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EXISTENCE OF HOMOCLINIC SOLUTIONS FOR TWO CLASSES OF DIFFERENTIAL SYSTEMS WITH $p-$LAPLACIAN

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In this paper, we are concerned with a class of periodic differential systems with $p-$Laplacian when the potential is with superquadratic or asymptotically quadratic growth at infinity in the second variable. Using the monotonicity trick of Jeanjean and the concentration compactness principle, we prove the existence of homoclinic solution. Some recent results in the literature are generalized and significantly improved.

1. Introduction

Laplacian and $p-$Laplacian systems are mathematical models used to describe a wide range of phenomena in fields like physics, biology, and engineering. Laplacian systems, being linear, are typically applied in problems such as heat conduction, fluid dynamics, image processing (e.g., image denoising), and geometric modeling. Conversely, $p-$Laplacian systems are nonlinear and are particularly relevant in situations where nonlinear effects play a significant role, such as in modeling the flow of non-Newtonian fluids, phase transitions in materials, and the analysis of complex biological systems. In this paper, we will

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investigate the existence of homoclinic solutions for the following differential $p-$Laplacian system
$(\mathcal{DV})$
$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right)+q(t)|\dot{u}(t)|^{p-2}\dot{u}(t)-V(t)|u(t)|^{p-2}u(t)+\nabla W(t,u(t))=0$$

where $p>1$, $q,V\in C(\mathbb{R},\mathbb{R})$ and $W:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a continuous function, differentiable with respect to the second variable with continuous derivative $\nabla W(t,x)=\frac{\partial W}{\partial x}(t,x)$. A solution u of $(\mathcal{DV})$ is said to be homoclinic (to 0) if $u(t)\to 0$ as $|t|\to+\infty$ and $u\neq 0$. Furthermore, if u minimizes the energy functional of $(\mathcal{DV})$ among all possible nontrivial homoclinic solutions then u is called a ground state homoclinic solution.

Many problems arising in science and engineering call for the solving of partial or ordinary differential equations and systems. These equations or systems are difficult to solve, and there are very few general techniques that can be applied to solve them. In the last fourth decades, critical point theory and variational methods have been highly successful in solving nonlinear problems in partial and ordinary differential equations and systems.

If $p=2$, the $p-$Laplacian system $(\mathcal{DV})$ reduces to the following Laplacian system
$$\ddot{u}(t)+q(t)\dot{u}(t)-V(t)u(t)+\nabla W(t,u(t))=0,\ t\in\mathbb{R},$$
which is a special case of the classical differentrial system
$$\ddot{u}(t)+q(t)\dot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0,\ t\in\mathbb{R}, \tag{1}$$
where $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetrick matrix. We should mention that only a few authors have studied homoclinic solutions for system (1), see [1,5,17,18,37-41,44,45].

When $p=2$ and $q=0$, then (1) reduces to the following second-order Hamiltonian system
$$\ddot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0,t\in\mathbb{R}. \tag{2}$$
With the development of critical point theory, the existence and multiplicity of homoclinic solutions for system (2) have been widely investigated by many authors, see [2-4,6-9,11-13,15,16,19-21,23,25,27-28,30-34,,43,46,47] and the references cited therein.

However, when $p>1$ is arbitrary and $q=0$, the system $(\mathcal{DV})$ takes the form
$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right)-V(t)|u(t)|^{p-2}u(t)+\nabla W(t,u(t))=0. \tag{3}$$
During the last decades there has been a growing interest in studying the existence and multiplicity of homoclinic orbits for system (3), see for example

[24,26,29,35,36,42,48,49,50] and the references therein.
In recent years, Du [10] studied the existence of nontrivial homoclinic solutions for system $(\mathcal{DV})$ when $q(t) = c$ is a constant, V is coercive and $W(t,x) = a(t)U(x)$ with $a \in C(\mathbb{R},\mathbb{R})$ and $U \in C^1(\mathbb{R}^N,\mathbb{R})$ is subquadratic.
Motivated by the above papers, we are interested in the present paper to the existence of homoclinic solutions for $(\mathcal{DV})$ when the nonlinearity $W(t,x)$ is superquadratic or asymptotically quadratic at infinity in the second variable, by using the monotonicity trick of Jeanjean and the concentration compactness principle. To the best of our knowledge, it seems that no similar results are obtained in the literature for differentrial systems.
The remaining of this paper is organized as follows. Section 2 is devoted to some preliminary results. In Section 3, we study the existence of ground state homoclinic solution for $(\mathcal{DV})$ under superquadratic growth. In the last Section, we prove the existence of homoclinic solution for $(\mathcal{DV})$ under asymptotically quadratic growth.

