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HOLOMORPHIC VECTOR BUNDLES ON HOLOMORPHICALLY CONVEX COMPLEX SURFACE

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Here we study holomorphic vector bundles on a two-dimensional holomorphically convex complex manifold Y. We present some characterizations of Y under which torsion free sheaves are filtrable. In the case when $Y = X \times C$, where X is a connected compact Riemann surface, we study in what sense every holomorphic vector bundle may be approximated by a sequence of algebraic vector bundles.

0. Introduction.

Let X be a smooth connected complex projective curve of genus $g \geq 0$. Set $Y := X \times C$. Hence Y is both an algebraic surface (neither affine nor projective) and a complex surface. Let $\pi : Y \to C$ be the projection. Usually we will see Y and π in the analytic category and use $Y_{\text{alg}}, \pi_{\text{alg}}$ and so on if we consider the corresponding objects in the algebraic category. Since π is proper, C is Stein and X is connected. Y is holomorphically convex and $\pi : Y \to C$ is the Remmet reduction of Y, i.e. $\pi_*(\mathbf{0}_Y) = \mathbf{0}_C$. In particular for every holomorphic function h on Y there is a unique holomorphic function h' on C with $h = h' \circ \pi$ ([7], p. 229).

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In this paper we study holomorphic vector bundles on Y and on other holomorphically convex complex surfaces whose cohomological properties are similar to the ones of y. In particular, we consider the problem of approximation of a holomorphic vector bundle by algebraic vector bundles. For the case of line bundles on $X \times C$, see Theorem 1.5 and Remark 1.6. For the case of rank 2 vector bundles, see 4.5. On suitable holomorphically convex surfaces Y every holomorphic vector bundle E admits an increasing filtration $\{E_i\}_{,0 \leq i \leq r}$, $r :=$ rank(E), with $E_0 = \{0\}$, $E_r = E$ and rank$(E_{i+1}/E_i) = 1$ for every $i < r$ (see 2.5, 2.6, 3.1, 3.2 and 3.3). We use this type of filtration to give a description of rank 2 holomorphic vector bundles on surface of type $X \times U$ with U an open Riemann Surface (see 4.5 and 4.6). In some cases the description is complete and give a recipe to construct every such holomorphic vector bundle (see 4.7). In general the description at least allows us to construct several interesting examples.

We now briefly state some of the basic concepts that we will be using and mention our previous work in this area. Recall that a complex space Y is holomorphically convex if one of the following equivalent conditions is satisfied:

i) there is a Stein space A and a proper holomorphic surjective map $u : Y \to A$;

ii) for every compact subset $K \subseteq Y$ the holomorphic hull $K^\wedge$ of K is compact.

If a pair $(u.A)$ satisfies i), then there is a Stein space B, a proper holomorphic surjective map $\pi : Y \to B$ and a holomorphic finite map $v : B \to A$ such that $\pi_*(\mathbf{0}_Y) = \mathbf{0}_B$ and $u = v \circ \pi$. The pair (B, π) is uniquely determined by Y and it is called the Remmert reduction of Y. The fibers of π are connected.
We studied the case when Y is 0-convex in [1]. Holomorphic convexity is a much more general concept. For reader's sake we compare the two situations. According to our convention Y is said to be 0-convex if there exists a C^∞ weakly pseudoconvex function $h : Y \to \mathbf{R}$ such that for every $c \in \mathbf{R}$ the set $\{x \in Y : f(x) < c\}$ is relatively compact and there exists a compact subset K of Y such that f is strictly 0-convex in $X \setminus K$. It is known that Y is 0-convex if and only if there exist a Stein space B, a finite subset S of B and a proper surjective holomorphic map $f : Y \to B$ such that $f|f^{-1}(Y \setminus S) : f - 1(Y \setminus S) \to Y \setminus S$ is a isomorphism. Hence compact implies 0-convex (just take B a point): a Stein space is 0-convex (take $B = Y$ and $f =$ identity); 0-convex implies holomorphically convex; a holomorphically convex space is 0-convex if and only if the union of all positive dimensional analytic subspaces of Y is analytic. Hence if X is a compact complex space, U is a Stein space, dim$(U) > 0$ and

$\dim(X) > 0$, then $X \times U$ is holomorphically convex but not 0-convex. For more details see [7] or [14]: we note however that there are two conventions for convexity and our concept of 0-convexity corresponds to their concept of 1-convexity.

1. Holomorphic line bundles on X×C.

In this section we take $Y := X \times \mathbf{C}$, where X is a smooth connected complex curve of genus g.

(1.1) The topological or C^∞ classification of vector bundles on Y is easy because Y is homotopically equivalent to X. Hence every topological or C^∞ vector bundle on Y is equivalent (in the topological or C^∞ category) to a vector bundle $\pi^*(F)$ with F vector bundle on X ([11], Ch. 3, 4). Every topological or C^∞ vector bundle on X is uniquely determined by its rank and its degree: for all integers $r.d$ with $r > 0$ there is a unique (up to topological or C^∞ equivalence) vector bundle F on X with $\mathrm{rank}(F) = r$ and $\deg(F) = d$ ([8]).

