

In this section, we consider the converse of the previous section. We assume as follows:

Assumption 4.1. Let $f : S \to \mathbb{P}^1$ be a nonhyperelliptic E-H special fibration of genus 4. Assume f is not isotrivial. Furthermore, we assume $\lambda(f) = 4$, $q(S) = 1$ and that S is minimal (and hence, f is relatively minimal).

Let $\alpha : S \to C := \mathrm{Alb}(S)$ be the Albanese map of S. C is an elliptic curve. Denote by g the fiber genus of α.

Lemma 4.2. *Let the notations and the conditions be as above. Let $(f \times \alpha) : S \to \mathbb{P}^1 \times C$ be the natural morphism. Then we have* $\deg(f \times \alpha) = 2$ *and* $p_g(S) = 3g$.

PROOF Since $\lambda(f) = 4$, we have $K_S^2 = 4p_g(S) - 12$. Hence,

$$\lambda(\alpha) = \frac{K_S^2}{p_g(S)} = 4 - \frac{12}{p_g(S)},$$

which leads us to

$$4 - \frac{12}{p_g(S)} \geq 4 - \frac{4}{g},$$

(cf. [13]), and we obtain

$$p_g(S) \geq 3g. \tag{1}$$

$E_\alpha := \alpha_* \omega_{S/C}$ is a locally free sheaf over C with $\mathrm{rk} E_\alpha = g$ and $\deg E_\alpha = p_g(S)$. Let $\mathcal{F}$ be a fiber of f and $\mathbb{F}$ a fiber of α. We have $\deg(f \times \alpha) = \mathcal{F}\mathbb{F}$.

If we assume $\mathcal{F}\mathbb{F} > 3$, then $(K_S - 3\mathbb{F})\mathcal{F} < 0$, and $K_S - 3\mathbb{F}$ cannot be effective. Hence, any indecomposable component E' of E_α satisfies $3\mathrm{rk}E' > \deg E'$ (cf. [2]), and we have $3\mathrm{rk}E_\alpha > \deg E_\alpha$, namely, $3g > p_g(S)$ holds, which contradicts to (1).

The case $\mathcal{F}\mathbb{F} = 1$ must be excluded because in this case, we have $S \cong \mathbb{P}^1 \times C$ which contradicts the assumption that f is not isotrivial.

Therefore, $\mathcal{F}\mathbb{F} = 2$ holds and $(f \times \alpha) : S \to \mathbb{P}^1 \times C$ is a double cover. If D_0 is a fiber of the natural projection $\iota : \mathbb{P}^1 \times C \to \mathbb{P}^1$, and if $\widetilde{\iota} : \mathbb{P}^1 \times C \to C$ is the natural projection, then $\mathcal{B} \sim 2(g+1)D_0 + \widetilde{\iota}^* m_0$ holds for some $m_0 \in \mathrm{Div}(C)$ with $\deg m_0 = 6$. Let $m_1 \in \mathrm{Div}(C)$ be a divisor with $2m_1 \sim m_0$ and $h : S' \to \mathbb{P}^1 \times C$ the double cover branched along $\mathcal{B}$ and constructed in the total space of $\mathcal{O}_{\mathbb{P}^1 \times C}((g+1)D_0 + \widetilde{\iota}^* m_1)$. Then any singularity of S' dominates a singularity of $\mathcal{B}$. Let

$$\begin{array}{ccc} \widehat{S} & \longrightarrow & S' \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f\times\alpha} \\ W & \xrightarrow[\nu']{} & \mathbb{P}^1 \times C \end{array}$$

be the canonical resolution of S'. ν' is a composition of blow-ups, and h is a double cover whose branch locus is smooth. The canonical divisor of $\widehat{S}$ satisfies

$$K_{\widehat{S}} \sim h^*(\nu'^*((g-1)D_0 + \widetilde{\iota}^* m_1) - \widetilde{E})$$

for some effective divisor $\widetilde{E}$ whose components are exceptional divisors. Hence, we have

$$p_g(S) = p_g(\widehat{S}) \le \dim H^0(\mathcal{O}_{\mathbb{P}^1 \times C}((g-1)D_0 + \widetilde{\iota}^* m_1) = 3g.$$

By combining this inequality with (1), we obtain $p_g(S) = 3g$.