2. Preliminaries

In order to introduce the concept of fast homoclinic solutions for $(\mathcal{DV})$ conveniently, we firstly describe some properties of the weighted Sobolev space E on which the certain variational functional associated with $(\mathcal{DV})$ is defined and the homoclinic solutions of $(\mathcal{DV})$ are the critical points of such functional. We shall use $L^p_Q(\mathbb{R})$ to denote the Banach space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the norm

$$\|u\|_{L^p_Q} = \Big(\int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^p\,dt\Big)^{\frac{1}{p}},$$

where $Q(t) = \int_0^t q(s)ds$. Similarly, $L^s_Q(\mathbb{R})$ $(1 \le s < \infty)$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^s_Q} = \Big(\int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^s\,dt\Big)^{\frac{1}{s}}$$

and $L^\infty_Q(\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^\infty_Q} = ess\sup\Big\{e^{\frac{Q(t)}{2}}\,|u(t)|\,/t \in \mathbb{R}\Big\}.$$

In the present paper, we consider the following condition
(C_1) $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetrick and positive definite matrix, and L,Q are $T-$periodic,
and we introduce the Banach space

$$E = \Big\{u \in L^p_Q(\mathbb{R})/\int_{\mathbb{R}} e^{Q(t)}\Big[\,|\dot{u}(t)|^p + V(t)\,|u(t)|^p\Big]dt < \infty\Big\}$$

equipped with the norm

$$\|u\|^p = \int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}(t)|^p + V(t) |u(t)|^p \right] dt$$

for $u \in E$. It is well known that E is continuously embedded into $L_Q^s(\mathbb{R})$ for $p \leq s \leq \infty$ and $E \hookrightarrow L_{Q,loc}^s(\mathbb{R})$ is compact for all $p \leq s < \infty$. Here, $L_{Q,loc}^s(\mathbb{R})$ is the space of measurable functions $u : \mathbb{R} \to \mathbb{R}^N$ such that $\int_K |u(t)|^s dt < \infty$ for all compact K of $\mathbb{R}$. Hence, there exists a constant $\eta_s > 0$ such that

$$\|u\|_{L_Q^s} \leq \eta_s \|u\|, \ \forall u \in E. \tag{4}$$

We shall prove that system $(\mathcal{DV})$ possesses a mountain pass type solution. For this purpose, we will apply a monotonicity trick due to Jeanjean [14] together with the concentration compactness principle [22].

Lemma 2.1. [14] *Let E be a Banach space and $I \subset \mathbb{R}^+$ be an interval. Consider a family $(f_\lambda)_{\lambda \in I}$ of continuously differentiable functionals on E of the form*

$$f_\lambda(u) = A(u) - \lambda B(u), \ \forall \lambda \in I,$$

where $B(u) \geq 0$ for all $u \in E$, $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. Assume that there exist two points $v_1, v_2 \in E$ such that

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f_\lambda(\gamma(t)) > \max\{f_\lambda(v_1), f_\lambda(v_2)\}, \ \forall \lambda \in I,$$

where

$$\Gamma = \{\gamma \in C([0,1], E) / \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost $\lambda \in I$, there is a sequence $(v_n) \subset E$ such that
(i) (v_n) is bounded in E,
(ii) $f_\lambda(v_n) \to c_\lambda$,
(iii) $f'_\lambda(v_n) \to 0$ on E'.
Moreover, the map $\lambda \to c_\lambda$ is continuous from the left.

