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FORMAL (q-)EULER INTEGRALS OVER THE UNIT HYPERCUBE AND OVER TRIANGLES IN HIGHER DIMENSIONS FOR MULTIPLE (q-)HYPERGEOMETRIC FUNCTIONS

T. ERNST

This article contains both multiple hypergeometric functions and corresponding q-analogues. First we present integral expressions for multiple hypergeometric functions over the unit hypercube and over triangles in higher dimensions.

Then we extend these integrals to the q-case by using the q-real number $\mathbb{R}_{\boxplus_q}$. The q-binomial theorem, the q-beta integral and their generalizations to higher dimensions are used in the proofs. Also confluent forms with the Euler q-exponential function are proved. Reduction formulas for Kampé de Fériet functions are proved by using Euler integrals, Beta integrals and hypergeometric transformations. Finally, Euler integral representations for Horn functions and q-Euler integral representations of q-Kampé de Fériet functions are proved.

Contents

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1. Introduction and basic definitions

The classical theory for the hypergeometric functions and basic hypergeometric functions (or q-series) are important and well-known. There are many extensions of this classical theory, for example, by by H.M. Srivastava [18] and by H. Exton [13], [14].

This paper, which is based on the notes of the late Per W. Karlsson, who together with Hari Srivastava wrote a book on the subject [19], is part of a series of papers on multiple (q-)hypergeometric functions. The other papers were about q-difference equations, q-integral representations [9], meromorphic continuations [12] and other solutions to corresponding systems of q-difference equations [11]. The general approach by Karlsson for multiple hypergeometric functions, which is introduced in this paper has not been used before, expect for by Exton in his books [13] and [14]. It leads to proofs with a common notation, which simplifies the understanding of the long formulas.

The paper is organized as follows: In section 1 we repeat the necessary q-calculus from [7]. In section 2 we summarize the theory of q-real numbers and convergence regions defined by them. In section 3 we find formal q-analogues of double integrals over the triangle $u+v \leq 1$ in the first quadrant. In section 4 we extend the previous formulas to dimension n. The purpose of section 5 is to prove a number of (q-)Euler integrals for higher-order functions of two variables by using the q-beta integral, the q-binomial theorem and related formulas. Then several reduction formulas for Kampé de Fériet functions are proved.

In section 6 Euler integrals for Horn functions are proved in a similar way. Finally, in section 7 we shall q-deform some formulas from Exton's book [14].

We now repeat some notation from [7]. In many formulas we put $B \equiv \sum_{i=1}^{n} b_i$ etc..

Definition 1.1. Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The power function is defined by

$$q^a \equiv e^{a \log(q)}. \tag{1}$$

The following notation is often used when we have long exponents.

$$\mathrm{QE}(x) \equiv q^x. \tag{2}$$

Definition 1.2. [7, p.19] The q-analogues of a complex number a, a natural number n and the factorial are defined as follows:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{0,1\}, \tag{3}$$

$$\{n\}_q \equiv \sum_{k=1}^{n} q^{k-1},\ \{0\}_q = 0,\ q \in \mathbb{C}\backslash\{0,1\}, \tag{4}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}\backslash\{0,1\}. \tag{5}$$

Definition 1.3. The q-shifted factorial [7] is defined by

$$\langle a; q\rangle_n \equiv \prod_{m=0}^{n-1} (1 - q^{a+m}). \tag{6}$$

Sometimes we also use

$$(a; q)_n \equiv \prod_{m=0}^{n-1} (1 - aq^m). \tag{7}$$

There are three other types of q-shifted factorials [7]: in the equations (11) to (16) we assume that $(m,l) = 1$.

Definition 1.4. In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod$\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R} \times e^{2\pi i\theta},\ \theta \in \mathbb{R}$. The operator

$$\widetilde{}: \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by the 2-torsion

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{8}$$

By (8) it follows that

$$\widetilde{\langle a;q\rangle_n} = \prod_{m=0}^{n-1}(1+q^{a+m}), \tag{9}$$

where this time the tilde denotes an involution which changes a minus sign to a plus sign in all the n factors of $\langle a;q\rangle_n$. Furthermore we define

$$\widetilde{\langle a;q\rangle_n} \equiv \langle \widetilde{a};q\rangle_n. \tag{10}$$

The generalized tilde operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l\log q}. \tag{11}$$

We also need another generalization of the tilde operator.

$$\widetilde{{}_k\langle a;q\rangle_n} \equiv \prod_{m=0}^{n-1}\Big(\sum_{i=0}^{k-1} q^{i(a+m)}\Big). \tag{12}$$

Formula (12) is used in (17).

The following, simple congruence rules [7] follow from (11).

Theorem 1.5.

$$\widetilde{{}^{\frac{m}{l}}a} \pm b \equiv \widetilde{{}^{\frac{m}{l}}(a\pm b)} \left(\operatorname{mod} \frac{2\pi i}{\log q}\right), \tag{13}$$

$$\sum_{k=1}^{n} \widetilde{{}^{\frac{1}{n}}\pm a_k} \equiv \sum_{k=1}^{n} \pm a_k \left(\operatorname{mod} \frac{2\pi i}{\log q}\right), \tag{14}$$

$$\frac{m}{l} \times \widetilde{a} \equiv \widetilde{{}^{\frac{m}{2l}}\frac{am}{l}} \left(\operatorname{mod} \frac{2\pi i}{\log q}\right), \tag{15}$$

$$\mathrm{QE}(\widetilde{{}^{\frac{m}{l}}a}) = \mathrm{QE}(a)e^{\frac{2\pi i m}{l}}, \tag{16}$$

where the second equation is a consequence of the fact that we work $\operatorname{mod}\frac{2\pi i}{\log q}$.

Definition 1.6.

$$\langle\lambda;q\rangle_{kn} \equiv \langle\triangle(q;k;\lambda);q\rangle_n \equiv \prod_{m=0}^{k-1}\langle\frac{\lambda+m}{k};q\rangle_n \times_k \langle\frac{\widetilde{\lambda+m}}{k};q\rangle_n. \tag{17}$$

We also use the notation $\triangle(q;k;\lambda)$ as a parameter in q-hypergeometric functions.

If λ is a vector, we mean the corresponding product of vector elements. If λ is replaced by a sequence of numbers, separated by commas, we mean the corresponding product, as in the case of q-factorials.

The last factor in (17) corresponds to k^{nk}.

With this notation, q-hypergeometric function- and hypergeometric function equations become very similar.

Definition 1.7. The q-derivative is defined by

$$(\mathrm{D}_q\varphi)(x) \equiv \begin{cases} \dfrac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{when } q\in\mathbb{C}\backslash\{1\},\ x\neq 0; \\ \dfrac{d\varphi}{dx}(x), & \text{when } q=1; \\ \dfrac{d\varphi}{dx}(0), & \text{when } x=0. \end{cases} \tag{18}$$

Definition 1.8. The q-integral is the inverse of the q-derivative.

$$\int_a^b f(t,q)\,d_q(t) \equiv \int_0^b f(t,q)\,d_q(t) - \int_0^a f(t,q)\,d_q(t),\ a,b\in\mathbb{R}, \tag{19}$$

where

$$\int_0^a f(t,q)\,d_q(t) \equiv a(1-q)\sum_{n=0}^{\infty} f(aq^n,q)q^n,\ 0<|q|<1,\ a\in\mathbb{R}. \tag{20}$$

Definition 1.9. The q-binomial coefficients [7] are defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}},\ k=0,1,\ldots,n. \tag{21}$$

Theorem 1.10. *[7]: The q-binomial coefficient* $\binom{n}{k}_q$ *is a polynomial of degree* $k(n-k)$ *in q with integer coefficients, whose sum equals* $\binom{n}{k}$.

