LE MATEMATICHE
Vol. LXXVIII (2023) – Issue II, pp. 405–414
doi: 10.4418/2023.78.2.8

SOME APPLICATIONS OF TWO MINIMAX THEOREMS

M. AIT MANSOUR - J. LAHRACHE - N. ZIANE

In this note, we present further applications of two results of Ricceri ([3, Theorem 1.1]) and [4, Theorem 2.4]). In particular, we prove the following: Let $(T,\mathcal{F},\mu)$ be a finite non-atomic measure space, let $[c,d] \subset \mathbf{R}$ be a compact interval and let $\omega,\psi : [c,d] \to [0,+\infty[$ be two continuous concave functions such that $\omega(d) = 0$, $\psi(c) < \psi(d)$ and

$$\sup_{x\in]c,d[} \frac{\omega(x)}{\psi(x)} = 1 \ .$$

Set

$$\delta := \frac{\omega(c)}{\psi(d)-\psi(c)}$$

and, if $\psi(c) > 0$, assume that

$$\sqrt{\delta^2+1}-\delta \leq \frac{\omega(c)}{\psi(c)} \ .$$

Denote by X the set of all measurable functions $u : T \to]c,d[$. Then, we have

$$\inf_{u\in X}\left(\frac{(\int_T \omega(u(t))d\mu)^2+(\int_T \psi(u(t)d\mu)^2}{\int_T \psi(u(t)d\mu}\right) = 2\mu(T)\delta(\sqrt{\delta^2+1}-\delta)\psi(d) \ .$$

Received on November 12, 2023

Keywords: Ricceri's Minimax Theorems, Applications, L^p spaces

1. Introduction and preliminaries

Let X,Y be two topological spaces and $f : X \times Y \to \mathbf{R}$ be a given function. In [2], Ricceri obtained the equality $\sup_Y \inf_X = \inf_X \sup_Y$ assuming, for the first time, that the sub-level sets of $f(\cdot, y)$ are connected. However, because of the great generality of such an assumption, a price has necessarily to be paid: Y must be a real interval. In any case, despite such a restriction, Ricceri's result has successfully been applied to obtain many significant consequences. In particular, in [3] and [4], Ricceri applied his minimax theorem to two specific classes of functions: the functions of the type $\varphi + \psi$, where φ is a non-zero continuous linear functional on a Banach space and ψ is a Lipschitzian functional whose Lipschitz constant is equal to the norm of φ; functionals on L^p spaces. In turn, consequences of the results of [3] and [4] have been obtained very recently by D. Giandinoto in [1].

The aim of this note is to present new applications of the results of [3] and [4], in the spirit of the ones of [1].

2. Infimum of certain functionals on Banach spaces

Throughout this section, X is a real Banach space whose norm is denoted by $\|\,.\,\|$, $\varphi : X \to \mathbb{R}$ is a non-zero continuous linear functional and $\psi : X \to \mathbb{R}$ is a Lipschitzian functional whose Lipschitzian constant L is equal to $\|\varphi\|_{X^*}$, X^* being the dual space of X whose norm is denoted by $\|\,.\,\|_{X^*}$.

Now, we will apply the following result established in [4] with concrete examples :

Theorem 2.1. *Let $\gamma : [-1,1] \to \mathbb{R}$ be a continuous function which is derivable in $]-1,1[$. Assume that γ' is strictly increasing in $]-1,1[$, with $\gamma'(]-1,1[) = \mathbb{R}$. Denote by η the inverse of the function γ'. Then, one has*

$$\max\left\{\inf_{x\in X}\big(\varphi(x)-\psi(x)-\gamma(-1)\big), \inf_{x\in X}\big(\varphi(x)+\psi(x)-\gamma(1)\big)\right\} = \inf_{x\in X}\Big(\varphi(x)+\eta(\psi(x))\psi(x)-\gamma(\eta(\psi(x)))\Big).$$

We start by the following result

Theorem 2.2. *We have*

$$\inf_{x\in X}(\varphi(x)+|\psi(x)|) = \inf_{x\in X}\Big(\varphi(x)-\psi(x)+2\log(e^{\psi(x)}+1)\Big).$$