(1.2) Let $\mathrm{Pic}(Y)$ be the group of all holomorphic line bundles on Y. up to isomorphism. Consider the exponential sequence

$$0 \to \mathbf{Z} \to \mathbf{0}_Y \to \mathbf{0}_Y^* \to 0 \tag{1}$$

Since Y is homotopically equivalent to X, we have $H^1(Y, \mathbf{Z}) \cong H^1(X, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 2g}$ and $H^2(Y, \mathbf{Z}) \cong H^2(X, \mathbf{Z}) \cong \mathbf{Z}$. Since Y is a two-dimensional connected complex manifold and Y is not compact, we have $H^2(Y, \mathbf{0}_Y) = \{0\}$ (see e.g. [13]). Thus the map $\delta : \mathrm{Pic}(Y) \to H^2(Y, \mathbf{Z})$ which sends a line bundle on Y into the degree of its restriction to any slice $X \times \{t\}$ is surjective. Call $\mathrm{Pic}^0(Y) = Ker(\delta)$: we will often write $\deg(L)$ instead of $\delta(L)$. for every $L \in \mathrm{Pic}(Y)$, the integer $\deg(L) = \delta(L)$ is called the degree of L. Since π is the Remmert reduction of Y, we have $H^0(Y, \mathbf{0}_Y) \cong H^0(\mathbf{C}, \mathbf{0}_\mathbf{C})$ and $H^0(Y, \mathbf{0}_Y^*) \cong H^0(\mathbf{C}, \mathbf{0}_\mathbf{C}{}^*)$. Thus the exponential map $H^0(Y, \mathbf{0}_Y) \to H^0(Y, \mathbf{0}_Y^*)$ is surjective. Since $\mathbf{C}$ is Stein and $\pi_*(\mathbf{0}_Y)$ is coherent, we have $H^1(\mathbf{C}, \pi_*(\mathbf{0}_Y)) = 0$. By relative duality we have $R^1\pi_*(\mathbf{0}_Y \cong \mathbf{0}_\mathbf{C}{}^{\oplus 2g}$; this is obvious in the algebraic category and the $2g$ linearly independent and spanning algebraic sections are obviously holomorphic, too; a very general theory of duality for proper holomorphic maps is given in [12] but in our paper the situation is simpler because the holomorphic map π is projective and hence we are in the set up of [2]. Hence we obtain $H^1(Y, \mathbf{0}_Y) \cong H^0(\mathbf{C}, \mathbf{0}_C)^{\oplus 2g}$. Alternatively, one can prove that $H^1(Y, \mathbf{0}_Y) \cong H^0(\mathbf{C}, \mathbf{0}_C)^{\oplus 2g}$ using a Künneth formula because the natural quasi-Frechet topologies on the vector spaces $H^0(\mathbf{C}, \mathbf{0}_C)$, $H^1(\mathbf{C}, \mathbf{0}_C) =$

$\{0\}$, $H^1(X, \mathbf{0}_X) \cong \mathbf{C}^{\oplus 2g}$, $H^0(X, \mathbf{0}_X) \cong \mathbf{C}$ are all separated and the same is true for a good family of open subsets of Y. Summing up, we have checked the following result.

Proposition 1.3. *We have* $\mathrm{Pic}(Y) \cong \mathrm{Pic}^0(Y) \oplus \mathbf{Z}$ *as abelian groups.* $\mathrm{Pic}^0(Y)$ *is the quotient of* $H^0(\mathbf{C}, \mathbf{0}_\mathbf{C}) \otimes_\mathbf{C} \mathbf{C}^{\oplus 2g}$ *by the group* $\{0\} \times \mathbf{Z}^{\oplus 2g} \cong \mathbf{Z}^{\oplus 2g}$ *acting on the second factor of the tensor product.*

From the statement of 1.3 we obtain at once the following result.

Proposition 1.4. *Fix* $P \in X$ *and* $L \in \mathrm{Pic}(Y)$ *and let* $d := \deg(L)$. *L is algebraizable if and only if* $L \otimes \mathbf{0}_Y(-d\{P\} \times \mathbf{C}) \in H^0(\mathbf{C}_{\mathrm{alg}}, \mathbf{0}_{\mathbf{C}\,\mathrm{alg}}) \otimes (\mathbf{C}^{2g}/\mathbf{Z}^{2g})$.

