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WARDROP EQUILIBRIA IN AN INFINITE NETWORK

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In a finite network, there is a classical theory of traffic flow, which gives existence of a Wardrop equilibrium for given OD demands and affine route costs. In this note, the existence theory is extended to infinite networks.

1. Introduction.

We define a classical model of Wardrop equilibrium for traffic flow [2]. Let (N, B) be a finite directed graph, with node set N and branch or link set B. A path in which all links are similarly directed is called a route, with the initial and final nodes forming an origin/destination or O/D pair. Consider a nonempty set W of O/D pairs, and for each $w \in W$, suppose a flow demand $d_w > 0$ to be given. Let R_w be the set of routes joining w. For each $w \in W$, consider $F_r \geq 0$ for each $r \in R_w$, such that $\sum_{r \in R_w} F_r = d_w$, giving a route flow vector $F = (F_r)_{r \in R_w, w \in W}$. This route flow induces a link flow $f = (f_b)_{b \in B}$, by $f_b = \sum_{r \ni b} F_r$ for each b, where we identify a route with the set of its links. For each link a, suppose a link cost $c_a = \sum_{b \in B} g_{ab} f_b + h_a$, where g_{ab} and h_a are given. For $r \in R_w$, $w \in W$, define a route cost by $C_r = \sum_{a \in r} c_a$.

A route flow H is a Wardrop equilibrium if it satisfies the condition that for all $r, s \in R_w$, $w \in W$, if $C_r < C_s$ then $H_s = 0$. In other words there is, for

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each w, a common route cost γ_w for all routes $r \in R_w$ with nonzero H_r. There exists a Wardrop equilibrium [4].

In this paper we shall let (N, B) be an infinite directed graph. Our mission is to extend our concepts in a natural way, so that we find that Wardrop equilibria exist. In Section 2 we take the most naive extensions, and give counterexamples to show the need for infinte routes. In Section 3 we show that allowing infinite routes is still not enough, and end with the promise to give Wardrop flows as measures. Some background on infinite routes is given in Section 4. Section 5 addresses a well known variational inequality in the setting of the pairing between Borel measures and continuous functions on the closure of the routes. In Section 6 we define a Wardrop equilibrium, and show that it exists, by taking the completion of the measure given by the variational inequality, and restricting this to the routes.

2. The need for a 1-network.

Recall an infinite graph is locally finite if each node has only finitely many links incident to it. We shall consider an infinite directed graph, allowing only finite routes, but a possibly infinite set of O/D pairs W. Of course, even with a single O/D pair, o, d, it may be that there are infinitely many routes from "o" to "d".

Assumption 1. Let $G = (N, B)$ be an infinite, locally finite, and connected directed graph.

Assumption 2. $d \in \ell l_1(W)$, i.e $\sum_{w \in W} d_w < \infty$.

Remark 1. Let $R = \bigcup_{w \in W} R_w$. Under Assumption 2, define $K = \{F : R \to [0, \infty) :$ for each w, $\sum_{r \in R_w} F_r = d_w\}$. Later we shall replace this definition. If $F \in K$ then link flows are finite, since for $b \in B$,

$$f_b = \sum_{r \ni b} F_r \leq \sum_{r \in R} F_r = \sum_{w \in W} d_w.$$

We keep the same definition of link costs, but it involves an infinite sum.

Temporary Assumption 3. Assume for each link b, $\sum_{a \in B} |g_{b,a}| < \infty$.

Suppose $F \in K$. Then, for each link b,

$$c_b = e_b + \sum_{a \in B} g_{b,a} f_a^F$$

is finite, by Remark 1. Given F, we may write f^F, c^F, and C^F to remind us that they are induced by F.

For each route r, the route cost $C_r^F = \sum_{b \in r} c_b$ is finite, since r is finite. Hence the definition of Wardrop equilibrium needs no change, under the above assumptions.

The first question is whether there exists a Wardrop equilibrium, considering only finite routes.