Definition 1.11. If $|q|>1\vee 0<|q|<1, |z|<|1-q|^{-1}$, the q-exponential function $\mathrm{E}_q(z)$ is defined by

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty}\frac{1}{\{k\}_q!}z^k. \tag{22}$$

Definition 1.12. We shall define a q-hypergeometric series by

$$
\begin{aligned}
&{}_{p+p'}\phi_{r+r'}\left[\begin{array}{c}\hat{a}_1,\ldots,\hat{a}_p\\ \hat{b}_1,\ldots,\hat{b}_r\end{array}|q;z||\begin{array}{c}\prod_i f_i(k)\\ \prod_j g_j(k)\end{array}\right]\equiv\\
&\sum_{k=0}^{\infty}\frac{\langle\hat{a}_1;q\rangle_k\ldots\langle\hat{a}_p;q\rangle_k}{\langle 1,\hat{b}_1;q\rangle_k\ldots\langle\hat{b}_r;q\rangle_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+r+r'-p-p'}z^k\frac{\prod_i f_i(k)}{\prod_j g_j(k)},
\end{aligned}
\tag{23}
$$

where

$$
\widehat{a}\equiv a\vee\widetilde{a}\vee\widetilde{\tfrac{m}{l}a}\vee_k\widetilde{a}\vee\triangle(q;l;\lambda). \tag{24}
$$

In case of $\triangle(q;l;\lambda)$ the index is adjusted accordingly. It is assumed that the denominator contains no zero factors, i.e. $\widehat{b_k}\neq -l+\frac{2m\pi i}{\log q}$, $k=1,\ldots,r,l,m\in\mathbb{N}$. We assume that the $f_i(k)$ and $g_j(k)$ contain p' and r' factors of the form $\langle\widehat{a(k)};q\rangle_k$ or $(s(k);q)_k$ respectively. In a few cases the parameter $\widehat{a}$ in (23) will be the real plus infinity
$(0<|q|<1)$. They correspond to multiplication by 1.

The following definition, like in the one-varible case, allows easy limits for parameters to $\pm\infty$.

Definition 1.13. [7, p.367 f]. Let the vectors $\vec{x}$ and $\vec{q}$ have lengths n. To simplify notation for the q-exponential, let m denote $m\vee k$. In these cases, m does not equal $\sum m_i$. Let the vectors

$$
(a),(b),(g_i),(h_i),(a'),(b'),(g'_i),(h'_i)
$$

have lengths

$$
A,B,G_i,H_i,A',B',G'_i,H'_i.
$$

Let

$$
1+B+B'+H_i+H'_i-A-A'-G_i-G'_i\geq 0, i=1,\ldots,n.
$$

Then the generalized q-Kampé de Fériet function is defined by

$$
\begin{aligned}
&\Phi^{A+A':G_1+G'_1;\ldots;G_n+G'_n}_{B+B':H_1+H'_1;\ldots;H_n+H'_n}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\ldots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\ldots;(\hat{h_n})\end{array}|\vec{q};\vec{x}||\begin{array}{c}(a'):(g'_1);\ldots;(g'_n)\\(b'):(h'_1);\ldots;(h'_n)\end{array}\right]\equiv\\
&\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,k)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g'_j)(q_j,k_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,k)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h'_j)(q_j,k_j)\langle 1;q_j\rangle_{m_j})}\times\\
&(-1)^{\sum_{j=1}^n m_j(1+H_j+H'_j-G_j-G'_j+B+B'-A-A')}\times\\
&\mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)\\
&\times\prod_{j=1}^{n}\mathrm{QE}\left((1+H_j+H'_j-G_j-G'_j)\binom{m_j}{2},q_j\right),
\end{aligned}
\tag{25}
$$

where

$$\widehat{a} \equiv a \vee \widetilde{a} \vee \widetilde{\tfrac{m}{l} a} \vee_k \widetilde{a} \vee \triangle(q;l;\lambda). \tag{26}$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0,k),(g'_j)(q_j,k_j),(b')(q_0,k),(h'_j)(q_j,k_j)$ contain factors of the form $\langle a(\hat{k});q\rangle_k$, $(s;q)_k,(s(k);q)_k$ or $\mathrm{QE}\,(f(\vec{m}))$.
The numbers in front of the colon denote the number of q-shifted factorials with index $m+k$ in numerator and denominator. The numbers after the colon denote the number of q-shifted factorials with index m_i in numerator and denominator. Equally, the numbers after the semicolon denote the number of q-shifted factorials with index k_i in numerator and denominator. We can skip G_2 if it is equal to G_1 for two variables etc. Every ∞ corresponds to multiplication with 1.

2. Survey of q-real numbers

The q-real numbers give a convenient notation for q-additions in formal power series, in particular for q-exponential and q-trigonometric functions. There is a one-to-one correspondence between the convergence regions of the two q-Lauricella functions $\Phi_{\mathrm{A}}^{(n)}$ and $\Phi_{\mathrm{C}}^{(n)}$ [8], and the existence of q-real numbers with n letters (or variables). There are three types of q-real numbers [10] $\mathbb{R}_{\oplus_q}$, $\mathbb{R}_q$ and $\mathbb{R}_{\boxplus_q}$. The q-real number $\mathbb{R}_{\oplus_q}$ is used in (52), (54), (59), (63), (70), (89) and (90).

Definition 2.1. [7, p. 24].
Let $a,b \in \mathbb{R}$. Then the NWA q-addition is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n=0,1,2,\ldots,\ a \oplus_q b \in \mathbb{R}_{\oplus_q}. \tag{27}$$

In particular, $(a \oplus_q b)^0 \equiv 1$. Furthermore,we put

$$(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k},\ n=0,1,2,\ldots. \tag{28}$$

Definition 2.2. Let $a,b \in \mathbb{R}$. The Jackson–Hahn–Cigler q-addition (JHC) is the function

$$\begin{aligned}(a \boxplus_q b)^n &\equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}}\, b^k a^{n-k} \\ &= a^n(-\frac{b}{a};q)_n,\ n=0,1,2,\ldots,\ a \boxplus_q b \in \mathbb{R}_q.\end{aligned} \tag{29}$$

The JHC q-subtraction is defined analogously:

$$(a \boxminus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}}\, (-b)^k a^{n-k},\ n=0,1,2,\ldots. \tag{30}$$

The JHC q-addition is neither commutative nor associative. For the commutative monoid $\mathbb{R}_{\oplus_q}$ we note the following definitions and formulas:

Theorem 2.3. *Assume that $\sim$ means equality on formal power series (see [7, p. 101]). The q-addition (27) has the following properties, for $\alpha,\beta,\gamma\in\mathbb{R}_{\oplus_q}$:*

Commutativity:

$$\alpha\oplus_q\beta\sim\beta\oplus_q\alpha. \tag{31}$$

Associativity

$$(\alpha\oplus_q\beta)\oplus_q\gamma\sim\alpha\oplus_q(\beta\oplus_q\gamma). \tag{32}$$

Definition 2.4. Let the JHC q-real numbers $\mathbb{R}_{\boxplus_q}$ with $n+1$ letters be defined as follows:

$$\mathbb{R}_{\boxplus_q}\equiv\{1\boxminus_q q^a x_1\boxminus_q\cdots\boxminus_q q^a x_n\},\ \{x_k\}_1^n\in\mathbb{R},\ a\in\mathbb{R}^\star,\ |x_k|<1,\ 0<q<1. \tag{33}$$

When any x_k is negative, we replace $\boxminus_q$ by $\boxplus_q$. This means that the JHC q-real numbers in (33) are functions of $n+1$ real numbers $\{x_k\}_1^n$, a.

The following formula applies for a q-deformed hypercube of length 1 in $\mathbb{R}^n$.

Definition 2.5. Assuming that the right hand side converges, and $a\in\mathbb{R}^\star$:

$$(1\boxminus_q q^a x_1\boxminus_q\cdots\boxminus_q q^a x_n)^{-a}\equiv\sum_{m_1,\cdots,m_n=0}^{\infty}\prod_{j=1}^{n}(-x_j)^{m_j}\binom{-a}{\vec{m}}_q^{\star} q^{\binom{\vec{m}}{2}+am}. \tag{34}$$

Corollary 2.6. *A generalization of the q-binomial theorem. The following formula [9] applies to a q-deformed hyper-rhombus of length 1 in $\mathbb{R}^n$.*

$$(1\boxminus_q q^a x_1\boxminus_q\cdots\boxminus_q q^a x_n)^{-a}=\sum_{\vec{m}=\vec{0}}^{\vec{\infty}}\frac{\langle a;q\rangle_m\vec{x}^{\vec{m}}}{\langle\vec{1};q\rangle_{\vec{m}}},\ a\in\mathbb{R}^\star. \tag{35}$$

The q-real number in (33) only exists when the series (34) or (35) converges.