Corollary 4.3. *Let the notations be as in Lemma* 4.2. *Then the branch locus* $\mathcal{B}$ *of* $(f \times \alpha)$ *has at most negligible singularities as its singularities.*

PROOF Since any fiber $\mathbb{F}$ of α is mapped onto $\mathbb{P}^1$ as a double cover, α is a hyperelliptic fibration of genus $p_g(S)/3$. Since

$$\lambda(\alpha) = \frac{K_S^2}{p_g(S)} = 4 - \frac{12}{p_g(S)} = \frac{4(g-1)}{g},$$

there is no fiber with positive H-index for hyperelliptic fibration in the sense of [13]. Hence, $\mathcal{B}$ has at most negligible singularities as its singularities.

Lemma 4.4. *Let the notations be as above. Then we have*

$$f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 3},$$

where $n = p_g(S)/3+1$.

PROOF From Corollary 4.3, the branch locus $\mathcal{B}$ of $\widetilde{f} := (f \times \alpha)$ has at most negligible singularities. Hence, we have

$$\begin{aligned} f_*\omega_{S/\mathbb{P}^1} &\cong \iota_*\widetilde{f}_*\omega_{S/\mathbb{P}^1} \\ &\cong \iota_*(\mathcal{O}_{\mathbb{P}^1\times C}(nD_0+\widetilde{\iota}^*m_1)\oplus \mathcal{O}_{\mathbb{P}^1\times C}) \\ &\cong \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 3}\oplus \mathcal{O}_{\mathbb{P}^1}. \end{aligned}$$

Put $E := f_*\omega_{S/\mathbb{P}^1}$, and consider the exact sequence

$$0 \to L \to \mathrm{Sym}^2 E \xrightarrow{\eta} f_*\omega_{S/\mathbb{P}^1}^{\otimes 2} \longrightarrow \mathcal{T} \to 0,$$

where η is the multiplication map. η can be written as

$$\mathcal{O}_{\mathbb{P}^1}\oplus \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 3}\oplus \mathcal{O}_{\mathbb{P}^1}(2n)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 3}\oplus \mathcal{O}_{\mathbb{P}^1}(2n)^{\oplus 6},$$

and hence, $L \cong \mathcal{O}_{\mathbb{P}^1}$ holds and η is surjective.

Let $\psi : S \cdots \to W := \mathbb{P}(E)$ be the rational map defined by the natural sheaf homomorphism $f^*E \to \omega_{S/\mathbb{P}^1}$. ψ is called the *relative canonical map*. Let T be the tautological divisor of W and F a fiber of $\pi : W \to \mathbb{P}^1$. By [9], there exists an irreducible relative hyperquadric Q containing $S' := \psi(S)$ and Q satisfies $Q \sim 2T$.

Since a general fiber of f is EH-special, Q has the relative vertex V_0. See Remark 2.6.

In the proof of the next lemma, we use the knowledge of H-index for the nonhyperelliptic E-H special fibration of genus 4. See [6] for detail. See also [12].

Lemma 4.5. *Let the notations be as above. Then we have*

$$V_0 = \mathbb{P}\left(E/\mathcal{O}_{\mathbb{P}^1}\oplus \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 2}\right).$$

PROOF Let $X_0 \in H^0(\mathcal{O}_W(T))$ and $X_1, X_2, X_3 \in H^0(\mathcal{O}_W(T-nF))$ be the global sections defining the homogeneous coordinates of each fiber of π. Let $\Psi \in H^0(\mathcal{O}_W(2T))$ be the global section defining Q. Then Ψ can be written as

$$\Psi = cX_0^2 + \sum_{\substack{i\geq 0, j\geq 0\\ i+j\leq 2}} \psi_{i,j}X_1^{2-i-j}X_2^iX_3^j, \tag{2}$$

where $c \in \mathbb{C}$ is a constant and $\psi_{i,j} \in H^0(\mathcal{O}_{\mathbb{P}^1}(2n))$. The rank of the 2-form of X_1, X_2 and X_3 in the right hand side of (2) is 3 when $c = 0$ while 2 when $c \neq 0$.