Proof. Consider the function $\gamma : [-1,1] \to \mathbb{R}$ defined by

$$\gamma(\lambda) = \begin{cases} (1+\lambda)\log(1+\lambda)+(1-\lambda)\log(1-\lambda) \;\; if \;\; |\lambda|<1 \\ \gamma(-1)=\gamma(1)=\log(4). \end{cases}$$

Clearly, γ is continuous in $[-1,1]$ and twice derivable in $]-1,1[$, also we have, for each $\lambda \in]-1,1[$,

$$\begin{cases} \gamma'(\lambda) = \log\left(\dfrac{1+\lambda}{1-\lambda}\right) \\ \\ \gamma''(\lambda) = \dfrac{2}{1-\lambda^2}. \end{cases}$$

Hence, the function γ' is strictly increasing in $]-1,1[$, with $\gamma'(]-1,1[) = \mathbb{R}$. Moreover, η, the inverse of γ', is given by

$$\eta(\mu) = \frac{e^\mu - 1}{e^\mu + 1}.$$

So, for each $x \in X \setminus \psi^{-1}(0)$, we have

$$\begin{aligned} \eta(\psi(x))\psi(x) - \gamma(\eta(\psi(x))) &= \frac{e^{\psi(x)}-1}{e^{\psi(x)}+1}\psi(x) - (1+\eta(\psi(x)))\log(1+\eta(\psi(x))) - (1-\eta(\psi(x)))\log(1-\eta(\psi(x))) \\ &= \frac{e^{\psi(x)}-1}{e^{\psi(x)}+1}\psi(x) - \frac{2e^{\psi(x)}}{e^{\psi(x)}+1}\log(\frac{2e^{\psi(x)}}{e^{\psi(x)}+1}) - \frac{2}{e^{\psi(x)}+1}\log(\frac{2}{e^{\psi(x)}+1}) \\ &= \frac{e^{\psi(x)}-1}{e^{\psi(x)}+1}\psi(x) - \frac{2e^{\psi(x)}}{e^{\psi(x)}+1}\log(\frac{2}{e^{\psi(x)}+1}) - \frac{2e^{\psi(x)}}{e^{\psi(x)}+1}\psi(x) - \frac{2}{e^{\psi(x)}+1}\log(\frac{2}{e^{\psi(x)}+1}) \\ &= -\psi(x) - \frac{2}{e^{\psi(x)}+1}\log(\frac{2}{e^{\psi(x)}+1})(1+e^{\psi(x)}) \\ &= -\psi(x) - \log 4 + 2\log(e^{\psi(x)}+1). \end{aligned}$$

Consequently, by Theorem 2.1, taking account that $\gamma(-1) = \gamma(1) = \log(4)$, we get

$$\max\left\{\inf_{x\in X}(\varphi(x) - \psi(x)) - \log 4, \inf_{x\in X}(\varphi(x) + \psi(x)) - \log 4\right\}$$

$$= \inf_{x\in X}(\varphi(x) + \eta(\psi(x))\psi(x) - \gamma(\eta(\psi(x)))).$$

Taking into account Theorem 4 of [4], this implies that

$$\inf_{x\in X}(\varphi(x) + |\psi(x)|) - \log 4 = \inf_{x\in X}(\varphi(x) - \psi(x) - \log 4 + 2\log(e^{\psi(x)}+1),$$

which is the conclusion.

$\square$

Remark 2.3. In [4], it was observed that

$$\inf_{x\in X}(\varphi(x)+|\psi(x)|)=\inf_{x\in X}(\varphi(x)+|\psi(x)|+e^{-|\psi(x)|}) .$$

From Theorem 2.2 we get

$$\inf_{x\in X}(\varphi(x)+|\psi(x)|)=\inf_{x\in X}\left(\varphi(x)+|\psi(x)|+2\log(e^{-|\psi(x)|}+1)\right)$$

This is an improvement of the result in [4] since $e^{-t}\leq 2\log(e^{-t}+1)$ for all $t\geq 0$.