2.1. Series definitions

We shall only define our q-hypergeometric functions. The corresponding (multiple) hypergeometric functions are defined by taking $\lim_{q\to1}$.

Definition 2.7. The q-Appell functions are defined by

$$\Phi_1(a;b,b';c|q;x_1,x_2)\equiv\sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}x_1^{m_1}x_2^{m_2},$$

$$\max(|x_1|,|x_2|)<1. \tag{36}$$

$$\Phi_2(a;b,b';c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2},$$
$$|x_1| \oplus_q |x_2| < 1. \tag{37}$$

$$\Phi_3(a,a';b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2},$$
$$\max(|x_1|,|x_2|) < 1. \tag{38}$$

$$\Phi_4(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \tag{39}$$
$$|\sqrt{x_1}| \oplus_q |\sqrt{x_2}| < 1.$$

Definition 2.8. The convergence regions are extended depending on q. E.g. an octahedron is increased to a q-octahedron for $n = 3$. The q-Lauricella functions [8] are

$$\Phi_{\mathrm{A}}^{(n)}(a,\vec{b};\vec{c}|q;\vec{x}) \equiv \sum_{\vec{m}} \frac{\langle a;q\rangle_m \langle\vec{b};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\vec{c},\vec{1};q\rangle_{\vec{m}}},\ |x_1| \oplus_q \ldots \oplus_q |x_n| < 1, \tag{40}$$

$$\Phi_{\mathrm{B}}^{(n)}(\vec{a},\vec{b};c|q;\vec{x}) \equiv \sum_{\vec{m}} \frac{\langle\vec{a},\vec{b};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle c;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}},\ \max(|x_1|,\cdots,|x_n|) < 1, \tag{41}$$

$$\Phi_{\mathrm{C}}^{(n)}(a,b;\vec{c}|q;\vec{x}) \equiv \sum_{\vec{m}} \frac{\langle a,b;q\rangle_m\vec{x}^{\vec{m}}}{\langle\vec{c},\vec{1};q\rangle_{\vec{m}}},\ |\sqrt{x_1}| \oplus_q \ldots \oplus_q |\sqrt{x_n}| < 1, \tag{42}$$

$$\Phi_{\mathrm{D}}^{(n)}(a,\vec{b};c|q;\vec{x}) \equiv \sum_{\vec{m}} \frac{\langle a;q\rangle_m\langle\vec{b};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle c;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}},\ \max(|x_1|,\ldots,|x_n|) < 1. \tag{43}$$

3. Formulas with the double $\triangle_q$ integral

The purpose of this short section is to find formal equations for q-analogues of the following double integral.

Theorem 3.1. *[2, p.29]. Let* $\int_\triangle \int\ du\, dv$ *denote the double integral for* $0 \le u \le 1$, $0 \le v \le 1$ *and* $u + v < 1$. *Then we have*

$$\begin{aligned}&\int_\triangle \int u^{p-1}v^{s-1}(1-u-v)^{r-1}\, du\, dv\\ &= \Gamma\begin{bmatrix} p,s,r \\ p+s+r \end{bmatrix}.\end{aligned} \tag{44}$$

Proof. Change variable and use the Beta integral. □

Definition 3.2. A q-analogue of (44). The $\triangle_q$ integral in dimension 2 is defined as follows:

$$\int_{\triangle,q}\int u^{p-1}v^{s-1}(1\boxminus_q q^{1-r}u\boxminus_q q^{1-r}v)^{r-1}\,d_qu\,d_qv \\ \equiv\Gamma_q\left[\begin{array}{c} p,s,r \\ p+s+r \end{array}\right]. \tag{45}$$

Theorem 3.3. *Integral representations of two q-Appell functions. A q-analogue of [2, p.28].*

$$\int_{\triangle,q}\int u^{b-1}v^{b'-1}(1\boxminus_q q^{a}ux\boxminus_q q^{a}vy)^{-a} \\ \times(1\boxminus_q q^{1+b+b'-c}u\boxminus_q q^{1+b+b'-c}v)^{c-b-b'-1}\,d_qu\,d_qv \\ \equiv\Gamma_q\left[\begin{array}{c} b,b',c-b-b' \\ c \end{array}\right]\Phi_1(a;b,b';c|q;x,y). \tag{46}$$

$$\int_{\triangle,q}\int u^{b-1}v^{b'-1}(qux;q)_{-a}(qvy;q)_{-a'} \\ \times(1\boxminus_q q^{1+b+b'-c}u\boxminus_q q^{1+b+b'-c}v)^{c-b-b'-1}\,d_qu\,d_qv \\ \equiv\Gamma_q\left[\begin{array}{c} b,b',c-b-b' \\ c \end{array}\right]\Phi_3(a,a';b,b';c|q;x,y). \tag{47}$$

Proof. Use (45). □

4. Formulas with the multiple $\triangle_q$ integral after Exton

The purpose of this section is to find formal equations with q-analogues of a multiple integral over a hypertriangle. All vectors in this section have length n.

Theorem 4.1. *[13, (2.3.2)] Let $\int_{\triangle_n}\int\,d\vec{u}$ denote the multiple integral for $0\leq u_i\leq 1$ and $\sum_{i=1}^{n}u_i<1$. Then we have*

$$\int_{\triangle,n}\int u_1^{p_1-1}\dots u_n^{p_n-1}(1-u_1-\dots-u_n)^{r-1}\,d\vec{u} \\ =\Gamma\left[\begin{array}{c} p_1,\dots,p_n,r \\ P+r \end{array}\right]. \tag{48}$$

Definition 4.2. A q-analogue of (48). The $\triangle_{n,q}$ integral in dimension n is defined as follows:

$$\int_{\triangle,n,q}\int u_1^{p_1-1}\dots u_n^{p_n-1}(1\boxminus_q q^{1-r}u_1\boxminus_q\dots\boxminus_q q^{1-r}u_n)^{r-1}\,d_q\vec{u} \\ \equiv\Gamma_q\left[\begin{array}{c} p_1,\dots,p_n,r \\ P+r \end{array}\right]. \tag{49}$$

Theorem 4.3. *q-Integral representations of two q-Lauricella functions. A q-analogue of Exton [13, p.48-49].*

$$\begin{aligned}
&\int_{\triangle,n,q}\int u_1^{b_1-1}\dots u_n^{b_n-1}(1\boxminus_q q^{1+B-c}u_1\boxminus_q\dots\boxminus_q q^{1+B-c}u_n)^{c-B-1}\\
&\times(1\boxminus_q q^a u_1x_1\boxminus_q q^a\dots\boxminus_q q^a u_nx_n)^{-a}\,d_q\vec{u}\\
&\equiv\Gamma_q\left[\begin{matrix}\vec{b},c-B\\ c\end{matrix}\right]\Phi_{\mathrm{D}}^{(n)}(a,\vec{b};c|q;\vec{x}).
\end{aligned}\tag{50}$$

$$\begin{aligned}
&\int_{\triangle,n,q}\int u_1^{a_1-1}\dots u_n^{a_n-1}(qu_1x_1;q)_{-b_1}\dots(qu_nx_n;q)_{-b_n}\\
&\times(1\boxminus_q q^{1+A-c}u_1\boxminus_q\dots\boxminus_q q^{1+A-c}u_n)^{c-A-1}\,d_q\vec{u}\\
&=\Gamma_q\left[\begin{matrix}\vec{a},c-A\\ c\end{matrix}\right]\Phi_{\mathrm{B}}^{(n)}(\vec{a},\vec{b};c|q;\vec{x}).
\end{aligned}\tag{51}$$

Proof. Use (49). □

The following integral formulas are proved in the same way by using (49).