If $c = 0$, then we have $V_0 = \mathbb{P}(E/\mathcal{O}_{\mathbb{P}^1}(2n)^{\oplus 3})$, while if $c \neq 0$, we have $V_0 = \mathbb{P}(E/\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2n))^{\oplus 2})$. By considering (2), the discriminant locus of Q is defined by $\det(\widetilde{\psi}_{ij})$ when $c = 0$, where $\widetilde{\psi}_{ii} = \psi_{ii}$ and $\widetilde{\psi}_{ij} = (1/2)\psi_{ij}$ for $i \neq j$. Namely, the degree of the discriminant locus of Q is $6n$, and hence, the sum of H-index is larger than or equal to $(18/7)n$. On the other hand, the sum of H-index is

$$K^2_{S/\mathbb{P}^1} - \frac{24}{7}\Delta(f) = \frac{12}{7}n,$$

a contradiction. By considering similarly, the degree of the discriminant locus of Q is $4n$ when $c \neq 0$, and we obtain the sum of H-index is $(12/7)n$. Furthermore, we have $V_0 = \mathbb{P}(E/\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2n)^{\oplus 2})$.

Remark 4.6. (1) By the proof of Lemma 4.5 and by the result of [12], H-index of a fiber of f is arising from the discriminant locus of Q. Namely, the relative canonical image of S is disjoint from the relative vertex of Q.

(2) Let $\rho : \widetilde{W} \to W$ be the blow-up along V_0. Then we have the following commutative diagram:

$$\begin{tikzcd} \widetilde{W} \arrow[r, "\rho"] \arrow[d, "\widetilde{\pi}"'] & W \arrow[d, "\pi"] \\ \mathbb{P}(E_0) \arrow[r, "\xi"'] & \mathbb{P}^1, \end{tikzcd}$$

where $E_0 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2n)^{\oplus 2}$, $\widetilde{\pi}$ is a $\mathbb{P}^1$-bundle and ξ is a $\mathbb{P}^2$-bundle. If we denote by T_{E_0} the tautological divisor of $\mathbb{P}(E_0)$, and by $\mathfrak{F}$ a fiber of ξ, then we have

$$\widetilde{W} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \oplus \mathcal{O}_{\mathbb{P}(E_0)}(n\mathfrak{F})).$$

(cf. [11].) If we put $\widetilde{T} \sim \rho^* T$, and if we let $\mathfrak{E}$ be the exceptional divisor of ρ, then we have $\widetilde{T} \sim \widehat{\pi}^* T_{E_0} + \mathfrak{E}$. If we let $\widetilde{Q}$ be the proper transform of Q by ρ, there exists a relative hyperquadric Q_0 of $\mathbb{P}(E_0)$ with $\widetilde{Q} = \widehat{\pi}^{-1}(Q_0)$.

If we put $\widehat{B}_0 := \mathbb{P}(E_0/\mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 2}) (\subset \mathbb{P}(E_0))$, we have $Q_0 \cap \widehat{B}_0 = \emptyset$. Let $\rho_0 : \widetilde{P} \to \mathbb{P}(E_0)$ be the blow-up along $\widehat{B}_0$. We have the following commutative diagram:

$$\begin{tikzcd} \widetilde{P} \arrow[r, "\rho_0"] \arrow[d, "\widehat{\pi}"'] & \mathbb{P}(E_0) \arrow[d, "\xi"] \\ \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\tau"'] & \mathbb{P}^1, \end{tikzcd}$$

where $\widehat{\pi}$ is a $\mathbb{P}^1$-bundle and τ is the natural projection. If we let Γ_0 be a fiber of τ, and if we let $\Delta_{0,0}$ be a fiber of another natural projection $\widehat{\tau} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, then we have

$$\widetilde{P} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\Delta_{0,0} + n\Gamma_0)).)$$

We can consider as $Q_0 \subset \widetilde{P}$, and the restriction map $\widehat{\pi}_0 := \widehat{\pi}|_{Q_0} : Q_0 \to \mathbb{P}^1 \times \mathbb{P}^1$ is a double cover whose branch locus $\mathfrak{B}_0$ is linearly equivalent to $2\Delta_{0,0} + 2n\Gamma_0$.

Q_0 has a linear pencil $|\widehat{\pi}_0^*\Delta_{0,0}|$ that is pulled-back to the elliptic pencil of S. This pencil gives α. Moreover, the triple cover $\gamma : C \to \mathbb{P}^1$ is naturally defined.