Here of course $d\{P\} \times \mathbf{C}$ is a Cartier divisor of Y, $\mathbf{0}_Y(-d\{P\} \times \mathbf{C})$ is algebraizable, and $H^0(\mathbf{C}_{\mathrm{alg}}, \mathbf{0}_{\mathbf{C}\,\mathrm{alg}})$ is the set of all polynomials. Since every element of $H^0(\mathbf{C}, \mathbf{0}_\mathbf{C})$, i.e. every entire function, is the limit of a sequence of polynomials (e.g. of its Taylor expansion at the origin) and $H^0(\mathbf{C}, \mathbf{0}_\mathbf{C}) \otimes_\mathbf{C} \mathbf{C}^{\oplus 2g}$ as topological vector spaces, we may say that every holomorphic line bundle on Y may be approximated by a sequence of algebraic line bundles on Y_{alg} in the following form.

Theorem 1.5. *For every integer d fix an algebraic line bundle* $L(d)$ *of degree d on Y; for instance fix* $P \in X$ *and take* $L(d) := \mathbf{0}_Y(d\{P\} \times \mathbf{C})$. *Take* $L \in \mathrm{Pic}(Y)$ *and set* $d := \deg(L)$. *Fix a lift* $f \in H^0(\mathbf{C}, \mathbf{0}_\mathbf{C} \otimes_\mathbf{C} \mathbf{C}^{2g}$ *of the isomorphism class of* $L \otimes L(d)^*$. *Then there exists a sequence* $L_n \in \mathrm{Pic}_{\mathrm{alg}}(Y)$ *of algebraic line bundles of degree d and lifts* $f_n \in H^0(\mathbf{C}_{\mathrm{alg}}, \mathbf{0}_{\mathbf{C}\,\mathrm{alg}}) \otimes_\mathbf{C} \mathbf{C}^{2g}$ *of the isomorphism class of* $L_n \otimes L(d)^*$ *such that for every compact subset K of* $\mathbf{C}$ *the sequence* $\{f_n|K\}_{n \in N}$ *converges uniformly to* $f|K$.

Remark 1.6. We believe that 1.5 is a strong result in the following sense. Although it does not give an approximation theorem for the analytic properties of an analytic line bundle, we believe that at least in the case of positive genus no such result is true. Assume $g > 0$. The group $\mathbf{Z}^{\oplus 2g}$ which appears in the description of $\mathrm{Pic}^0(Y)$ is the lattice Γ such that $\mathbf{C}^{2g}/\Gamma \cong \mathrm{Pic}^0(X)$ as complex tori. Since the affine line is rational, there is no non-constant rational map from the affine line into $\mathrm{Pic}^0(X)_{\mathrm{alg}}$. in particular every algebraic map from $\mathbf{C}_{\mathrm{alg}}$ into $\mathrm{Pic}^0(Y)_{\mathrm{alg}}$ is constant. However, there is a huge number of non-constant holomorphic maps from $\mathbf{C}$ into the torus $\mathbf{C}^{2g}/\Gamma \cong \mathrm{Pic}^0(X)$: any such holomorphic map $\mathbf{C} \to \mathrm{Pic}^0(X)$ is uniquely determined by its lifting $\mathbf{C} \to \mathbf{C}^{2g}$ to the universal covering $\mathbf{C}^{2g}$ of $\mathrm{Pic}^0(X)$; conversely any holomorphic map $\mathbf{C} \to \mathbf{C}^{2g}$ gives a holomorphic map $\mathbf{C} \to \mathrm{Pic}^0(X)$. a very nice case occurs when X has genus 1. Consider the universal line bundle, $\mathbf{L}$. over $X \times \mathrm{Pic}^0(X)$. Fix the map $u : \mathbf{C} \to \mathbf{C}/\Gamma \cong \mathrm{Pic}^0(X)$ induced by the identity $\mathbf{C} \to \mathbf{C}$ and use it to obtain a line bundle. L. over Y. Varying t in $\mathbf{C}$ the restrictions $L|X \times \{t\}$ run through every isomorphism class of degree 0 line bundles on X and each isomorphism class appears for infinitely (but countably) many t. We just saw that for every algebraic line bundle. L. on Y the line bundles $L|X \times \{t\}$ are isomorphic for every t.

2. General definitions and the case $Y = X \times U$.

First we give a few definitions for an arbitrary smooth complex surface Y. In the latter part of the section we will consider the case $Y = X \times U$, X a compact Riemann Surface and U an open Riemann Surface. For every analytic coherent sheaf F on Y. set $F^* := Hom(F.\mathbf{0}_Y)$. There is a natural map $j_F : F \to F^{**}.F$ is torsion free if and only if j_F is injective. F is said to be reflexive if j_F is an isomorphism. Since Y is smooth and $\dim(Y) = 2$, every reflexive sheaf on Y is locally free ([10]. Cor. 1.4). Set $\text{rank}(F) := \text{rank}(F^{**})$. Since Y is connected and F^{**} is a vector bundle, this is a well-defined notion of rank. If A is any analytic coherent sheaf, every map $u : A \to F$ induces a bidual map $u^{**} : A^{**} \to F^{**}$.