In particular, since $\inf_{x\in X}(\varphi(x)+\|\varphi\|_{X^*}\|x\|)=0$, from Theorem 2.2 we get:

Corollary 2.4. *We have*

$$\begin{aligned}\inf_{x\in X}(\varphi(x)-\|\varphi\|_{X^*}\|x\|+2\log(e^{\|\varphi\|_{X^*}\|x\|}+1)) &= \inf_{x\in X}(\varphi(x)+\|\varphi\|_{X^*}\|x\|\\ &\quad +2\log(e^{-\|\varphi\|_{X^*}\|x\|}+1))=0.\end{aligned}$$

3. Infimum of functionals in L^p spaces

Let $(T,\mathcal{F},\mu)$ be a σ-finite non-atomic measure space, E a real Banach space, whose norm is denoted by $\|\,.\,\|$, $p\in[1,+\infty[$. As usual, $L^p(T,E)$ denotes the space of all (equivalence classes of) strongly μ-measurable functions $u:T\to E$ such that $\int_T\|u(t)\|^p d\mu<+\infty$, equipped with the norm

$$\|u\|_{L^p(T,E)}=\left(\int_T\|u(t)\|^p d\mu\right)^{\frac{1}{p}} .$$

A set $D\subset L^p(T,E)$ is said to be decomposable if, for every $u,v\in D$ and every $A\in\mathcal{F}$, the function

$$t\to\chi_A(t)u(t)+(1-\chi_A(t))v(t)$$

belongs to D, where χ_A denotes the characteristic function of A. A real-valued function on $T\times E$ is said to be a Carathéodory function if it is measurable in T and continuous in E.

Theorem 3.1. *, [4, Theorem 2.4]). Let $(T,\mathcal{F},\mu)$ be a σ-finite non-atomic measure space, E a real Banach space, $p\in[1,+\infty[$, $X\subset L^p(T,E)$ a decomposable set, $[a,b]$ a compact real interval, and $\gamma:[a,b]\to\mathbb{R}$ a convex (res. concave) and continuous function. Moreover, let $\varphi,\psi,\omega:T\times E\to\mathbb{R}$ be three Caratheodory functions such that, for some $M\in L^1(T),k\in\mathbb{R}$, one has*

$$\max\{|\varphi(t,x)|,|\psi(t,x)|,|\omega(t,x)|\}\leq M(t)+k\|x\|^p$$

for all $(t,x) \in T \times E$ *and*

$$\gamma(a)\int_T \psi(t,u(t))d\mu + a\int_T \omega(t,u(t))d\mu \neq \gamma(b)\int_T \psi(t,u(t))d\mu + b\int_T \omega(t,u(t))d\mu,$$

for all $u \in X$ *such that,* $\int_T \psi(t,u(t))d\mu > 0$ *(resp.* $\int_T \psi(t,u(t))d\mu < 0$*). Then, one has*

$$\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T(\varphi(t,u(t))d\mu+\gamma(\lambda)\psi(t,u(t))+\lambda\omega(t,u(t)))d\mu\right)=\inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\int_T(\varphi(t,u(t))+\gamma(\lambda)\psi(t,u(t))+\lambda\omega(t,u(t)))d\mu\right).$$

From now on, we assume that $\mu(T) < +\infty$. Let $I \subset E$ be a non-empty set. We denote by $\mathcal{A}_I$ the class of all pairs of continuous functions $\omega, \psi : I \to \mathbb{R}$ with $\omega(x) \geq 0$ and $\psi(x) > 0$ for all $x \in I$, such that

$$\sup_{x\in I}\frac{|\omega(x)|+|\psi(x)|}{1+\|x\|^p} < +\infty.$$

Moreover, we denote by $\mathcal{B}_I$ the family of all decomposable subsets X of $L^p(T,E)$ such that $u(T) \subseteq I$ for all $u \in X$, and containing each constant function taking its value in I.

Remark 3.2. Notice that, if $(\omega,\psi) \in \mathcal{A}_I$, we have

$$\inf_{x\in I}\frac{\omega(x)}{\psi(x)} \leq \frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu} \leq \sup_{x\in I}\frac{\omega(x)}{\psi(x)} \quad \text{for all} \quad u \in X.$$

Now, we apply Theorem 3.1 to get the following result:

Theorem 3.3. *Let* $(\omega,\psi) \in \mathcal{A}_I$ *and* $X \in \mathcal{B}_I$*. Then, one has*

$$\inf_{u\in X}\left(2\log\left(\frac{e^{\frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu}}+1}{2}\right)\int_T \psi(u(t))d\mu - \int_T \omega(u(t))d\mu\right)$$

$$= \mu(T)\sup_{\lambda\in[0,1]}\inf_{x\in I}\left(\lambda\omega(x) - ((1+\lambda)\log(1+\lambda)+(1-\lambda)\log(1-\lambda))\psi(x)\right).$$