Theorem 4.7. *Let the notations be as above. Then S is a minimal resolution of $S' := Q_0 \times_{\mathbb{P}^1\times\mathbb{P}^1} (\mathbb{P}^1 \times C)$. The singularity of S' is a rational double point dominating the negligible singularity of $\mathfrak{B}_0$.*

PROOF The statement for the fiber product can be proved by the same argument as Theorem 3.3. The statement for the singularity is clear.

Corollary 4.8. *Let the notaions and the conditions be as above.*

(i) *Let $\mathfrak{B}$ be a divisor of $\mathbb{P}^1 \times C$ that is linearly equivalent to $kD_0 + \widetilde{\imath}^* m_0$ for some integer k and some divisor m_0 of degree 6. Then the restriction of $\mathfrak{B}$ to any fiber of $\mathbb{P}^1 \times C \to \mathbb{P}^1$ is a pull-back of the divisor of degree 2 on $\mathbb{P}^1$ by a triple cover $C \to \mathbb{P}^1$ if and only if $\mathfrak{B}$ is a pull-back of a bisection of $\mathbb{P}^1 \times \mathbb{P}^1$ by $\mathrm{id}_{\mathbb{P}^1} \times \gamma$.*

(ii) *If $\mathfrak{B}$ is not a pull-back of a bisection of $\mathbb{P}^1 \times \mathbb{P}^1$, the nonhyperelliptic fibration of genus 4 on the double cover of $\mathbb{P}^1 \times C$ branched along $\mathfrak{B}$ is E-H general.*

Proposition 4.9. *Let the notations be as before. Then there exists a relative hypercubic $Y \in |3T - 3nF|$ such that S' is a complete intersection of Q and Y, where F is a fiber of π.*

PROOF Since $\widetilde{\pi}_{\widetilde{Q}} : \widetilde{Q} \to Q_0$ is a $\mathbb{P}^1$-bundle, we have

$$\mathrm{Pic}(\widetilde{Q}) \cong \mathbb{Z}\widetilde{T} \oplus \widetilde{\pi}_{\widetilde{Q}}^* \mathrm{Pic}(Q_0).$$

Since $S' \cap V_0 = \emptyset$ in W, we obtain $S' \sim 3\widetilde{T} - 3n\widetilde{F}$ as the divisor of $\widetilde{Q}$, where $\widetilde{F}$ is a fiber of $\pi \circ \rho$. Hence, it is sufficient to prove that the restriction map

$$H^0(\mathcal{O}_{\widetilde{W}}(3\widetilde{T} - 3n\widetilde{F})) \to H^0(\mathcal{O}_{\widetilde{Q}}(3\widetilde{T} - 3n\widetilde{F}))$$

is surjective.

The followings are easily calculated:

$$\dim H^0(\mathcal{O}_{\widetilde{W}}(3\widetilde{T}-\widetilde{\pi}^*(2T_{E_0}+3n\mathbb{F})))=0,$$
$$\dim H^0(\mathcal{O}_{\widetilde{W}}(3\widetilde{T}-3n\widetilde{F}))=10,$$

where $\mathbb{F}$ is a fiber of ξ. From Remark 4.6, we have

$$\widetilde{Q}\cong\mathbb{P}(\mathcal{O}_{Q_0}(T_{E_0})\oplus\mathcal{O}_{Q_0}(n\Gamma_0)),$$

where Γ_0 is a fiber of $\xi|_{Q_0}$. We have

$$H^i(\mathcal{O}_{\widetilde{Q}}(3\widetilde{T}-\widetilde{\pi}^*(3n\mathbb{F})))\cong\bigoplus_{j=0}^{3}H^i(\mathcal{O}_{Q_0}(jT_{E_0}-jn\mathbb{F})).$$

Consider the exact sequence

$$0\to\mathcal{O}_{\mathbb{P}(E_0)}((j-2)T_{E_0})\to\mathcal{O}_{\mathbb{P}(E_0)}(jT_{E_0})\to\mathcal{O}_{Q_0}(jT_{E_0})\to 0,$$

for $j=1,2,3$. Note that $R^1\xi_*\mathcal{O}_{\mathbb{P}(E_0)}((j-2)T_{E_0})=0$ holds. Similarly, we have $\xi_*\mathcal{O}_{\mathbb{P}(E_0)}(-T_{E_0})=0$, and we obtain $\xi_*\mathcal{O}_{Q_0}(T_{E_0})\cong\xi_*\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0})\cong\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 2}$. Hence, equality $\dim H^0(\mathcal{O}_{Q_0}(T_{E_0}-n\mathbb{F}))=2$ holds.