Definition 2.1. Let F be a torsion free analytic coherent sheaf on Y and set $r := \text{rank}(F)$. If $r = 1$ we will say that F is semi-filtrable. Assume $r \geq 2$. Then the concept of semi-filtrability is defined by induction on r. If there is no pair (L, u) with L coherent torsion free rank 1 sheaf on Y and $u : L \to F$ injective map of sheaves, we will say that F is not semi-filtrable. Assume the existence of L and $u : L \to F$. Let $Tors(F/u(L))$ be the torsion part of $F/u(L)$ and let M be the kernel of the composition of the two surjections $F \to F/u(L) \to (F/u(L))/Tors(F/u(L))$. We have a natural inclusion of L in M and M is a rank 1 torsion free sheaf. F/M is a torsion free sheaf with $\text{rank}(F/M) = r - 1$. We will say that F is semi-filtrable if there exist L and u such that the associated torsion free sheaf F/M is semi-filtrable. If $\text{rank}(F) = 1$ we will say that F is weakly filtrable. If $r := \text{rank}(F) \geq 2$ we will define inductively that F is weakly filtrable if there exists L and u with M line bundle and such that F/M is torsion free and weakly filtrable.

Remark 2.2. Notice that F is semi-filtrable if and only if F^{**} is semi-filtrable and that if F is weakly filtrable, then F^{**} is weakly filtrable.

Remark 2.3. If Y is compact the notion of semi-filtrability is called filtrability in [3]. p. 91. If Y is projective, then every vector bundle on Y is weakly filtrable

([3]. p. 91) but there are complex compact surface (even Kähler ones) which admits non semi-filtrable rank two vector bundles ([3]. Ex 4.26).

From now on in this section we will study the case $Y = X \times U$ with X compact connected Riemann Surface of genus g and U open connected Riemann Surface. Thus the projection $\pi : Y \to U$ is the Remmert reduction of Y. We will study again this type of surface in section 4.

Lemma 2.4. *Let E be a rank r vector bundle on X. If* $\deg(F) > r(g-1)$*, then* $h^0(X, F) \neq 0$.

Proof. By Riemann - Roch (see e. g. [8]) we have $h^0(X, F) - h^1(X, F) = \deg(F) + r(1-g) > 0$.

Proposition 2.5. *Every coherent torsion free sheaf on $Y := X \times U$ is semi-filtrable. Every rank 2 holomorphic vector bundle on Y is weakly filtrable.*

Proof. Fix a coherent torsion free sheaf F on Y and set $r := \text{rank}(F)$. If $r = 1$ the result is obvious. Hence we may assume $r \geq 2$ and assume the result for all torsion free coherent sheaves of lower rank. By Remark 2.2 we may assume F locally free. Furthermore, it is equivalent to prove the result for $F \otimes L$ with $L \in \text{Pic}(Y)$. Fix $P \in X$ and set $d := \deg(det(F))$. By twisting by a power of the line bundle $\mathbf{0}_Y(\{P\} \times \mathbf{C})$, we may assume $d > r(g-1)$. Hence $H^0(X, F|X \times \{t\}) \neq 0$ for every t. Furthermore, we see that the coherent sheaf $\pi_*(F)$ is not zero. Thus $H^0(Y, F) \neq \{0\}$. Take a non-zero homomorphism $u : \mathbf{0}_Y \to F$ and let M be the kernel of the quotient map $F \to F/u(\mathbf{0}_Y)/(Tors(F/u(\mathbf{0}_Y)))$. Since F/M is torsion free of rank $r-1$, F/M is semi-filtrable by the inductive assumption. Hence F is semi-filtrable by its very definition. If F is locally free and $\text{rank}(F) = 2$, then the bidual of the inclusion map $M \to F$ shows that M is reflexive and hence a line bundle ([10], Prop. 1.9). Hence F is weakly filtrable.

Remark 2.6. By 2.5 every rank 2 vector bundle F on Y fits in an exact sequence

$$0 \to M \to F \to R \otimes \mathbf{I}_Z \to 0 \tag{2}$$

with $M \in \text{Pic}(Y)$, $R \in \text{Pic}(Y)$ and Z zero-dimensional closed complex subspace (in general unreduced) of Y. Since F is locally free of rank 2, for every $P \in Z_{red}$ the ideal sheaf of Z is locally generated by two elements. Since Y is smooth, this is equivalent to the condition that Z is locally a complete intersection in Y.