Counterexample 1. Consider the infinite ladder with nodes at points on the plane $(n, 0)$ and $(n, 1)$ for $n \geq 0$. The nth top link from $(n-1, 1)$ to $(n, 1)$ is denoted nt, the nth bottom one nb, and the nth vertical one from $(n-1, 1)$ to $(n-1, 0)$, $(n-1)v$. Let the top links point right, the vertical ones down, and the bottom ones left. Let the link cost be given in terms of link flow f by

$$c = a_n + b_n f$$

on each of nt, nb and nv, where a_n and b_n are assumed non-negative. Suppose we have one OD pair, from $(0, 1)$ to $(0, 0)$, with demand $d = 1$. Suppose that for all n, $3(a_{n+1} + b_{n+1}) < a_n$. Then there is no Wardrop equilibrium.

Let the routes be indexed by the vertical links nv. Suppose F is a Wardrop equilibrium, given by F_n on each nv. For any $n \geq 0$, let the route cost on route nv be C_n. Let the link cost on any link a be c_a. Then

$$\begin{aligned} C_{n+1} &= C_n - c_{nv} + c_{(n+1)t} + c_{(n+1)b} + c_{(n+1)v} \\ &\leq C_n + 3(a_{n+1} + b_{n+1}) - a_n \\ &< C_n. \end{aligned}$$

Since F is a Wardrop equilibrium, $F_n = 0$. But then summing the F_n gives zero instead of d, a contradiction.

What we shall do is define a generalized Wardrop equilibrium, in this case being a flow of 1 out along the top infinite route and back along the bottom route. To formalize this we may use a little of the concept of "1-networks" from [1]. In this case, we merely add on an ideal node n_1, (the end) and regard the top infinite route as being from $(0, 1)$ to n_1, and the bottom from n_1 back to $(0, 0)$. That is, we have allowed infinite routes.

3. The need for route flows as measures.

Even this construction of 1-networks is not enough in general, and the next example and counterexample provoke a concept of flow which will give existence. We have just shown that it is possible that merely allowing infinite routes can give a Wardrop equilibrium. We now look at two almost identical situations: Example 1 for which it is enough to allow infinite routes and give a route flow as a summable function on these, and Counterexample 2 for which it is not enough.

Example 1. Consider the graph formed from the infinite ladder above by identifying nodes $(n, 0)$ and $(n, 1)$ for all n, and eliminating nv, and re-orienting links nb to point right. That is, we have two links, nt and nb in parallel from $(n-1, 0)$ to $(n, 0)$, for each $n \geq 1$. As above we have n_1, and we suppose we have one OD pair, from $(0, 0)$ to n_1, with demand $d = 1$, joined by infinite routes. Let the link cost be given in terms of link flow f by

$$c = e_b + g_b f$$

on each link b, where e_b and g_b are nonnegative. Suppose e and g are summable, so that route costs, obtained by summing their link costs, are finite. There does exist a Wardrop equilibrium.

Let us define $\delta_{a,r}$ for any link a and route r to be 1 if $a \in r$ and 0 otherwise. First, for for each n, there is a Wardrop equilibrium on the subnetwork with links nt and nb, giving two link flows f_{nt} and f_{nb}, summing to 1. Take route $r1$ such that for each n, nt is in it iff $f_{nt} \geq 1/2$, and take route flow $F_{r1} = 1/2$. Then take route $r2$ such that for each n, nt is in it iff $f_{nt} - \delta_{nt,r1} F_{r1} \geq 1/4$, and take route flow $F_{r2} = 1/4$. Continuing, we have a route flow vector $F = \{F_{rn} : n = 1, \ldots\}$. These give the link flow vector $\{f_b : b \in B\}$, and therefore route costs are all equal on the routes rn, and F is a Wardrop equilibrium.

This looks encouraging, and we are led to look for a flow F as a summable function on R, that is, as an element of K from Remark 1. Using this concept, we obtain a counterexample to a claim of existence of a Wardrop equilibrium.

Counterexample 2. Consider a directed graph as in the last example, but with n links in parallel from $(n-1, 0)$ to $(n, 0)$, for each n. As above we have n_1, and we suppose we have one OD pair, from $(0, 0)$ to n_1, with demand $d = 1$. Let the link cost be given in terms of link flow f by

$$c = e_b + g_b f$$

on each link b, where e_b and g_b are positive, with both e_b all equal and g_b all equal if b is from $(n-1,0)$ to $(n,0)$. Suppose e and g are summable, so that route costs are finite. There does not exist a Wardrop equilibrium.