Next, we consider the cases $j=2,3$. We use the notations of Remark 4.6. Since we can consider as $\mathcal{O}_{Q_0}(jT_{E_0})\cong\mathcal{O}_{Q_0}(\widehat{\pi}_0^*(j\Delta_0+jn\Gamma))$ for any positive integer j, we have

$$(\widehat{\pi}_0)_*\mathcal{O}_{Q_0}(jT_{E_0})\cong\mathcal{O}_{\Sigma_0}(j\Delta_0+jn\Gamma)\oplus\mathcal{O}_{\Sigma_0}$$

by the projection formula. Hence, we obtain

$$\begin{aligned}&H^i(\mathcal{O}_{Q_0}(jT_{E_0}-jn\mathbb{F}))\\ \cong\ &H^i(\mathcal{O}_{\Sigma_0}((j\Delta_0)))\oplus H^i(\mathcal{O}_{\Sigma_0}((j-1)\Delta_0-(a+d)\Gamma))\\ \cong\ &H^i(\mathcal{O}_{\mathbb{P}^1})^{\oplus(j+1)}\oplus H^i(\mathcal{O}_{\mathbb{P}^1}(-n))^{\oplus j},\end{aligned}$$

which leads us to

$$\dim H^0(\mathcal{O}_{Q_0}(2T_{E_0}-2n\mathbb{F}))=3,$$
$$\dim H^0(\mathcal{O}_{Q_0}(3T_{E_0}-3n\mathbb{F}))=4.$$

and combining with the above results, we obtain

$$\dim H^0(\mathcal{O}_{\widetilde{Q}}(3\widetilde{T}-\widetilde{\pi}^*(3n\mathbb{F})))=10.$$

Therefore, we obtain the isomorphism

$$H^0(\mathcal{O}_{\widetilde{W}}(3\widetilde{T}-3n\widetilde{F}))\cong H^0(\mathcal{O}_{\widetilde{Q}}(3\widetilde{T}-3n\widetilde{F})).$$

Remark 4.10. In [3], the following is proved:

Theorem 4.11. (cf. [3]) *Let S be a minimal surface of general type of maximal Albanese dimension. The equality $K_S^2 = 4\chi(\mathcal{O}_S)$ holds if and only if:*

(a) $q(S) = 2$ *and*

(b) *the canonical model of S is a double cover of the Albanese surface $Alb(S)$ whose branch divisor is ample and has at most nigligible singularities.*

Under the condition of the theorem, if $\mathrm{Alb}(S) = C_1 \times C_2$ for some two elliptic curves C_1 and C_2, then S has two pencils whose bases are C_1 and C_2. Moreover, the slope of each pencil is 4. (They do not depend on the fiber genus.) The result of our paper (containing the case of E-H general, because we do not use the condition that f is E-H special in the proof of Lemma 4.2 and Corollary 4.3) is similar to this theorem.

In fact, a surface like above exists. For $i = 1, 2$, let $\iota_i : C_1 \times C_2 \to C_i$ be the natural projection and F_i a fiber of ι_i. For integers m and n, there exists a curve $\mathfrak{B} \in |2mF_1 + 2nF_2|$ with at most negligible singularities. If S is a minimal resolution of the double cover over $C_1 \times C_2$ branched along $\mathfrak{B}$, then we have $K_S^2 = 4mn$ and $\chi(\mathcal{O}_S) = mn$. If $h : S \to C_1 \times C_2$ is the double cover and if $\alpha : S \to \mathrm{Alb}(S)$ is the Albanese map, then there exists a unique homomorphism $\varphi : \mathrm{Alb}(S) \to C_1 \times C_2$ of Abelian varieties with $h = \alpha \circ \varphi$. Clearly, φ is isomorphism and h is the Albanese map of S. This example is the special case of [4, Example 4.1] and [10, Example 7.1].

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