3. The general case.

Theorem 3.1. *Let Y be a two-dimensional irreducible and reduced holomorphically convex complex space and $\pi : Y \to U$ its Remmert reduction. Assume π projective in the sense of [2], Ch. IV. Fix a torsion free sheaf F on Y. Then for every relatively compact open subset Ω of U the sheaf $F|\pi^{-1}(\Omega)$ is semi-filtrable. If F is locally free, and* $\text{rank}(F) = 2$ *then $F|\pi^{-1}(\Omega)$ is weakly filtrable.*

Proof. If Y is compact, i.e. if U is a point, then Y is projective by the assumption on π. This case is well-known ([3]. p. 91.) Assume $\dim(U) = 1$. Thus U is a Stein irreducible curve. Every torsion free sheaf on any subset of U is spanned by its global sections. By definition of projective morphism there is a line bundle H on Y which is π-ample. Thus by a theorem of Grauert and Remmert ([2]. Th. IV. 2.1) for every relatively compact open subset Ω of U there in an integer $n(F, \Omega)$ such that for all integers $n \geq n(F, \Omega)$ the natural map $\pi^*((\pi_*|\pi^{-1}(\Omega))((F \otimes H^{\otimes n})|\pi^{-1}(\Omega))) \to F \otimes H^{\otimes n})|\pi^{-1}(\Omega)$ is surjective. Hence $\pi_*F \otimes H^{\otimes n})|\Omega$ is not zero. Thus $\pi_*(F \otimes H^{\otimes n})|\Omega$ has a non-trivial section and we may repeat the inductive proof of Proposition 2.5. Now assume $\dim(U) = 2$, i.e. assume Y0-convex. This case is well-known without any assumption on π and may be checked in a very similar way: here we have a stronger result, i.e. we may take U instead of Ω because π contracts the finite number of compact curves contained in Y, while outside these compact curves π is an isomorphism.

Proposition 3.2. *Let Y be a two-dimensional holomorphically convex irreducible complex space such that its Remmert reduction $\pi : Y \to U$ is a smooth map whose fibers are Riemann surface of genus $g \neq 1$. Then every torsion free coherent sheaf on Y is semi-filtrable.*

Proof. It is well-known that π projective ([2], Cor. IV. 4.4); indeed. if $g \geq 2$ the relative dualizing sheaf if π-ample, while if $g = 0$ the relative tangent sheaf is π-ample. Just applying Theorem 3.1 we would obtain a weaker statement. To obtain the full statement just notice that we may apply Lemma 2.4 and then the proofs of 2.5 and 3.1.

Proposition 3.3. *Consider a two-dimensional holomorphically convex irreducible complex space Y and assume that its Remmert reduction $\pi : Y \to U$ is projective smooth and with as fibers Riemann surface of genus 1. Then every torsion free coherent sheaf on Y is semi-filtrable.*

Proof. By assumption π is projective. Hence, as in the case of 3.2, we may apply 2.4 and copy the proofs of 2.5 and 3.1.

Remark 3.4. Let Y be a two-dimensional holomorphically convex irreducible complex space such that every fiber of its Remmert reduction $\pi : Y \to U$ is a smooth curve of genus 1. The map π is projective if and only if there is a finite surjective map $T \to U$ such that the fiber product map $Y \times_U T \to T$ has a holomorphic section.

Remark 3.5. If Y is smooth in 3.1, 3.2 or 3.3 and F is a rank 2 vector bundle on Y, then, as in Remark 2.6, an exact sequence such as (2) holds.

(3.6) We now assume that Y is smooth and study the exact sequence (2) in the case that Z is a locally complete intersection zero-dimensional analytic subspace of Y and M, R are analytic line bundles. Assume Y holomorphically convex, two-dimensional, connected and not compact and let $\pi : Y \to U$ be its Remmert reduction. Since the case Y0-convex is easy, we assume $\dim(U) = 1$. For every $P \in Z_{red}$ let $Z(P)$ be the connected component of the scheme Z containing $\{P\}$. Thus $Z(P)_{red} = \{P\}$ and Z is the disjoint union of all schemes $Z(P)$, $P \in Z_{red}$. Thus $H^0(Y, \mathbf{0}_Z) \cong H^0(Z, \mathbf{0}_Z) = \Pi_{P \in Z_{red}} H^0(P, Z(P))$; notice that here there is a product, not a direct sum: thus if Z_{red} is infinite, then $H^0(Y, \mathbf{0}_Z)$ considered as a vector space has not a countable basis. Consider the exact sequence