Any Wardrop equilibrium F gives an equilibrium on the links from $(n-1,0)$ to $(n,0)$. These give link flows which must be equal on the links from $(n-1,0)$ to $(n,0)$, and thus take the value $1/n$. For any route r, $F_r \leq 1/n$ for all n. Therefore $F_r = 0$, contradicting $\sum F_r = d$.

What we shall do is define a Wardrop equilibrium as a measure on a sigma field of subsets of R, to give existence. First we develop some theory.

4. Notation and definitions.

We take $s : B \to (0,\infty)$ a summable function, after noting that B is countable by Assumption 1.. Define the metric d^0 on N by $d^0(x,x) = 0$ and

$$d^0(x,y) \;=\; \inf\left\{\sum_{b\in B} s_b \,:\, P \text{ a path connecting } x \text{ and } y\right\}.$$

Write $(\widehat{N}, d^0)$ for the completion, and call any limit point or cluster point (i.e., any element of $\widehat{N}'$) an end of (N,B). We now allow infinite routes, which may be between a node $x \in N$ and an end $e \in \widehat{N}'$, or even between two ends, with the initial and final nodes or ends forming an O/D pair. We say that an infinite route is between two ends e_1 and e_2 if its nodes n_k converge to these in $(\widehat{N}, d^0)$ as k goes to plus or minus infinity. Likewise for an infinite route between a node x and an end e. We shall identify a route with its branch set. We also allow 1-routes, made by concatenating a finite number of infinite routes, the destination of one infinite route being the origin of the next, each node and each end only visited once on a 1-route. Counterexample 1 gives a typical 1-route, the top infinite route followed by the bottom infinite route. Let R stand for the union of all the routes, both finite and infinite, including the 1-routes.

Let S be the metric space of subsets of B with metric

$$d(U,V) = \sum\{s_b : b \in (U \setminus V) \cup (V \setminus U)\}$$

5. Solution of a variational inequality.

Proposition 1. *S is compact.*

Proof. S is a complete metric space [5], chapter II,2, Proposition. We need only show S is totally bounded. Let $\epsilon > 0$ be given. Take H so that $\sum\{s_b | b \notin H\} < \epsilon$. For $P \in S$, the distance $d(P, P \cap H) < \epsilon$. Note $\{P \cap H | P \in R\}$ is finite, and we have a finite ϵ-net. □

We recall that the dual of the Banach space of continuous real valued functions on a compact metric space is the space of signed Borel measures [5]. Now we update our definition of K. Noting $R \subset S$, let $cl(R)$ denote the closure in S of R.

Definition. Let $K = \{\mu \in C(cl(R); \mathbb{R})' : \mu \geq 0, \mu(cl(R_w) = d_w$ for all $w \in W\}$.

The next proposition extends a result of [4].

Proposition 2. *Let $H \in K$, $w \in W$, and $C \in C(cl(R); \mathbb{R})$. The following are equivalent.*

(1) $H\{r \in cl(R_w) : C(r) > inf\{C(x) : x \in cl(R_w)\}\} = 0.$

(2) $\displaystyle\int_{cl(R_w)} C(r)\, d(F - H)(r) \geq 0$ *for all* $F \in K$.

Proof. (1) $\Rightarrow$ (2) Let $\gamma_w = \inf\{C(x) : x \in cl(R_w)\}$. Let $F \in K$ be given. We want to show

$$\int_{cl(R_w)} C(r)\, dF(r) \geq \int_{cl(R_w)} C(r)\, dH(r),$$

or equivalently

$$\int_{cl(R_w)} (C(r) - \gamma_w)\ df(r) \geq \int_{cl(R_w)} (C(r) - \gamma_w)\, dH(r).$$

But $RHS = 0$ by (1), and $LHS \geq 0$ since $F \geq 0$.