Proof. First of all, to simplify the writing, for each $u \in X$, we put $\lambda_u = \frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu}$. We apply Theorem 2.1, with $[a,b] = [0,1]$, $\varphi = 0$ and

$$\gamma(\lambda) = \begin{cases} -(1+\lambda)\log(1+\lambda)-(1-\lambda)\log(1-\lambda) & if \;\; \lambda \in [0,1[\\ -\log 4 \;\; if \;\; \lambda = 1. \end{cases}$$

Since γ is concave and $\int_T \psi(u(t)d\mu > 0$ for all $u \in X$, all conditions of Theorem 3.1 are satisfied, and hence

$$\sup_{\lambda\in[0,1]}\inf_{u\in X}(\lambda \int_T \omega(u(t))d\mu + \gamma(\lambda)\int_T \psi(u(t))d\mu) = \inf_{u\in X}\sup_{\lambda\in[0,1]}(\lambda \int_T \omega(u(t))d\mu + \gamma(\lambda)\int_T \psi(u(t))d\mu). \qquad (1)$$

Fix $u \in X$. The function $F : \lambda \to \lambda \int_T \omega(u(t))d\mu + \gamma(\lambda)\int_T \psi(u(t))d\mu$ is concave in $[0,1]$ and its derivative is given by

$$F'(\lambda) = \int_T \omega(u(t))d\mu - \log(\frac{1+\lambda}{1-\lambda})\int_T \psi(u(t))d\mu,$$

which vanishes at the point $\lambda_0 = \frac{e^{\lambda_u}-1}{e^{\lambda_u}+1}$ which lies in $[0,1[$. Consequently, we have

$$\begin{aligned}
\inf_{u\in X}\sup_{\lambda\in[0,1]}\left(\lambda \int_T \omega(u(t))d\mu + \gamma(\lambda)\int_T \psi(u(t))d\mu\right) &= \inf_{u\in X}(\frac{e^{\lambda_u}-1}{e^{\lambda_u}+1}\int_T \omega(u(t))d\mu - (\frac{2e^{\lambda_u}}{e^{\lambda_u}+1}\log(\frac{2e^{\lambda_u}}{e^{\lambda_u}+1}) + \frac{2}{e^{\lambda_u}+1}\log(\frac{2}{e^{\lambda_u}+1}))\int_T \psi(u(t))d\mu) \\
&= \inf_{u\in X}(\frac{e^{\lambda_u}-1}{e^{\lambda_u}+1}\int_T \omega(u(t))d\mu - \frac{2e^{\lambda_u}}{e^{\lambda_u}+1}\int_T \omega(u(t))d\mu - \frac{2e^{\lambda_u}+2}{e^{\lambda_u}+1}\log(\frac{2}{e^{\lambda_u}+1})\int_T \psi(u(t))d\mu) \\
&= \inf_{u\in X}(\int_T -\omega(u(t))d\mu + (\log((e^{\lambda_u}+1)^2) - \log(4))\int_T \psi(u(t))d\mu).
\end{aligned}$$

On the other hand, X contains each constant function taking its value in I, which implies that for all $\lambda \in [0,1]$

$$\inf_{u\in X}\left(\lambda \int_T \omega(u(t))d\mu + \gamma(\lambda)\int_T \psi(u(t))d\mu\right) = \mu(T)\inf_{x\in I}(\lambda\omega(x) + \gamma(\lambda)\psi(x)).$$

Hence, we have

$$\sup_{\lambda\in[0,1]}\inf_{u\in X}(\lambda \int_T \omega(u(t))d\mu + \gamma(\lambda)\int_T \psi(u(t))d\mu) = \mu(T)\sup_{\lambda\in[0,1]}\inf_{x\in I}(\lambda\omega(x) + \gamma(\lambda)\psi(x)). \qquad (2)$$

Now, the conclusion follows directly from (1) and (2).