Definition 2.2. Let (u_n) be a bounded sequence in a Banach space. We say that (u_n) is vanishing if, for each $R > 0$,

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} e^{Q(t)} |u_n(t)|^p dt = 0$$

and (u_n) is nonvanishing if there exist $\sigma > 0$, $R > 0$ and $(y_n) \subset \mathbb{R}$ such that

$$\liminf_{n \to \infty} \int_{y_n-R}^{y_n+R} e^{Q(t)} |u_n(t)|^p dt \geq \sigma.$$

In the vanishing case, we have the following result, which is a special case of Lions [22].

Lemma 2.3. *Let (u_n) be a bounded sequence, if for each $R > 0$*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} e^{Q(t)} |u_n(t)|^p \, dt = 0,$$

then $u_n \to 0$ in $L^s_Q(\mathbb{R})$ for $p < s < \infty$.

3. Superquadratic growth

In this Section, we are concerned with the existence of ground state homoclinic solution for the differential system $(\mathcal{DV})$ when $W(t,x)$ is periodic in t and superquadratic with respect to the second variable not satisfying the global $(\mathcal{AR})$ superquadratic condition. More precisely, we take the following conditions
(W_1) $\nabla W(t,x) = o(|x|^{p-1})$ as $x \to 0$ uniformly in t, $W(t,0) = 0$ and $W(t,x) \geq 0$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$;
(W_2) there exist constants $\mu > p$ and $C_0 > 0$ such that

$$|\nabla W(t,x)| \leq C_0\big(1+|x|^{\mu-1}\big), \ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N;$$

$$(W_3) \qquad \lim_{|x|\to\infty} \frac{W(t,x)}{|x|^p} = +\infty, \ for \ a.e. \ t \in \mathbb{R};$$

(W_4) there exists a constant $\sigma \geq 1$ such that

$$\widetilde{W}(t,sx) \leq \sigma \widetilde{W}(t,x), \ \forall (s,t,x) \in [0,1]\times\mathbb{R}\times\mathbb{R}^N,$$

where $\widetilde{W}(t,x) = \frac{1}{p}\nabla W(t,x)\cdot x - W(t,x)$.
Our main result in this Section reads as follows

Theorem 3.1. *Assume that (C_1) and $(W_1)-(W_4)$ are satisfied. Then system $(\mathcal{DV})$ possesses at least one ground state homoclinic solution.*

It is very important to notice that conditions (W_2) and (W_4) imply that $W(t,x)$ is superquadratic both at the origin and at infinity, which is different from the $(\mathcal{AR})-$ condition.
Let us state the following example to illustrate our Theorem 3.1.

Example 3.2. Let $q(t) = sin\big(\frac{2\pi}{T}t\big)$, $V(t) = \big(\frac{3}{2}+cos\big(\frac{2\pi}{T}t\big)\big)I_N$ and $W(t,x) = \big(1+sin\big(\frac{2\pi}{T}t\big)\big)\,|x|^p\, ln\big(1+|x|^p\big)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. It is easy to check that Q, L and W satisfy all the conditions of Theorem 3.1. However, since $W(\frac{3T}{4},x) = 0$ for all $x\in\mathbb{R}^N$, it does not satisfy the $(\mathcal{AR})-$condition.

3.1. Proof of Theorem 3.1

Now we are going to establish the corresponding variational framework to obtain the existence of ground state homoclinic solution of $(\mathcal{DV})$. For this end, define the energy functional f associated to system $(\mathcal{DV})$

$$f(u)=\frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}\Big[|\dot{u}(t)|^p+V(t)\,|u(t)|^p\Big]dt-\int_{\mathbb{R}} e^{Q(t)}W(t,u(t))dt$$

defined on the Banach space E introduced in Section 2. By (W_1) and (W_2), for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$|\nabla W(t,x)|\leq \varepsilon\,|x|^{p-1}+C_\varepsilon\,|x|^{\mu-1} \tag{5}$$

and

$$0\leq W(t,x)\leq \frac{\varepsilon}{p}\,|x|^p+\frac{C_\varepsilon}{\mu}\,|x|^\mu \tag{6}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. Hence, it is well known that $f\in C^1(E,\mathbb{R})$ and

$$\begin{aligned} f'(u)v=&\int_{\mathbb{R}} e^{Q(t)}\Big[|\dot{u}(t)|^{p-2}\,\dot{u}(t)\cdot\dot{v}(t)+V(t)\,|u(t)|^{p-2}\,u(t)\cdot v(t)\Big]dt\\ &-\int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u(t))\cdot v(t)dt \end{aligned}$$