(2) $\Rightarrow$ (1) Suppose (1) does not hold. Let $T = \{r \in cl(R_w) : C(r) > \gamma_w\}$. Then take $b^* \in cl(R_w)$ with $C(b^*) = \gamma_w$. Define F by

$$F = H|_{cl(R)\setminus T} + 0.5H|_T + 0.5H(T)\delta_{b^*}.$$

Note $F \geq 0$, and

$$F(cl(R_w)) = H(cl(R_w) \setminus T) + 0.5H(T) + 0.5H(T)$$

$$= H(cl(R_w)).$$

Note that for $v \in W \setminus \{w\}$, since R_v and R_w are a positive distance apart,

$$F(cl(R_v)) = H(cl(R_v)),$$

and hence $F \in K$. Then

$$\int_{cl(R_w)} C(r)\, d(F - H)(r) = -0.5 \int_T C(r)\, d(H)(r)$$

$$+ 0.5 H(T) C(b^*)$$

$$= -0.5 \int_T (C(r) - \gamma_w)\, d(H)(r)$$

$$< 0 \text{ since } H(T) > 0, \text{ and } C(r) - \gamma_w > 0 \text{ on } T,$$

and (2) does not hold. $\square$

Note that for $b \in B$, $\{r \in cl(R) : r \ni b\}$ is open in $cl(R)$, in particular a Borel set, allowing the next definition.

Definition. For $F \in K$, let us define $f_b = f_b^F$ for $b \in B$ by $f_b = F\{r \in cl(R) : r \ni b\}$, then $c_b = c_b^F$ by $c_b = e_b + \sum_{a \in B} g_{b,a} f_a$, and then for $r \in S$, $C(r) = C^F(r)$ is defined as $C(r) = \sum_{b \in r} c_b$, giving $C^F : S \to \mathbb{R}$.

Note that the link flows are bounded, for

$$f_b = \sum_{w \in W} F\{r \in cl(R_w) : r \ni b\} \leq \sum_{w \in W} d_w.$$

The link costs are finite by Temporary Assumption 3. The route costs are now infinite sums, but are finite by the following Assumption 3, a stronger assumption than the temporary one.

Assumption 3. Both e and g are summable, g is nonnegative and all $e_b > 0$, and $\sum_{a,b \in B} g_{b,a} f_a f_b \geq 0$ for a bounded real valued function f on B.

Proposition 3. *Let $H \in K$. C^H is continuous on S.*

Proof. Let $\epsilon > 0$ be given. Using Assumption 3, we take a finite $B^f \subset B$, such that $\sum_{b \notin B^f} e_b < \epsilon_1$, and $\sum_{a \in B, b \notin B^f} g_{b,a} < \epsilon_1$, where ϵ_1 is to be chosen. Take δ to be $\min\{s_b : b \in B^f\}$. Let r and r^* be in S with $d(r, r^*) < \delta$. Then $r \cap B^f = r^* \cap B^f$. Therefore

$$C^H(r) - C^H(r^*) = \sum_{b \in r \setminus B^f} c_b - \sum_{b \in r^* \setminus B^f} c_b,$$

and as noted after the previous definition, for any b,

$$f_b^H \leq \sum_{w \in W} d_w,$$

giving

$$\begin{aligned} |C^H(r) - C^H(r^*)| &\leq \sum_{b \notin B^f} [e_b + \sum_{a \in B} g_{b,a} f_b^H] \\ &< \epsilon_1 + (\sum_{w \in W} d_w) \sum_{b \notin B^f, a \in B} g_{b,a} \\ &\leq \epsilon_1 + (\sum_{w \in W} d_w)\epsilon_1 \\ &= \epsilon, \end{aligned}$$

choosing ϵ_1 so that the last equality holds. Thus C^H is uniformly continuous on S. $\square$

Let us denote the restriction of C^H to $cl(R)$ by C^H.

Proposition 4. *The map $C : K \to C(cl(R); \mathbb{R})$, mapping H to C^H, is monotone.*

Proof. Let F and H be in K. For $a \in B$, let $S(a) = \{r \in cl(R) : r \ni a\}$. Then

$$\begin{aligned} (C^F - C^H, F - H) &= \int_{r \in cl(R)} (C^F(r) - C^H(r))\, d(F - H)(r) \\ &= \int_{r \in cl(R)} (\sum_{a,b \in B} g_{a,b} \chi_r(a)((F - H)S(b)))\, d(F - H)(r) \\ &= (\sum_{a,b \in B} g_{a,b} \int_{r \in cl(R)} \chi_r(a)((F - H)S(b))\, d(F - H)(r) \end{aligned}$$

by Fubini

$$= \sum_{a,b\in B} g_{a,b} \int_{r\in S(a)} (F-H)S(b)\, d(F-H)(r)$$

$$= \sum_{a,b\in B} g_{a,b}[(F-H)(S(a))][(F-H)(S(b))]$$

$$\geq 0.$$

□

Proposition 5. *For $F, H \in K$, the map $Z \mapsto (C^Z, F-H)$ is continuous on the line segment $[F, H]$.*