$$0 \to \mathbf{I}_Z \otimes R \to R \to \mathbf{0}_Z \to 0 \tag{3}$$

in which we have identified $R|Z$ with $\mathbf{0}_Z$ because Z is discrete and R is a line bundle. Since Z has codimension 2 in Y, we have $Ext^0(R \otimes \mathbf{I}_Z, M) \cong Hom^0(R \otimes \mathbf{I}_Z, M) \cong M \otimes R^*$; the latter isomorphism follows from Hartogs theorem because Y is smooth, $\dim(Y) > 1$, $M \otimes R^*$ is locally free and Z is discrete. We have $Hom(\mathbf{0}_Z, M) = 0$ because M has no torsion. We have $Ext^i(R, M) = 0$ for every $i \geq 1$ because R and M are locally free. We apply the global Ext-functor $Ext(-, M)$ to the exact sequence (3). We obtain an exact sequence $0 \to H^1(Y, M \otimes R^*) \to Ext^1(Y; R \otimes \mathbf{I}_Z, M) \to Ext^2(Y; R \otimes \mathbf{0}_Z, M) \to H^2(Y, M \otimes R^*)$. Since Y is not compact we have $H^2(Y, M \otimes R^*) = 0$ ([13]). Every locally complete intersection is Gorenstein and in particular Z is Gorenstein. Since Z is Gorenstein and of codimension 2 in Y, we have $Ext^1(R \otimes \mathbf{0}_Z, M) \cong \omega_Z \cong \mathbf{0}_Z$, the latter isomorphism being true because Z is Gorenstein and zero-dimensional. Thus we obtain an exact sequence

$$0 \to H^1(Y, M \otimes R^*) \to Ext^1(Y; R \otimes \mathbf{I}_Z, M) \to H^0(Y, \mathbf{0}_Z) \to 0 \tag{4}$$

Thus $Ext^1(Y; R \otimes \mathbf{I}_Z, M) \cong (\Pi_{P \in Z_{red}} H^0(P, Z(P))) \oplus H_1(Y, M \otimes R^*)$ as topological vector spaces. An extension (2) has middle term locally free if and only if its extension class has a component generating $\mathbf{0}_Z$ at every point of Z_{red}.

4. Rank 2 vector bundles.

In this section we use our techniques in the special case $Y := X \times U$ with X a compact connected Riemann Surface of genus g and U an open connected Riemann Surface. If we want Y to be algebraic we need to assume that $U = V \setminus F$, where V is a connected, smooth compact Riemann Surface and F is a finite and non-empty subset of V.
For every holomorphic map $u : U \to \mathbf{C}$ and every holomorphic vector bundle E on $X \times \mathbf{C}$, we obtain a holomorphic vector bundle $(Id_X, u)^*(E)$ on $X \times U$. If u is surjective and L is a line bundle on $X \times \mathbf{C}$ with the properties described in 1.6, then $(Id_X, u)^*(L)$ has the same properties. Since U is Stein, there are "many" surjective holomorphic maps from U onto $\mathbf{C}$. We recall that every analytic vector bundle on U is trivial ([6], Th. 30.1).

Remark 4.1. Assume U algebraic, say $U = V \setminus F$ with V of genus 0. Then 1.5 works verbatim for $X \times U$.

Remark 4.2. The proofs of 2.5 and 2.6 work verbatim for $X \times U$.

(4.3) Here we use the set-up of 3.6 with $Y := X \times U$. We fix a countable family of open subset $U_n, n \in \mathbf{N}$, of U with U_n relative compact in U_{n-1} for all $n \geq 0$ and $\cup_n U_n = U$. Set $Y_n := X \times U_n$ and $Z[n] := Z \cap Y_n$. First let us consider the case $\deg(M) \geq \deg(R) + 2g - 1$. Hence $\deg(M \otimes R^*) \geq 2g - 1$. Thus $R^1\pi_*(M \otimes R^*) = \{0\}$. Since $\pi_*(M \otimes R^*)$ is a coherent sheaf on U and U is Stein, we obtain $H^1(U, \pi_*(M \otimes R^*)) = 0$. Hence by the Leray spectral sequence of the proper map π we have $H^1(Y, M \otimes R^*) = 0$. Now we study $H^1(Y, M \otimes R^*)$ under the assumption that $\deg(R) > \deg(M)$. Hence for every $t \in U$ we have $h^1(X \times \{t\}, M \otimes R^*|X \times \{t\}) = \deg(R) - \deg(M) + g - 1$ and $h^0(X \times \{t\}, M \otimes R^*|X \times \{t\}) = 0$. Hence by a theorem of changing basis for the projective morphism π (see e.g. [2]; Th. III. 3.1) we have $\pi_*(M \otimes R^*) = 0$ and $R^1\pi_*(M \otimes R^*)$ is a locally free sheaf on U with $x := \mathrm{rank}(R^1\pi_*(M \otimes R^*)) = \deg(R) - \deg(M) + g - 1$. Since U is Stein and one-dimensional, we have $R^1\pi_*(M \otimes R^*) \cong \mathbf{0}_U^{\oplus X}$ ([6], p. 91). Thus $H^1(Y, M \otimes R^*) \cong H^0(U, \mathbf{00}_U)^{\oplus X}$. Now assume $\deg(R) \leq \deg(M) \leq \deg(R) + 2g - 2$. In particular we assume $g > 0$. Here the situation is wild, in the sense that it depends very much on the choice of $M \otimes R^*$, not just on $\deg(M) - \deg(R)$. Set $L := \deg(M \otimes R^*)$ and fix $a := \deg(L)$ with $0 \leq \mathrm{a} \leq 2g - 2$. By Riemann - Roch for every $z \in U$ and every $L \in \mathrm{Pic}(Y)$ with $\deg(L) = a$ we have $h^0(X \times \{t\}, L|X \times \{t\}) - h^1(X \times \{t\}, L|X \times \{t\}) = a + 1 - g$. For every L by the semicontinuity theorem for proper morphisms ([2], Ch. 3) there is a open subset U' of U with $U \setminus U'$ discrete and such that $h^0(X \times \{t\}, L|X \times \{t\})$ and $h^1(X \times \{t\}, L|X \times \{t\})$ are constant for $t \in U'$ while if $z \in (U \setminus U')$