for all $u,v\in E$. Moreover, the nontrivial critical points of f on E are homoclinic solutions of $(\mathcal{DV})$. Now, we define on E the family of functionals

$$f_\lambda(u)=A(u)-\lambda B(u),\ \lambda\in[1,2]$$

where

$$A(u)=\frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}\Big[|\dot{u}(t)|^p+V(t)\,|u(t)|^p\Big]dt$$

and

$$B(u)=\int_{\mathbb{R}} e^{Q(t)}W(t,u(t))dt$$

and we present some lemmas which will be used in the subsequent discussion.

Lemma 3.3. *Assume that (C_1) and $(W_1)-(W_3)$ are satisfied. Then*
(i) There exists $u_0\in E\setminus\{0\}$ such that $f_\lambda(u_0)<0$ for all $\lambda\in[1,2]$,

(ii) $$c_\lambda=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} f_\lambda(\gamma(t))>\max\{f_\lambda(0),f_\lambda(u_0)\},\ \forall\lambda\in[1,2],$$

where

$$\Gamma=\{\gamma\in C([0,1],E)/\gamma(0)=0,\gamma(1)=u_0\}.$$

Proof. (i) Let $e_0 \in C_0^\infty(\mathbb{R}) \setminus \{0\}$. By (W_3), the fact $W(t,x) \geq 0$ and Fatou's lemma, we have

$$\lim_{s\to\infty} \frac{f_\lambda(se_0)}{s^p} \leq \lim_{s\to\infty} \frac{f_1(se_0)}{s^p} = \frac{1}{p}\|e_0\|^p - \lim_{s\to\infty} \int_{e_0\neq 0} e^{Q(t)} \frac{W(t,se_0)}{|se_0|^p} |e_0|^p \, dt = -\infty,$$

for all $\lambda \in [1,2]$. Hence, there is $s_0 > 0$ large enough such that $f_1(s_0 e_0) < 0$. Then, setting $u_0 = s_0 e_0$, we obtain $f_\lambda(u_0) \leq f_1(u_0) < 0$ and (i) holds.
(ii) By (4) and (6), we have

$$\int_{\mathbb{R}} e^{Q(t)} W(t,u)dt \leq \frac{\varepsilon}{p} \|u\|^p_{L^p_Q} + \frac{C_\varepsilon}{\mu} \|u\|^\mu_{L^\mu_Q} \leq \frac{\varepsilon \eta_p^p}{p} \|u\|^p + \frac{\eta_\mu^\mu C_\varepsilon}{\mu} \|u\|^\mu,$$

hence

$$f_\lambda(u) \geq f_2(u) \geq \Big(\frac{1}{p} - \varepsilon \eta_p^p\Big) \|u\|^p - \frac{2\eta_\mu^\mu C_\varepsilon}{\mu} \|u\|^\mu.$$

By taking ε small enough, we deduce that there exist constants $\alpha > 0$ and $0 < \rho < \|u_0\|$ such that

$$f_{\lambda_{|\partial B_\rho}} \geq \alpha \textit{ for all } \lambda \in [1,2], \textit{ where } B_\rho = \{u \in E / \|u\| < \rho\}.$$

Letting $\Gamma = \{\gamma \in C([0,1],E)/\gamma(0) = 0, \gamma(1) = u_0\}$. Since, for any $\gamma \in \Gamma$, we have $\gamma(0) = 0 < \rho < \gamma(1) = \|u_0\|$, then there exists $t_\gamma \in]0,1[$ such that $\rho = \gamma(t_\gamma)$ and so

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f_\lambda(\gamma(t)) \geq \alpha > \max\{f_\lambda(0), f_\lambda(u_0)\}.$$

The proof of Lemma 3.3 is completed.