Proof. We put, for $t \in [0, 1]$, $Z = (1-t)F + tH$, and consider

$$t \mapsto \int_{r\in cl(R)} C^Z(r)\, d(H-F)(r) = \int_{r\in cl(R)} \sum_{a\in r} c_a^Z\, d(H-F)(r)$$

$$= \int_{r\in cl(R)} \sum_{a\in r} (e_a + \sum_{b\in B} g_{a,b} Z(S(b))\, d(H-F)(r)$$

$$= \int_{r\in cl(R)} \sum_{a\in B} \chi_r(a)(e_a + \sum_{b\in B} g_{a,b}[tH(S(b)) + (1-t)F(S(b)])\, d(H-F)(r)$$

$$= t \int_{r\in cl(R)} \sum_{a\in B} \chi_r(a)(e_a + \sum_{b} g_{a,b} H(S(b))\, d(H-F)(r)$$

$$+ (1-t) \int_{r\in cl(R)} \sum_{a\in B} \chi_r(a)(e_a + \sum_{b} g_{a,b} F(S(b))\, d(H-F)(r).$$

This is continuous. □

We recall the following corollary of the Debrunner-Flor Lemma [3].

Proposition 6. *Let E be a real locally convex topological vector space and let K be a nonempty compact convex subset of E. Suppose $C : K \to E'$ is monotone and for $x, y \in K$, the map $z \mapsto (C(z), y-x)$ is continuous on the line segment $[x, y]$. Then there exists $x \in K$ such that for all $y \in K$, $(C(x), y-x) \geq 0$.*

Proposition 7. *Suppose Assumptions 1,2, and 3 hold. There exists $H \in K$ such that H and $C^H : cl(R) \to \mathbb{R}$ satisfy (1) and (2) of Proposition 2.*

Proof. Use the Debrunner-Flor result with E the space of signed Borel measures on $cl(R)$, with the weak* topology on $E = C(cl(R); \mathbb{R})'$, and $K = \{\mu \in C(cl(R); \mathbb{R})' : \mu \geq 0, \mu(cl(R_w) = d_w$ for all $w \in W\}$. For $H \in K$, the map $F \mapsto (C^H, F)$ is a weak* continuous linear functional, and we have shown the required monotonicity and continuity. □

6. Wardrop equilibria and their existence.

We now look at the relationship between $R_{o,d}$ and its closure, for o, d an O/D pair. This allows us to consider the completion H_0 of H on R.

Notation. Let (N, B) be a connected locally finite directed graph, with finitely many ends. For F a finite subset of B, and e an end, we write $e(F)$ for the infinite component of $(N, B \setminus F)$ with e as limit point. If G is a subgraph of (N, B), we write $E(G)$ for its link set and $V(G)$ for its node set. A 1-walk will be like a 1-route, a concatenation of infinite routes, except we may visit ends more than once.

Theorem 8. *Let (N, B) be a connected locally finite directed graph, with finitely many ends. Let o and $d (d \neq o)$ be in $\widehat{N}$. For $r \in cl(R_{o,d})$, there exists $s \in R_{o,d}$ with $s \subset r$.*

Proof. Suppose for now that o and d are ends, to fix ideas. Take a finite $F \subset B$, such that $e_1(F) \notin e_2(F)$ if e_1 and e_2 are distinct ends. Take a sequence $r_k \to r$, with $r_k \in R_{o,d}$, and take k_0 such that for all $k \geq k_0$, $r_k \cap F = r \cap F$. This intersection is a finite union of finite routes p_i. We enlarge F to F^* by including all finite routes q contained in r with $q \cap F = \emptyset$, and origin and destination in F. By taking k_0 larger if necessary, assume $r_k \cap F^* = r \cap F^*$ for all $k \geq k_0$. There are now no finite routes q contained in r with $q \cap F^* = \emptyset$ and having origin and destination in F^*. Let $F^* \cap r_k$, for $k \geq k_0$, be the finite union of finite routes p_i, from o_i to d_i (all distinct), $i = 1, \ldots, L$. By taking a subsequence, we assume that all r_k traverse the routes p_i in the same order, say p_1 to p_L, and $k_0 = 1$.