Theorem 4.4. *q-Integral representation of a q-Kampé de Fériet function with $\mathbb{R}_{\oplus_q}$. A q-analogue of [14, p.121, (6.1.1.2)]. Assume $C=D+1$.*

$$\begin{aligned}
&\int_{\triangle,n,q}\int u_1^{a_1-1}\dots u_n^{a_n-1}(1\boxminus_q q^{1+A-b}u_1\boxminus_q\dots\boxminus_q q^{1+A-b}u_n)^{b-A-1}\\
&\times\ {}_C\Phi_D\left[\vec{c};\vec{d}|q;x_1u_1\oplus_q\dots\oplus_q x_nu_n\right]\,d_q\vec{u}\\
&\equiv\Gamma_q\left[\begin{matrix}\vec{a},b-A\\ b\end{matrix}\right]\Phi_{D+1:0}^{C:1}\left[\begin{matrix}\vec{c}:a_1;\dots;a_n\\ \vec{d},b:\cdot;\dots;\cdot\end{matrix}\middle|q;\vec{x}\right].
\end{aligned}\tag{52}$$

Theorem 4.5. *q-Integral representation of a q-Kampé de Fériet function. A q-analogue of [14, p.122, (6.1.2.2)]. Assume $C=F,\ D=G+1$.*

$$\begin{aligned}
&\int_{\triangle,n,q}\int u_1^{a_1-1}\dots u_n^{a_n-1}(1\boxminus_q q^{1+A-b}u_1\boxminus_q\dots\boxminus_q q^{1+A-b}u_n)^{b-A-1}\\
&\times\Phi_{F:G}^{C:D}\left[\begin{matrix}\vec{c}:\vec{d}^1;\dots;\vec{d}^n\\ \vec{f}:\vec{g}^1;\dots;\vec{g}^n\end{matrix}\middle|q;\vec{u}\vec{x}\right]\,d_q\vec{u}\\
&\equiv\Gamma_q\left[\begin{matrix}\vec{a},b-A\\ b\end{matrix}\right]\Phi_{F+1:G+1}^{C+1:D+1}\left[\begin{matrix}\vec{c},\infty:\vec{d}^1,a_1;\dots;\vec{d}^n,a_n\\ \vec{f},b:\vec{g}^1,\infty;\dots;\vec{g}^n,\infty\end{matrix}\middle|q;\vec{x}\right].
\end{aligned}\tag{53}$$

Theorem 4.6. *Integral representations of the fourth q-Lauricella function. A q-analogue of the corrected version of [14, p.233, (A.9.2.1)].*

$$\begin{aligned}&\int_{\triangle,n,q}\int u_1^{a_1-1}\ldots u_n^{a_n-1}(1\boxminus_q q^{1+A-b}u_1\boxminus_q\ldots\boxminus_q q^{1+A-b}u_n)^{b-A-1}\\&{}_2\Phi_1(c,b;d|q;x_1u_1\oplus_q\ldots\oplus_q x_nu_n)\,d_q\vec{u}\\&\equiv\Gamma_q\left[\begin{array}{c}\vec{a},b-A\\b\end{array}\right]\Phi_{\mathrm{D}}^{(n)}(c,\vec{a};d|q;\vec{x}).\end{aligned}\tag{54}$$

Remark 4.7. We note that formulas (52), (53) and (54) have the same Γ_q-factor.

The following formula illustrates a similar proof.

Theorem 4.8.

$$\begin{aligned}&\int_0^1\int_0^1(1\boxminus_q q^a u^2x\boxminus_q q^a uvy)^{-a}u^{b_1-1}(qu;q)_{c_1-b_1-1}v^{b_2-1}(qv;q)_{c_2-b_2-1}\,d_qud_qv\\&=\Gamma_q\left[\begin{array}{c}b_1,c_1-b_1,b_2,c_2-b_2\\c_1,c_2\end{array}\right]\sum_{m,n=0}^{\infty}\frac{\langle b_1;q\rangle_{2m+n}\langle b_2;q\rangle_n\langle a;q\rangle_{m+n}}{\langle c_1;q\rangle_{2m+n}\langle 1;q\rangle_m\langle 1,c_2;q\rangle_n}.\end{aligned}\tag{55}$$

Proof.

$$\text{LHS}=\sum_{m,n=0}^{\infty}\frac{\langle a;q\rangle_{m+n}x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\Gamma_q\left[\begin{array}{c}b_1+2m+n,b_2+n,c_1-b_1,c_2-b_2\\c_1,c_2\end{array}\right]=\text{RHS}.\tag{56}$$

□

5. q-Euler integrals for higher-order functions of two variables

In all theorems, except for a few exceptions, which are always pointed out, vectors with Greek letters have length p. The integration variables $\vec{u},\vec{s},\vec{t}$ also have length p. In these cases, the Greek letters appear as (vector) indices or expontents. Sometimes we write vector B_q for vector Beta functions (of length p) instead of vector Γ_q functions. Before each q-analogue, we repeat the corresponding hypergeometric formula from [2, p. 154] on three occations.

Theorem 5.1. *An integral for a Kampé de Fériet function [2, p. 154]. Assume that $\beta,\beta'\in\mathbb{R}^{\star}$. Then*

$$\begin{aligned}&\int_{\vec{s}=\vec{0}}^{\vec{1}}(1-xs_1\ldots s_p)^{-\beta}(1-ys_1\ldots s_p)^{-\beta'}\vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\vec{\gamma}-\alpha-1}d\vec{s}\\&=\mathrm{B}(\alpha,\vec{\gamma}-\alpha)\mathrm{F}_{p:0}^{p:1}\left[\begin{array}{c}\vec{\alpha}:\beta;\beta'\\\vec{\gamma}:\cdot;\cdot\end{array}\middle|x,y\right].\end{aligned}\tag{57}$$

Theorem 5.2. *A q-integral for a q-Kampé de Fériet function, a q-analogue of (57) Assume that* $\beta,\beta'\in\mathbb{R}^\star$. *Then*

$$\begin{aligned}&\int_{\vec{s}=\vec{0}}^{\vec{1}}\frac{1}{(xs_1\dots s_p;q)_\beta}\,\frac{1}{(ys_1\dots s_p;q)_{\beta'}}\vec{s}^{\vec{\alpha}-1}(q\vec{s};q)_{\vec{\gamma}-\vec{\alpha}-1}\,d_q\vec{s}\\&=\mathrm{B}_q(\vec{\alpha},\vec{\gamma}-\vec{\alpha})\Phi^{p:1}_{p:0}\left[\begin{matrix}\vec{\alpha}:\beta;\beta'\\ \vec{\gamma}:\cdot,\cdot\end{matrix}\middle|q;x,y\right].\end{aligned}\tag{58}$$

Proof. Use the q-binomial theorem. □

Corollary 5.3. *A confluent form.*

$$\begin{aligned}&\int_{\vec{s}=\vec{0}}^{\vec{1}}e_q(xs_1\dots s_p\oplus_q ys_1\dots s_p)\,\vec{s}^{\vec{\alpha}-1}(q\vec{s};q)_{\vec{\gamma}-\vec{\alpha}-1}\,d_q\vec{s}\\&=\mathrm{B}_q(\vec{\alpha},\vec{\gamma}-\vec{\alpha})\;{}_{p+1}\Phi_p\left[\begin{matrix}\vec{\alpha},\infty\\ \vec{\gamma}\end{matrix}\middle|q;x\oplus_q y\right].\end{aligned}\tag{59}$$

Proof. Let $\beta,\beta'\to\infty$ in (58). □

Theorem 5.4. *An integral for a Kampé de Fériet function [2, p. 154]. Assume that* $\alpha\in\mathbb{R}^\star$. *Then*

$$\begin{aligned}&\int_{\vec{s},\vec{t}=\vec{0}}^{\vec{1}}(1-xs_1\dots s_p-yt_1\dots t_p)^{-\alpha}\;\vec{s}^{\vec{\beta}-1}\\&\times(1-\vec{s})^{\vec{\delta}-\vec{\beta}-1}\vec{t}^{\vec{\beta'}-1}(1-\vec{t})^{\vec{\delta'}-\vec{\beta'}-1}\,d\vec{s}d\vec{t}\\&=\mathrm{B}(\vec{\beta},\vec{\delta}-\vec{\beta})\mathrm{B}(\vec{\beta'},\vec{\delta'}-\vec{\beta'})\mathrm{F}^{1:p}_{0:p}\left[\begin{matrix}\alpha:\vec{\beta};\vec{\beta'}\\ \cdot:\vec{\delta};\vec{\delta'}\end{matrix}\middle|x,y\right].\end{aligned}\tag{60}$$