Theorem 3.4. *Let $(\omega,\psi) \in \mathcal{A}_I$, $X \in \mathcal{B}_I$ and $q > 1$. Set*

$$a := \inf_{x\in I}\left(\frac{\omega(x)}{\psi(x)}\right)^{\frac{1}{q-1}} \quad , \quad b := \sup_{x\in I}\left(\frac{\omega(x)}{\psi(x)}\right)^{\frac{1}{q-1}}$$

and suppose that $b < +\infty$. Then, one has

$$\begin{aligned}
\inf_{u\in X}(\frac{(q-1)(\int_T \omega(u(t))d\mu)^{\frac{q}{q-1}} + (\int_T \psi(u(t)d\mu)^{\frac{q}{q-1}}}{(\int_T \psi(u(t)d\mu)^{\frac{1}{q-1}}}) &= \mu(T)\sup_{\lambda\in[a,b]}\inf_{x\in I}(q\lambda\omega(x) \\
&\quad +(1-\lambda^q)\psi(x)).
\end{aligned}$$

Proof. By Remark 3.2, we have

$$a \leq \left(\frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu} \right)^{\frac{1}{q-1}} \leq b \text{ for all } u \in X.$$

Since X contains each constant function taking its value in I, we clearly have for all $\lambda \in [a,b]$

$$\inf_{u\in X} \left(q\lambda \int_T \omega(u(t))d\mu + (1-\lambda^q) \int_T \psi(u(t))d\mu \right) = \mu(T) \inf_{x\in I}(q\lambda\omega(x) + (1-\lambda^q)\psi(x)).$$

Hence, we obtain

$$\sup_{\lambda\in[a,b]} \inf_{u\in X} \left(q\lambda \int_T \omega(u(t))d\mu + (1-\lambda^q) \int_T \psi(u(t))d\mu \right) = \mu(T) \sup_{\lambda\in[a,b]} \inf_{x\in I}(q\lambda\omega(x) + (1-\lambda^q)\psi(x)). \quad (3)$$

We can apply Theorem 2.1, with $\varphi = 0$, $\gamma(\lambda) = 1-\lambda^q$ (and $q\omega$ instead of ω), obtaining

$$\sup_{\lambda\in[a,b]} \inf_{u\in X} \left(q\lambda \int_T \omega(u(t)d\mu + (1-\lambda^q) \int_T \psi(u(t)d\mu \right) = \inf_{u\in X} \sup_{\lambda\in[a,b]} \left(q\lambda \int_T \omega(u(t)d\mu + (1-\lambda^q) \int_T \psi(u(t)d\mu \right). \quad (4)$$

Fix $u \in X$. The function $F : \lambda \to q\lambda \int_T \omega(u(t)d\mu + (1-\lambda^q) \int_T \psi(u(t)d\mu$ is concave in $[0,+\infty[$ and its derivative is given by

$$F'(\lambda) = q \int_T \omega(u(t)d\mu - q\lambda^{q-1} \int_T \psi(u(t)d\mu,$$

which vanishes at the point $\left(\frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu} \right)^{\frac{1}{q-1}}$ which lies in $[a,b]$. Consequently, we have

$$\inf_{u\in X} \sup_{\lambda\in[a,b]} (q\lambda \int_T \omega(u(t))d\mu + (1-\lambda^q) \int_T \psi(u(t))d\mu) = \inf_{u\in X} \left(\frac{(q-1)(\int_T \omega(u(t))d\mu)^{\frac{q}{q-1}} + (\int_T \psi(u(t)d\mu)^{\frac{q}{q-1}}}{(\int_T \psi(u(t)d\mu)^{\frac{1}{q-1}}} \right)$$

which, jointly with (3.3) and (3.4), gives the conclusion.

$\square$

Now, from Theorem 3.4, we get the following result

Corollary 3.5. *Let $E = \mathbf{R}$, $I =]c,d[$, and let $(\omega, \psi) \in \mathcal{A}_I$. Assume that ω, ψ are continuous and concave in $[c,d]$ and that $\omega(d) = 0$, $\psi(c) < \psi(d)$ and*

$$\sup_{x\in I} \frac{\omega(x)}{\psi(x)} = 1 \ .$$

Set

$$\delta := \frac{\omega(c)}{\psi(d) - \psi(c)}$$

and, if $\psi(c) > 0$, assume that

$$\sqrt{\delta^2+1} - \delta \leq \frac{\omega(c)}{\psi(c)} \ .$$

Then, for every $X \in \mathcal{B}_I$, one has

$$\inf_{u\in X}\left(\frac{(\int_T \omega(u(t))d\mu)^2 + (\int_T \psi(u(t)d\mu)^2}{\int_T \psi(u(t)d\mu}\right) = 2\mu(T)\delta(\sqrt{\delta^2+1} - \delta)\psi(d) \ .$$