We claim there is a 1-route in r from d_{i-1} to o_i for each i from 2 to L, and infinite routes from o to o_1 and from d_L to d. For all k, the links of r_k from d_{i-1} to o_i form a finite route or 1-route, denoted r_k^i. Since r_k traverses the p_j in order 1 to L, the links of r_k^i are all in $E(e(F^*))$ for some end e. Now suppose thare is a finite $H \subset B$, with $F^* \subset H$, such that for all k, r_k^i stays in $E(e(F^*)) \cap H$. Then there exists a finite route contained in $r \cap E(e(F^*))$, with origin and destination in F^*, which is impossible. Hence for each such H, r_k^i has some links $(r_k^i)_b$ in $E(e(H))$. This gives the existence of a route q^i contained in r from d_{i-1} to e, because given any finite subset H of B containing F^*, $H \cap r^i_k$ is constant for large k, and we take q_H to be the component of $H \cap r^i_k$ with origin d_{i-1}, and let q^i be the union of the q_H. Similarly, there exists a route contained in r from e to o_i. Together, these two infinite routes give a 1-route in r from d_{i-1} to o_i. Similarly there are infinite routes in r from o to o_1 and from d_L to d.

Note that an end e may be approximated more than once by nodes of the r_k. It follows that there exists a 1-walk w in r from o to d, noting all nodes are only visited once, since each link of w is in r_k for k large. Hence there exists a 1-route s in r from o to d. The cases where one or both of o and d are nodes can be dealt with by adjusting the preceding argument. □

Proposition 9. *Suppose Assumptions 1,2, and 3 hold. Let H and $C^H : cl(R) \to \mathbb{R}$ satisfy (1) and (2) of Proposition 2, and let $w \in W$ be given. Then R_w is measurable with respect to the completion $(cl(R), \mathcal{B}_0, H_0)$ of the measure space $(cl(R), \mathcal{B}, H)$, where $\mathcal{B}$ denotes the Borel sets, and $H_0(R_w) = d_w$.*

Proof. We claim $cl(R_w) \setminus R_w$ is contained in a Borel set of H measure zero. It suffices to show that if $r \in cl(R_w) \setminus R_w$, then $C^H(r) > C^H(s)$ for some $s \in cl(R_w)$. Now for $r \in cl(R_w) \setminus R_w$, r contains a 1-route $s \neq r$, by Theorem 7. Then

$$\begin{aligned} C(r) &= C(s) + \sum_{a \in r \setminus s} c_a \\ &\geq C(s) + \sum_{a \in r \setminus s} e_a \\ &> C(s). \end{aligned}$$

□

Definition. Suppose Assumptions 1,2 and 3 hold. Suppose there is a measure space $(R, \mathcal{B}_1, H_1)$ where B_1 contains the Borel σ - field of R, such that $H_1(R_w) = d_w$ and

$$H_1\{r \in R_w : C^{H_1}(r) > inf\{C^{H_1}(x) : x \in R_w\}\} = 0,$$

for all $w \in W$, where $C^{H_1} : R \to \mathbb{R}$ is given by: for all $b \in B$, $f_b = H_1\{r \in R : r \ni b\}$, $c_b = e_b + \sum_{a \in B} g_{b,a} f_a$, and for $r \in R$, $C^{H_1}(r) = \sum_{b \in r} c_b$. Then we say H_1 is a Wardrop equilibrium.

Theorem 10. *Suppose Assumptions 1,2 and 3 hold. There exists a Wardrop equilibrium, H_1.*

Proof. By Proposition 9, taking $(\mathcal{B}_1, H_1)$ to be the restriction to R of $(cl(R), \mathcal{B}_0, H_0)$. $\square$

Question. Is there a version of this with the map from link flows to link costs being non affine?

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