Theorem 5.5. *A q-integral for a q-Kampé de Fériet function, a q-analogue of (60). Assume that* $\alpha\in\mathbb{R}^\star$. *Then*

$$\begin{aligned}&\int_{\vec{s},\vec{t}=\vec{0}}^{\vec{1}}(1\boxminus_q q^\alpha xs_1\dots s_p\boxminus_q q^\alpha yt_1\dots t_p)^{-\alpha}\;\vec{s}^{\vec{\beta}-1}\\&\times(q\vec{s};q)_{\vec{\delta}-\vec{\beta}-1}\vec{t}^{\vec{\beta'}-1}(q\vec{t};q)_{\vec{\delta'}-\vec{\beta'}-1}\,d_q\vec{s}d_q\vec{t}\\&=\mathrm{B}_q(\vec{\beta},\vec{\delta}-\vec{\beta})\mathrm{B}_q(\vec{\beta'},\vec{\delta'}-\vec{\beta'})\Phi^{1:p+1}_{1:p}\left[\begin{matrix}\alpha:&\vec{\beta},\infty;\vec{\beta'},\infty\\ \infty:&\vec{\delta};\vec{\delta'}\end{matrix}\middle|q;x,y\right].\end{aligned}\tag{61}$$

Proof. By the q-Beta integral $\star$.

$$\begin{aligned}
&\text{LHS} \overset{\text{by}(35)}{=} \int_{\vec{s},\vec{t}=\vec{0}}^{\vec{1}} \sum_{m,n=0}^{\infty} \frac{\langle \alpha;q\rangle_{m+n} x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \vec{s}^{\beta-\vec{1}+m} \\
&\times \overrightarrow{(qs;q)}_{\delta-\beta-1} \vec{t}^{\beta'-\vec{1}+n} \overrightarrow{(qt;q)}_{\delta'-\beta'-1} \, d_q\vec{s} d_q\vec{t} \\
&\overset{\text{by}\star}{=} \sum_{m,n=0}^{\infty} \frac{\langle \alpha;q\rangle_{m+n} x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \mathrm{B}_q(\beta+\vec{m},\delta-\beta)\mathrm{B}_q(\beta'+\vec{n},\delta'-\beta') = \text{RHS}.
\end{aligned} \tag{62}$$

□

Corollary 5.6. *A confluent form.*

$$\begin{aligned}
&\int_{\vec{s},\vec{t}=\vec{0}}^{\vec{1}} e_q(xs_1\ldots s_p \oplus_q ys_1\ldots s_p)\ \vec{s}^{\beta-\vec{1}} \overrightarrow{(qs;q)}_{\delta-\beta-1} \vec{t}^{\beta'-\vec{1}} \overrightarrow{(qt;q)}_{\delta'-\beta'-1} \, d_q\vec{s} d_q\vec{t} \\
&= \mathrm{B}_q(\beta,\vec{\delta}-\beta)\mathrm{B}_q(\beta',\vec{\delta'}-\beta')\ {}_{p+1}\Phi_p \left[\begin{array}{c} \vec{\beta},\infty \\ \vec{\delta} \end{array} \middle| q;x \right] {}_{p+1}\Phi_p \left[\begin{array}{c} \vec{\beta'},\infty \\ \vec{\delta'} \end{array} \middle| q;y \right].
\end{aligned} \tag{63}$$

Proof. Let $\alpha \to \infty$ in (61). □

Theorem 5.7. *An integral for a Kampé de Fériet function [2, p. 154]. Assume that $\beta_0, \beta_0' \in \mathbb{R}^\star$ and $\vec{\beta}, \vec{\beta'}$ have lengths $p+1$. Then*

$$\begin{aligned}
&\int_{\triangle,p} \int (1-xs_1\ldots s_p)^{-\beta_0}\ (1-yt_1\ldots t_p)^{-\beta_0'} \vec{s}^{\beta-\vec{1}} \vec{t}^{\beta'-\vec{1}} \\
&\overrightarrow{(1-s-t)}^{\gamma-\beta-\beta'-1}\ d\vec{s}d\vec{t} \\
&= \Gamma\left[\begin{array}{c} \vec{\beta},\vec{\beta'},\gamma-\vec{\beta}-\beta' \\ \vec{\gamma} \end{array} \right] \mathrm{F}^{0:p+1}_{p:0} \left[\begin{array}{cc} \cdot : & \vec{\beta};\vec{\beta'} \\ \vec{\gamma}: & \cdot;\cdot \end{array} \middle| x,y \right].
\end{aligned} \tag{64}$$

Theorem 5.8. *A q-integral for a q-Kampé de Fériet function, a q-analogue of (64).*

Assume that $\beta_0, \beta_0' \in \mathbb{R}^\star$ and $\vec{\beta}, \vec{\beta'}$ have lengths $p+1$. Then

$$\begin{aligned}
&\int_{\triangle,p,q} \int \frac{1}{(xs_1\ldots s_p;q)_{\beta_0}} \frac{1}{(yt_1\ldots t_p;q)_{\beta_0'}} \vec{s}^{\beta-\vec{1}} \vec{t}^{\beta'-\vec{1}} \\
&\overrightarrow{(1 \boxminus_q q^{1-\gamma+\beta+\beta'} s \boxminus_q q^{1-\gamma+\beta+\beta'} t)}^{\gamma-\beta-\beta'-1}\ d_q\vec{s} d_q\vec{t} \\
&= \Gamma_q \left[\begin{array}{c} \vec{\beta},\vec{\beta'},\gamma-\vec{\beta}-\beta' \\ \vec{\gamma} \end{array} \right] \Phi^{p:p+1}_{p:p} \left[\begin{array}{cc} p\infty: & \vec{\beta};\vec{\beta'} \\ \vec{\gamma} & p\infty;p\infty \end{array} \middle| q;x,y \right].
\end{aligned} \tag{65}$$

Proof. By the q-binomial theorem $\star$.

$$\begin{aligned}&\text{LHS} \overset{\text{by}\star}{=} \int_{\triangle,p,q}\int \sum_{m,n=0}^{\infty} \frac{\langle\beta_0;q\rangle_m\langle\beta_0';q\rangle_n x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n} \vec{s}^{m+\vec{\beta}-1}\vec{t}^{n+\vec{\beta}'-1}\\ &\times(1\boxminus_q q^{1-\gamma+\beta+\beta'}s\boxminus_q \vec{q}^{1-\gamma+\beta+\beta'}t)^{\gamma-\beta-\beta'-1}\\ &\overset{\text{by}(35)}{=}\sum_{m,n=0}^{\infty}\frac{\langle\beta_0;q\rangle_m\langle\beta_0';q\rangle_n x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\Gamma_q\left[\begin{matrix}\beta\vec{+}m,\beta'\vec{+}n,\gamma-\vec{\beta}-\beta'\\ \gamma+\vec{m}+n\end{matrix}\right]\\ &=\text{RHS}.\end{aligned} \tag{66}$$

□

The following integral for a Kampé de Fériet function is new.

Theorem 5.9. *Assume that* $\alpha\in\mathbb{R}^\star$. *Then*

$$\begin{aligned}&\int_{\triangle,p}\int(1-xs_1\dots s_p-yt_1\dots t_p)^{-\alpha}\vec{s}^{\vec{\beta}-1}\vec{t}^{\vec{\beta'}-1}\\ &(1-s-\vec{t})^{\gamma-\beta-\beta'-1}\,d\vec{s}d\vec{t}\\ &=\Gamma\left[\begin{matrix}\vec{\beta},\vec{\beta}',\gamma-\vec{\beta}-\beta'\\ \vec{\gamma}\end{matrix}\right]\mathrm{F}^{1:p}_{p:0}\left[\begin{matrix}\alpha: & \vec{\beta};\vec{\beta}'\\ \vec{\gamma}: & \cdot;\cdot\end{matrix}\middle| x,y\right].\end{aligned} \tag{67}$$

Theorem 5.10. *A q-integral for a q-Kampé de Fériet function, a q-analogue of (67). Assume that* $\alpha\in\mathbb{R}^\star$. *The notation* $(p-1)\infty$ *just means* $p-1$ *factors 1. Then*

$$\begin{aligned}&\int_{\triangle,p,q}\int(1\boxminus_q q^\alpha xs_1\dots s_p\boxminus_q q^\alpha yt_1\dots t_p)^{-\alpha}\vec{s}^{\vec{\beta}-1}\vec{t}^{\vec{\beta'}-1}\\ &(1\boxminus_q q^{1-\gamma+\beta+\beta'}s\boxminus_q\vec{q}^{1-\gamma+\beta+\beta'}t)^{\gamma-\beta-\beta'-1}\,d_q\vec{s}d_q\vec{t}\\ &=\Gamma_q\left[\begin{matrix}\vec{\beta},\vec{\beta}',\gamma-\vec{\beta}-\beta'\\ \vec{\gamma}\end{matrix}\right]\\ &\Phi^{p:p}_{p:p-1}\left[\begin{matrix}\alpha,(p-1)\infty: & \vec{\beta};\vec{\beta}'\\ \vec{\gamma}: & (p-1)\infty;(p-1)\infty\end{matrix}\middle| q;x,y\right].\end{aligned} \tag{68}$$