Proof. Fix $\lambda \in [0,1]$. Since the function $2\lambda\omega + (1-\lambda^2)\psi$ is concave in $[c,d]$ its infimum is attained either at c or at d. That is to say (recalling that $\omega(d) = 0$)

$$\inf_{x\in I}(2\lambda\omega(x) + (1-\lambda^2)\psi(x)) = \min\{2\lambda\omega(c) + (1-\lambda^2)\psi(c), (1-\lambda^2)\psi(d)\} \ .$$

On the other hand, we have

$$2\lambda\omega(c) + (1-\lambda^2)\psi(c) \leq (1-\lambda^2)\psi(d)$$

if and only if $\lambda \leq -\delta + \sqrt{\delta^2+1}$. Consequently

$$\inf_{x\in I}(2\lambda\omega(x) + (1-\lambda^2)\psi(x)) = \begin{cases} 2\lambda\omega(c) + (1-\lambda^2)\psi(c) \ \ if \ \ \lambda \in \left[0, -\delta + \sqrt{\delta^2+1}\right] \\ \\ (1-\lambda^2)\psi(d) \ \ if \ \ \lambda \in \left[-\delta + \sqrt{\delta^2+1}, 1\right] \ . \end{cases}$$

From this, it clearly follows that

$$\sup_{\lambda\in[0,1]} \inf_{x\in I}(2\lambda\omega(x) + (1-\lambda^2)\psi(x)) = 2\delta(\sqrt{\delta^2+1} - \delta)\psi(d) \ .$$

Now, the conclusion follows directly from Theorem 3.4 applied with $q = 2$.

□

Remark 3.6. Concerning Corollary 3.1, it is very important to observe that the infimum of the restriction of functional $u \to \frac{(\int_T \omega(u(t))d\mu)^2+(\int_T \psi(u(t)d\mu)^2}{\int_T \psi(u(t)d\mu}$ to the set of all constant functions taking their values in $]c,d[$ (say $\tilde{X}$) can be strictly larger than $2\mu(T)\delta(\sqrt{\delta^2+1}-\delta)\psi(d)$. To see this, it is enough to consider the following setting: $[c,d]=[0,1]$, $\omega(x)=1-x^2$, $\psi(x)=x+1$. Indeed, in this case, we have $\delta=1$ and

$$\inf_{u\in\tilde{X}}\left(\frac{(\int_T(1-u(t))d\mu)^2+(\int_T(u(t)+1)d\mu)^2}{\int_T(u(t)+1)d\mu}\right)=\mu(T)\frac{50}{27}>4\mu(T)(\sqrt{2}-1)\ .$$

Acknowledgements

The authors would like to address warm thanks to Professor Biagio Ricceri for his very nice suggestions and discussions on the topic of this paper. In particular, the third author formulates to him sincere gratitude for his guidance and illuminating ideas.

REFERENCES

[1] D. Giandinoto, *Further Applications of two minimax theorems*, Le Matematiche, **77** (2022), 449-463.

[2] B. Ricceri, *Some topological mini-max theorems via an alternative principle for multifunctions*, Arch. Math. (Basel), **60** (1993), 367-377.

[3] B. Ricceri, *On the infimum of certain functionals*, in ”Essays in Mathematics and its Applications - In Honor of Vladimir Arnold”, Th. M. Rassias and P. M. Pardalos eds., 361-367, Springer (2016).

[4] B. Ricceri, *Minimax theorems in a fully non-convex setting*, J. Nonlinear Var. Anal. **3** (2019), 45-52.

M. AIT MANSOUR
Département de Physiques, LPFAS Laboratory, Faculté Poly-disciplinaire, Safi, Université Cadi Ayyad, Morocco
e-mail: `ait.mansour.mohamed@gmail.com`

J. LAHRACHE
Département de Mathématiques, LMFA Laboratory, Faculté des Sciences El Jadida, Université Chouaib Doukkali, B.P 20, El Jadida Morocco.
e-mail: jaafarlahrache@yahoo.fr

N. ZIANE
Département de Mathématiques, LMFA Laboratory, Faculté des Sciences El Jadida, Université Chouaib Doukkali, B.P 20, El Jadida Morocco.
e-mail: zianenour@gmail.com