and $t \in U'$ we have $h^0(X \times \{z\}, L|X \times \{z\}) > h^0(X \times \{t\}, L|X \times \{t\})$ and $h^1(X \times \{z\}, L|X \times \{z\}) > h^0(X \times \{t\}, L|X \times \{t\})$. Furthermore, it is easy to find such L with $U' \neq U$ because $\mathrm{Pic}^a(X)$ is a representable functor and hence there is a natural bijection between degree a holomorphic line bundles on $X \times U$ and holomorphic maps $U \to \mathrm{Pic}^a(X)$. Since there are many surjective holomorphic maps $U \to \mathbf{C}$ we may repeat the discussion made in 1.6. In particular for every integer a with $0 \leq a \leq 2g-2$ there exists L such that the discrete set $U \setminus U'$ is infinite and we may prescribe this discrete set.

Remark 4.4. Fix a rank 2 holomorphic vector bundle F on Y. The proofs of 2.5 and 2.6 show that F fits in an exact sequence (2) in which $\deg(R) - \deg(M)$ is a large as we want. In particular we may apply the discussion made in 4.3 to this exact sequence. However, in this way we do not obtain a "canonical" exact sequence or, at least a "minimal" exact sequence and E may fit in different exact sequences with different (or even with equal) M, R and Z.

(4.5) Here we specialize the set-up of 4.3 to the case $U = \mathbf{C}$. We assume also $\deg(M) \geq \deg(R) + 2g - 1$. Hence $H^1(Y, M \otimes R^*) \cong H^0(\mathbf{C}, \mathbf{0_C})^{\oplus x}$ with $x := \deg(M) - \deg(R) + 1 - g$. Fix the class $\mathbf{e} \in Ext^1(Y : R \otimes \mathbf{I}_Z, M) \cong (\Pi_{P \in Z_{red}} H^0(P, Z(P))) \oplus H^1(Y, M \otimes R^*)$ of (2) and call $\mathbf{f}$ (resp.$\mathbf{t}$) its component in $\Pi_{P \in Z_{red}} H^0(P, Z(P))$ (resp. $H^1(Y, M \otimes R^*)$). Let $\Delta(0, r)$ be the open disk of $\mathbf{C}$ with center 0 and radius r. For every integer $n > 0$, set $Y_n := X \times \Delta(0, n)$ and $Z[n] := Z \cap Y_n$. The component $\mathbf{f}$ is a "projective limit" of the components $\Pi_{P \in Z[n]_{red}} H^0(P, Z(P))$, $n > 0$, in the sense of truncation of the components taking 0 in the components not in $Z[n]_{red}$. We may approximate the element $\mathbf{t} \in H^0(\mathbf{C}, \mathbf{0}_C)^{\oplus}_X$ by a sequence whose components are x polynomials. Notice that if $M \otimes R^*$ is algebraic, we obtain, just as in 3.6. that $H^1(Y_{\mathrm{alg}}, (M \otimes R^*)_{\mathrm{alg}}) \cong H^0(\mathbf{C}_{\mathrm{alg}}, \mathbf{0}_{\mathbf{C}\,\mathrm{alg}})^{\oplus x} \cong \mathbf{C}[z]^{\oplus x}$, i.e. we may approximate the extension class $\mathbf{t}$ by a sequence of algebraic extensions. If $M \otimes R^*$ is not algebraic, we approximate M and R by algebraic line bundles in the sense of 1.5. Then the approximation of the corresponding extension is associated to a family of rank 2 algebraic vector bundles which approximates the original analytic vector bundle. We need only to justify why the approximated extensions, say $\{F_\alpha\}$, give locally free sheaves and not just torsion free sheaf. This is a problem only on Z_{red}. Fix $P \in Z_{red}$. Now we explain why in our set-up we have a Cayley - Bacharach type condition for the local freeness of F and F_α at P and that this condition sits in the component over $Z(P)$ of $\Pi_{P \in Z[n]_{red}} H^0(P, Z(P))$. For the case of a compact surface, see [5]. Th. 1.4. In our set-up we use that $H^2(Y, A) = 0$ for every coherent analytic sheaf A on Y ([13]) and in particular $H^2(Y, M \otimes R^*) = 0$. Fix $P \in Z_{red}$ and set $z := h^0(Z(P), \mathbf{O}_{Z(P)}) = lengh(Z(P))$. As $\mathbf{C}$-vector spaces $\omega_{Z(P)}$ and $\mathbf{O}_{Z(P)}$ are isomorphic to $\mathbf{C}^z$ while their socle, $\sum$, is a one-dimensional linear subspace of $\omega_{Z(P)} \cong \mathbf{O}_{Z(P)}$ because $Z(P)$ is Gorenstein. The component of the extension over $Z(P)$ is given by a linear map $\lambda : \mathbf{C}^Z \to \mathbf{C}$ (see [4]. p. 15 or [5]). The corresponding extension is locally free at P if and only if $\lambda(\sum) \neq 0$ (see [5], lines 3-5 of p. 69). By assumption this condition is satisfied by F. Call $Z[n]$ the subscheme of Z used for F_α; if $P \in Z[n]_{red}$, then F_α is an extension of two line bundles near P and hence it is locally free near P, if $P \in Z[n]_{red}$, then the extension of F and F_α have the same component in the product $\Pi_{P\in Z[n]_{red}} H^0(P, Z(P))$ and hence F_α is locally free, too.