Proof.

$$\begin{aligned}&\text{LHS}\overset{\text{by}(35)}{=}\int_{\triangle,p,q}\int\sum_{m,n=0}^{\infty}\frac{\langle\alpha;q\rangle_{m+n}x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\vec{s}^{\beta-\vec{1}+m}\vec{t}^{\vec{\beta'}-1}\\ &(1\boxminus_q q^{1-\gamma+\beta+\beta'}s\boxminus_q\vec{q}^{1-\gamma+\beta+\beta'}t)^{\gamma-\beta-\beta'-1}\,d_q\vec{s}d_q\vec{t}\\ &\overset{\text{by}(49)}{=}\sum_{m,n=0}^{\infty}\frac{\langle\alpha;q\rangle_{m+n}x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\mathrm{B}_q(\beta+\vec{m},\delta-\beta)\mathrm{B}_q(\beta'+\vec{n},\delta'-\beta')\\ &=\text{RHS}.\end{aligned} \tag{69}$$

□

Corollary 5.11. *A confluent form.*

$$\begin{aligned}&\int_{\triangle,p,q} e_q(xs_1\dots s_p \oplus_q ys_1\dots s_p)\ \vec{s}^{\vec{\beta}-1}\vec{t}^{\vec{\beta'}-1}\\&(1\boxminus_q q^{1-\gamma+\beta+\beta'}s\boxminus_q q^{1-\gamma+\beta+\beta'}t)^{\gamma-\beta-\beta'-1}\ d_q\vec{s}d_q\vec{t}\\&\equiv\Gamma_q\left[\begin{matrix}\vec{\beta},\vec{\beta'},\gamma-\vec{\beta}-\beta'\\ \vec{\gamma}\end{matrix}\right]\Phi^{1:p}_{p:0}\left[\begin{matrix}\infty:\vec{\beta};\vec{\beta'}\\ \vec{\gamma}:(p+1)\infty;(p+1)\infty\end{matrix}\middle|q;x,y\right].\end{aligned}\tag{70}$$

Proof. Let $\beta_0,\beta_0'\to\infty$ in (65) or $\alpha\to\infty$ in (68). □

Remark 5.12. We note that all the confluent corollaries have the function $e_q(xs_1\dots s_p\oplus_q ys_1\dots s_p)$ as first factor in the multiple q-integrals. Formula (58) used the q-binomial theorem and the q-beta integral exactly in this order in the proof. Furthermore, the proofs of formulas (61) and (65) used the q-beta integral and the q-binomial theorem and (35) in opposite order, which led to different formulas, but formulas (65) and (68) have the same Γ_q factors from formula (35). Finally, formula (68) has two q-real numbers in the integrand.

Remark 5.13. In formulas (61), (65) and (68), the numbers of ∞ in numerator and denominator are equal, a characteristic of q-Kampé de Fériet functions.

For $p=1$ these four formulas reduce to q-analogues of known integral expressions for F_1, F_2, F_3 and F_1 in [2].

Theorem 5.14. *A reduction formula for a Kampé de Fériet function.*

$$\begin{aligned}&F^{p:2}_{p:1}\left[\begin{matrix}\vec{\alpha}: & \beta,\tilde{\beta}; & \beta,\tilde{\beta}\\ \vec{\gamma}: & \beta+\tilde{\beta}-\frac{1}{2}; & \beta+\tilde{\beta}+\frac{1}{2}\end{matrix}\middle|z,z\right]\\&={}_{p+3}F_{p+2}\left[\begin{matrix}\vec{\alpha},2\beta,2\tilde{\beta},\beta+\tilde{\beta}\\ \vec{\gamma},2\beta+2\tilde{\beta}-1,\beta+\tilde{\beta}+\frac{1}{2}\end{matrix}\middle|z\right].\end{aligned}\tag{71}$$

Proof. Use [6, §4.3 (8)] to obtain

$$\begin{aligned}&B(\alpha,\vec{\gamma}-\alpha)\times\text{LHS}\\&=\int_{\vec{s}=\vec{0}}^{\vec{1}}{}_2F_1\left[\begin{matrix}\beta,\tilde{\beta}\\ \beta+\tilde{\beta}-\frac{1}{2}\end{matrix}\middle|z\prod s_j\right]\\&{}_2F_1\left[\begin{matrix}\beta,\tilde{\beta}\\ \beta+\tilde{\beta}+\frac{1}{2}\end{matrix}\middle|z\prod s_j\right]\vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\&\overset{\text{by}[6,\S4.3(8)]}{=}\int_{\vec{s}=\vec{0}}^{\vec{1}}{}_3F_2\left[\begin{matrix}2\beta,2\tilde{\beta},\beta+\tilde{\beta}\\ 2\beta+2\tilde{\beta}-1,\beta+\tilde{\beta}+\frac{1}{2}\end{matrix}\middle|z\prod s_j\right]\vec{s}^{\vec{\alpha}-1}(1-\vec{s})_{\gamma-\alpha-1}\,\vec{ds}\\&=B(\alpha,\vec{\gamma}-\alpha)\times\text{RHS}.\end{aligned}\tag{72}$$

□

Theorem 5.15. *A reduction formula for a Kampé de Fériet function.*

$$\begin{aligned} &\mathrm{F}^{p:2}_{p:1}\left[\begin{matrix}\vec{\alpha}: & \beta,\tilde{\beta};\beta,\tilde{\beta}\\ \vec{\gamma}: & \beta+\tilde{\beta}+\frac{1}{2};\beta+\tilde{\beta}+\frac{1}{2}\end{matrix}\,\middle|\, z,z\right]\\ &= {}_{p+3}\mathrm{F}_{p+2}\left[\begin{matrix}\vec{\alpha},2\beta,2\tilde{\beta},\beta+\tilde{\beta}\\ \vec{\gamma},2\beta+2\tilde{\beta},\beta+\tilde{\beta}+\frac{1}{2}\end{matrix}\,\middle|\, z\right]. \end{aligned} \tag{73}$$

Proof. Use Clausen's formula. □

Theorem 5.16. *Another reduction formula for a Kampé de Fériet function.*

$$\mathrm{F}^{p:0}_{p:1}\left[\begin{matrix}\vec{\alpha}:\cdot;\cdot\\ \vec{\gamma}:\rho;\sigma\end{matrix}\,\middle|\, z,z\right] = {}_{p+2}\mathrm{F}_{p+3}\left[\begin{matrix}\vec{\alpha},\frac{1}{2}(\rho+\sigma),\frac{1}{2}(\rho+\sigma-\frac{1}{2})\\ \vec{\gamma},\rho,\sigma,\rho+\sigma-1\end{matrix}\,\middle|\, 4z\right]. \tag{74}$$

Proof. Use Bailey [6, §4.3 (2)] □

Remark 5.17. Other multiplication formulas give $z^2/4$ and can therefore not be used immediately. Series transformation shows that you get more general results with $\mathrm{F}^{p:0}_{q:1}$ of the same form. Then you get

1. for $p = 2, q = 0$ reduction of $\mathrm{F}_4(\alpha,\alpha';\rho,\sigma;z,z)$ to ${}_4\mathrm{F}_3$ (Burchnall [4])

2. for $p = 1, q = 0$ reduction of Ψ_2 to ${}_3\mathrm{F}_3$ (Burchnall, Chaundy II [3, (66)]

Theorem 5.18. *A reduction formula for a multiple hypergeometric function of n equal variables z. $\vec{\beta}$ has length n.*

$$\begin{aligned} &\mathrm{F}^{p:2}_{p:1}\left[\begin{matrix}\vec{\alpha}: & \frac{1}{2}\vec{\beta},\frac{1}{2}(\vec{\beta+1})\\ \vec{\gamma}: & 1+\vec{\beta}\end{matrix}\,\middle|\, z,\ldots,z\right]\\ &= {}_{p+2}\mathrm{F}_{p+1}\left[\begin{matrix}\vec{\alpha},\frac{1}{2}\sum\beta_i,\frac{1}{2}(1+\sum\beta_i)\\ \vec{\gamma},1+\sum\beta_i\end{matrix}\,\middle|\, z\right]. \end{aligned} \tag{75}$$

Proof.

$$\begin{aligned}
&\mathrm{B}(\alpha,\vec{\gamma}-\alpha)\times\mathrm{LHS}\\
&=\int_{\vec{s}=\vec{0}}^{\vec{1}} {}_2\mathrm{F}_1\left[\begin{matrix}\frac{1}{2}\beta_1,\frac{1}{2}(\beta_1+1)\\ \beta_1+1\end{matrix}\middle| z\prod s_j\right]\cdots {}_2\mathrm{F}_1\left[\begin{matrix}\frac{1}{2}\beta_n,\frac{1}{2}(\beta_n+1)\\ \beta_n+1\end{matrix}\middle| z\prod s_j\right]\\
&\times \vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\
&=\int_{\vec{s}=\vec{0}}^{\vec{1}}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-z\prod s_j}\right)^{-\sum\beta_i}\vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\
&=\int_{\vec{s}=\vec{0}}^{\vec{1}} {}_2\mathrm{F}_1\left[\begin{matrix}\frac{1}{2}\sum\beta_i,\frac{1}{2}+\frac{1}{2}\sum\beta_i\\ 1+\sum\beta_i\end{matrix}\middle| z\prod s_j\right]\vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\
&=\mathrm{B}(\alpha,\vec{\gamma}-\alpha)\times\mathrm{RHS}.
\end{aligned}\tag{76}$$

□

Remark 5.19. Formula (75) generalizes Srivastava's three results [17].

Corollary 5.20. *A summation formula for a Kampé de Fériet function.*

$$\begin{aligned}
&\mathrm{F}_{2:1}^{2:2}\left[\begin{matrix}-N,1+a-b: & c_1,c_1+\frac{1}{2};\ldots;c_n,c_n+\frac{1}{2}\\ \frac{1+a-b-N}{2},\frac{2+a-b-N}{2}: & 2c_1+1;\ldots;2c_n+1\end{matrix}\middle|1,\ldots,1\right]\\
&=\frac{(b+2\sum c_i-a)_N}{(b-a)_N}.
\end{aligned}\tag{77}$$

Proof.

$$\begin{aligned}
\mathrm{LHS}&\overset{\text{by}(75)}{=}{}_4\mathrm{F}_3\left[\begin{matrix}-N,1+a-b,\sum c_i,\frac{1}{2}+\sum c_i\\ \frac{1+a-b-N}{2},\frac{2+a-b-N}{2},1+2\sum c_i\end{matrix}\middle|1\right]\\
&\overset{\text{by}[16,III(20)]}{=}\frac{(1+a-b-N-2\sum c_i)_N}{(1+a-b-N)_N}=\mathrm{RHS}.
\end{aligned}\tag{78}$$

□

Remark 5.21. One can compare with Exton [13, (4.9.14)].

Theorem 5.22. *A reduction formula for a multiple hypergeometric function of n equal variables z. $\vec{\beta}$ has length n.*

$$\mathrm{F}_{p:1}^{p:2}\left[\begin{matrix}\vec{\alpha}: & \vec{\beta},1+\frac{1}{2}\vec{\beta}\\ \vec{\gamma}: & \frac{1}{2}\vec{\beta}\end{matrix}\middle|z,\ldots,z\right]=\mathrm{F}_{p:0}^{p:1}\left[\begin{matrix}\vec{\alpha}: & n+\sum\beta_i;-n\\ \vec{\gamma}: & \cdot;\cdot\end{matrix}\middle|z,-z\right].\tag{79}$$

Proof. We use the Beta integral n times twice.

$$\begin{aligned}
&\mathrm{B}(\alpha,\vec{\gamma}-\alpha)\times\mathrm{LHS}\\
&=\int_{\vec{s}=\vec{0}}^{\vec{1}} {}_2\mathrm{F}_1\left[\begin{matrix}\beta_1,1+\frac{1}{2}\beta_1\\ \frac{1}{2}\beta_1\end{matrix}\middle| z\prod s_j\right]\cdots {}_2\mathrm{F}_1\left[\begin{matrix}\frac{1}{2}\beta_n,1+\frac{1}{2}\beta_n\\ \frac{1}{2}\beta_n\end{matrix}\middle| z\prod s_j\right]\\
&\times \vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\
&=\int_{\vec{s}=\vec{0}}^{\vec{1}} \left(1-\vec{z}\prod s_j\right)^{-\vec{1}-\beta}\left(1+\vec{z}\prod s_j\right)\vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\
&=\int_{\vec{s}=\vec{0}}^{\vec{1}} \left(1-z\prod s_j\right)^{-n-\Sigma\beta_i}\left(1+z\prod s_j\right)^{n}\vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\
&=\mathrm{B}(\alpha,\vec{\gamma}-\alpha)\times\mathrm{RHS}.
\end{aligned}\tag{80}$$

□

Theorem 5.23. *A reduction formula for a multiple hypergeometric function of n equal variables z. $\vec{\beta}$ has length n.*

$$\mathrm{F}^{p:2}_{p:1}\left[\begin{matrix}\vec{\alpha}: & \vec{\beta},1+\vec{2\beta}\\ \vec{\gamma}: & \vec{2\beta}\end{matrix}\middle| z,\ldots,z\right]=\mathrm{F}^{p:1}_{p:0}\left[\begin{matrix}\vec{\alpha}: & n+\sum\beta_i;-n\\ \vec{\gamma}: & \cdot;\cdot\end{matrix}\middle| z,\frac{1}{2}z\right].\tag{81}$$

Proof.

$$\begin{aligned}
&\mathrm{B}(\alpha,\vec{\gamma}-\alpha)\times\mathrm{LHS}\\
&=\int_{\vec{s}=\vec{0}}^{\vec{1}} {}_2\mathrm{F}_1\left[\begin{matrix}\beta_1,1+2\beta_1\\ 2\beta_1\end{matrix}\middle| z\prod s_j\right]\cdots {}_2\mathrm{F}_1\left[\begin{matrix}\beta_n,1+2\beta_n\\ 2\beta_n\end{matrix}\middle| z\prod s_j\right]\\
&\times \vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\
&=\int_{\vec{s}=\vec{0}}^{\vec{1}} \left(1-\vec{z}\prod s_j\right)^{-\vec{1}-\beta}\left(1-\frac{\vec{1}}{2}z\prod s_j\right)\vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\
&=\int_{\vec{s}=\vec{0}}^{\vec{1}} \left(1-z\prod s_j\right)^{-n-\Sigma\beta_i}\left(1-\frac{1}{2}z\prod s_j\right)^{n}\vec{s}^{\vec{\alpha}-1}(1-\vec{s})^{\gamma-\alpha-1}\,\vec{ds}\\
&=\mathrm{B}(\alpha,\vec{\gamma}-\alpha)\times\mathrm{RHS}.
\end{aligned}\tag{82}$$

□

6. Euler integrals for Horn functions

Single Euler integral representations of Horn functions were found in [5]. The following presentation shows that more such formulas can easily be found.