Remark 4.6. Consider now the set-up of 3.6 (i.e. Y smooth and Z a zero-dimensional locally complete intersection) but use the description of the Cayley - Bacharach condition given in 4.5. For every $P \in Z_{red}$, set $z(P) := length(Z(P))$ and let $\sum(p) \cong \mathbf{C} \subseteq \mathbf{O}_{Z(P)} \cong \mathbf{C}^{z(P)}$ be the socle of $Z(P)$ (see [4], p. 15). Notice that if $\lambda = (\lambda_p)_{P\in Z_{red}}$ in the product $\Pi_{P\in Z_{red}} H^0(P, Z(P))$ is an extension class, where $\lambda p : \mathbf{C}^{z(P)} \to \mathbf{C}$ is a linear map with $\lambda_p(\sum(p)) \neq \{0\}$, then the extension is locally free. The "construction" of all locally complete intersection discrete analytic subspaces of Y is elementary. The data of the extension classes λ are just trivial algebra. Hence we obtain in this way a huge amount of analytic rank 2 vector bundles. In the next remark we will see how to use the description given in 4.5 to control their geometric properties.

(4.7) Here we take the set-up of 4.3 with $\deg(M) \geq \deg(R) + 2g - 1$. Hence we have $H^1(Y, M \otimes R^*) = 0$. Here we fix any coherent sheaf F fitting in (2). Set $\Omega := U \setminus \pi(Z_red)$. Since π is proper and Z_{red} is discrete in Y, Ω is open and dense in U. Consider the restriction of (2) to $X \times \Omega$. By the assumption on $\deg(M) - \deg(R)$ for every $Q \in \Omega$ the restriction to $X \times \{Q\}$ of (2) splits and $F|X \times \{Q\} \cong (M|X \times \{Q\}) \oplus (R|X \times \{Q\})$. As in (3.6) (or using the local-to-global spectral sequence for the Ext-functors) we obtain $Ext^1(X \times \Omega : R|X \times \Omega, M|X \times \Omega) = 0$. Hence $F|X \times \Omega \cong (M|X \times \Omega) \oplus (R|X \times \Omega)$. now fix $P \in \pi(Z_{red})$. If F is locally free in a neighborhood of $X \times \{P\}$, then the inclusion $L|X \times \{P\} \to F|X \times \{P\}$ is not an embedding of line bundles, i.e. its cokernel is not locally free and the length of the torsion of the support of the cokernel is at least the length of the zero-dimensional scheme $Z \cap X \times \{P\}$. In the particular case in which Z is reduced, then these two integers are the same and if $Z \cap X \times \{P\} = \cup_{1\leq i\leq t}(P_1, P)$, then $F|X \times \{P\}$ is an extension of $(R|X \times \{P\})(-\sum_{1\leq i\leq t} P_1)$ by $(M|X \times \{P\})(\sum_{1\leq i\leq t} P_1)$. Since $\deg(M) \geq \deg(R) + 2g - 1$, we have $H^1(X \times \{P\}, (M \otimes R^*|X \times \{P\})(\sum_{1\leq i\leq t} 2P_1)) = 0$. Hence this extension splits. Thus taking Z reduced we may find a rank 2 holomorphic vector bundle F fitting in (2) and with essentially all possible splitting types for its restriction to a discrete (but perhaps infinite) subset of

U. Under the assumption $\deg(M) \geq (R) + 2g - 1$ if there is no jumping in the degree of the splitting types of the bundles $\{F|X \times \{t\}\}_{t \in U}$, we must have $Z = \emptyset$ and hence $F \cong M \oplus R$.

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