Theorem 6.1. *An integral representation of the third Horn function.*

$$\begin{aligned} \mathrm{H}_3(\alpha,\beta,\gamma;x,y)=\Gamma\left[\begin{array}{c}2\gamma-1\\ \alpha,\beta,2\gamma-\alpha-\beta-1\end{array}\right]\\ \times\int_{\triangle}\int(1-4u^2x-4uvy)^{\frac{1}{2}-\gamma}u^{\alpha-1}v^{\beta-1}(1-u-v)^{2\gamma-\alpha-\beta-2}\,dudv. \end{aligned} \tag{83}$$

Proof.

$$\begin{aligned} &\int_{\triangle}\int(1-4u^2x-4uvy)^{\frac{1}{2}-\gamma}u^{\alpha-1}v^{\beta-1}(1-u-v)^{2\gamma-\alpha-\beta-2}\,dudv\\ &\overset{\text{by}(45)}{=}\Gamma\left[\begin{array}{c}\alpha,\beta,2\gamma-\alpha-\beta-1\\ 2\gamma-1\end{array}\right]\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n(\gamma-\frac{1}{2})^{m+n}x^my^n}{(2\gamma-1)_{2m+2n}m!n!}. \end{aligned} \tag{84}$$

Now equate coefficients to conclude the proof. □

Theorem 6.2. *An integral representation of the fourth Horn function.*

$$\begin{aligned} \mathrm{H}_4(\alpha,\beta,\gamma,\delta;x,y)=\frac{1}{2}\Gamma\left[\begin{array}{c}\gamma,\delta\\ \frac{1}{2},\gamma-\frac{1}{2},\delta-\beta\end{array}\right]\int_0^1\int_0^1 u^{-\frac{1}{2}}(1-u)^{\gamma-\frac{3}{2}}\\ \times\left[(1-2\sqrt{ux}-vy)^{-a}+(1+2\sqrt{ux}-vy)^{-a}\right]v^{\beta-1}(1-v)^{\delta-\beta-1}\,dudv. \end{aligned} \tag{85}$$

Proof.

$$\begin{aligned} &\int_0^1\int_0^1\left[(1-2\sqrt{ux}-vy)^{-a}+(1+2\sqrt{ux}-vy)^{-a}\right]\\ &u^{-\frac{1}{2}}(1-u)^{\gamma-\frac{3}{2}}v^{\beta-1}(1-v)^{\delta-\beta-1}\,dudv\\ &=\int_{\vec{0}}^{\vec{1}}\sum_{k,n=0}^{\infty}\frac{(a)_{k+n}x^{\frac{k}{2}}y^n2^k(1+(-1)^k)}{k!n!}\\ &u^{\frac{k}{2}-\frac{1}{2}}(1-u)^{\gamma-\frac{3}{2}}v^{\beta+n-1}(1-v)^{\delta-\beta-1}\,dudv\\ &=2\int_0^1\int_0^1\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}x^my^n2^{2m}(1+(-1)^k)}{(2m)!n!}\\ &u^{m-\frac{1}{2}}(1-u)^{\gamma-\frac{3}{2}}v^{\beta+n-1}(1-v)^{\delta-\beta-1}\,dudv\\ &=2\Gamma\left[\begin{array}{c}\frac{1}{2},\gamma-\frac{1}{2},\beta,\delta-\beta\\ \gamma,\delta\end{array}\right]\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(\beta)_nx^my^n2^{2m}}{m!n!2^{2m}(\frac{1}{2},\gamma)_m(\delta)_n}. \end{aligned} \tag{86}$$

Now solve for H_4. □

Theorem 6.3. *Another integral representation of the third Horn function.*

$$\begin{aligned}&\mathrm{H}_3(\alpha,\beta,\gamma;x,y)=\frac{1}{2}\Gamma\left[\begin{array}{c}\gamma\\ \frac{1}{2},\beta,\gamma-\beta-\frac{1}{2}\end{array}\right]\int_{\triangle}\int\\ &\times\left[(1-2\sqrt{ux}-vy)^{-a}+(1+2\sqrt{ux}-vy)^{-a}\right]\\ &u^{-\frac{1}{2}}v^{\beta-1}(1-u-v)^{\gamma-\beta-\frac{3}{2}}\,dudv.\end{aligned}\tag{87}$$

Proof.

$$\begin{aligned}&\int_{\triangle}\int\left[(1-2\sqrt{ux}-vy)^{-\alpha}+(1+2\sqrt{ux}-vy)^{-\alpha}\right]\\ &u^{-\frac{1}{2}}v^{\beta-1}(1-u-v)^{\gamma-\beta-\frac{3}{2}}\,dudv\\ &\overset{\text{by}(45)}{=}2\Gamma\left[\begin{array}{c}\frac{1}{2},\beta,\gamma-\frac{1}{2}-\beta\\ \gamma\end{array}\right]\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+n}(\frac{1}{2})_m(\beta)_n x^m y^n}{(\gamma)_{m+n}(2m)!n!}.\end{aligned}\tag{88}$$

Now solve for H_3. □

7. q-analogues of slightly corrected versions of Exton

In this section we shall q-deform some formulas from [14]. The proofs are simple. Assume $C=D$ and $\vec{a},\vec{b},\vec{x}$ are vectors of length n.

Theorem 7.1. *A q-analogue of [14, A 9.1.2]. An integral representation of a q-Kampé de Fériet function.*

$$\begin{aligned}&\int_0^1\ldots(n)\ldots\int_0^1 u_1^{a_1-1}\ldots u_n^{a_n-1}(qu_1;q)_{b_1-1}\ldots(qu_n;q)_{b_n-1}\\ &\times_{C+1}\Phi_D\left[\begin{array}{c}\vec{c},\infty\\ \vec{d}\end{array}\middle|q;x_1u_1\oplus_q\ldots\oplus_q x_nu_n\right]d_q(u_1)\ldots d_q(u_n)\\ &=\Gamma_q\left[\begin{array}{c}\vec{a},\vec{b}\\ a\vec{+}b\end{array}\right]\Phi_{D:1}^{C:2}\left[\begin{array}{cc}\vec{c}: & \infty,a_1;\ldots;\infty,a_n\\ \vec{d}: & a_1+b_1;\ldots;a_n+b_n\end{array}\middle|q;\vec{x}\right].\end{aligned}\tag{89}$$

Theorem 7.2. *A q-analogue of [14, A 9.1.3]. An integral representation of a q-Kampé de Fériet function. We assume that we can also do NWA q-addition for q-shifted factorials.*

$$\int_0^1 \ldots(n)\ldots\int_0^1 u_1^{a_1-1}\ldots u_n^{a_n-1}(qu_1;q)_{b_1-1}\ldots(qu_n;q)_{b_n-1}$$
$$\times {}_{C+1}\Phi_D\left[\begin{array}{c}\vec{c},\infty\\ \vec{d}\end{array}\middle| q;x_1u_1(q^{b_1}u_1;q)_{m_1}\oplus_q\ldots\oplus_q x_nu_n(q^{b_n}u_n;q)_{m_n}\right]$$
$$d_q(u_1)\ldots d_q(u_n)$$
$$=\Gamma_q\left[\begin{array}{c}\vec{a},\vec{b}\\ a\vec{+}b\end{array}\right]\Phi^{C:5}_{D:4}\left[\begin{array}{cc}\vec{c}: & 3\infty,a_1,b_1;\ldots;3\infty,a_n,b_n\\ \vec{d}: & \triangle(q;2;a_1+b_1);\ldots;\triangle(q;2;a_n+b_n)\end{array}\middle| q;\vec{x}\right]. \tag{90}$$

Proof. For simplicity, by NWA on the LHS, we just equate the coefficients of $\sum_{m_i=0}^{\infty}\frac{x^i}{\langle 1;q\rangle_{m_i}}$. By the q-beta integral $\star$,

$$\begin{aligned}&\text{LHS}=\int_0^1 u_i^{a_i-1+m_i}(qu_i;q)_{b_i-1+m_i}d_q(u_i)\overset{\text{by}\star}{=}\Gamma_q\left[\begin{array}{c}a_i+m_i,b_i+m_i\\ a_i+b_i+2m_i\end{array}\right]\\ &=\Gamma_q\left[\begin{array}{c}a_i,b_i\\ a_i+b_i\end{array}\right]\frac{\langle a_i,b_i;q\rangle_{m_i}}{\langle\triangle(q;2;a_i+b_i);q\rangle_{m_i}}=\text{RHS}.\end{aligned} \tag{91}$$

□

8. Conclusion

In this paper we have shown that Euler integrals for multiple hypergeometric functions and multiple q-hypergeometric functions belong together. By using different q-real numbers we have q-deformed some of our formulas. We repeated some formulas from Appell and Kampé de Fériet [2], since this book is not well-known nowadays. Most of the formulas in our paper are new and generalize some well-known ones.

9. Discussion

In future papers we will continue on similar themes, the plan is already clear. We will develop a nomenclature which resemles Kampé de Fériet functions and also study multiple hypergeometric functions of even number of variables.

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T. ERNST
Department of Mathematics
Uppsala University
P.O. Box 480, SE-751 06 Uppsala, Sweden
e-mail: thomas@math.uu.se