□

Combining Lemmas 2.1,3.3, we obtain

Lemma 3.4. *Assume that (C_1) and $(W_1)-(W_3)$ are satisfied. Then, for any $\lambda \in [1,2]$, there exists a bounded sequence $(u_n) \subset E$ such that $f_\lambda(u_n) \to c_\lambda$ and $f'_\lambda(u_n) \to 0$ as $n \to \infty$.*

Lemma 3.5. *Assume that (W_1) and (W_2) are satisfied. Then, for any bounded vanishing sequence $(u_n) \subset E$, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t,u_n)dt = 0.$$

Proof. Using (5) and (6), we obtain

$$\int_{\mathbb{R}} e^{Q(t)} W(t,u_n)dt \le \frac{\varepsilon}{p} \|u_n\|_{L_Q^p}^p + \frac{C_\varepsilon}{\mu} \|u_n\|_{L_Q^\mu}^\mu$$

and

$$\left| \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u_n) \cdot u_n dt \right| \le \varepsilon \|u_n\|_{L_Q^p}^p + C_\varepsilon \|u_n\|_{L_Q^\mu}^\mu .$$

Since (u_n) is vanishing, Lemma 2.3 implies that

$$\int_{\mathbb{R}} e^{Q(t)} W(t,u_n)dt \to 0 \text{ and } \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u_n) \cdot u_n dt \to 0$$

as $n \to \infty$, and the proof of Lemma 3.5 is completed.

□

Lemma 3.6. *Assume that* (W_1), (W_2) *and* (W_4) *are satisfied. Then, for all bounded sequence* $(u_n) \subset E$ *satisfying*

$$0 < \lim_{n\to\infty} f_\lambda(u_n) \le c_\lambda \text{ and } \lim_{n\to\infty} f'_\lambda(u_n) = 0,$$

there is $(y_n) \subset \mathbb{Z}$ *such that, up to a subsequence,* $\widetilde{u}_n(t) = u_n(t + y_n T)$ *satisfies*

$$\widetilde{u}_n \rightharpoonup u_\lambda \neq 0,\ f_\lambda(u_\lambda) \le c_\lambda \text{ and } f'_\lambda(u_\lambda) = 0.$$

Proof. Since $f'_\lambda(u_n)u_n \to 0$, one has

$$\lim_{n\to\infty} \lambda \int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t,u_n)dt = \lim_{n\to\infty} \left(f_\lambda(u_n) - \frac{1}{p} f'_\lambda(u_n)u_n \right) = \lim_{n\to\infty} f_\lambda(u_n) > 0$$

which with Lemma 3.5 implies that (u_n) is nonvanishing. Hence, there exist constants $\sigma > 0$, $R > 0$ and a subsequence $(\widetilde{y}_n) \subset \mathbb{R}$ such that

$$\liminf_{n\to\infty} \int_{\widetilde{y}_n - R}^{\widetilde{y}_n + R} e^{Q(t)} |u_n|^p dt \ge \sigma > 0.$$

Choose $(y_n) \subset \mathbb{Z}$ such that, letting $\widetilde{u}_n(t) = u_n(t + y_n T)$,

$$\liminf_{n\to\infty} \int_{-2R}^{2R} e^{Q(t)} |\widetilde{u}_n|^p dt \ge \frac{\sigma}{2} > 0. \tag{7}$$

Since $Q(t)$, $V(t)$ and $W(t,x)$ are $T-$periodic in t, then $\|\widetilde{u}_n\| = \|u_n\|$, $f_\lambda(\widetilde{u}_n) = f_\lambda(u_n)$ and

$$f'_\lambda(\widetilde{u}_n) \to 0 \text{ as } n \to \infty. \tag{8}$$

Indeed, for any $v \in E$, set $v_n(t) = v(t - y_n T)$. It is clear that $\|v_n\| = \|v\|$ and

$$
\begin{aligned}
\left|f'_\lambda(\widetilde{u}_n)v\right| &= \left|\int_{\mathbb{R}} e^{Q(t)}\left[\left|\dot{\widetilde{u}}_n\right|^{p-2}\dot{\widetilde{u}}_n\cdot\dot{v} + V(t)\left|\widetilde{u}_n\right|^{p-2}\widetilde{u}_n\cdot v - \lambda\nabla W(t,\widetilde{u}_n)\cdot v\right]dt\right| \\
&= \left|\int_{\mathbb{R}} e^{Q(t)}\left[\left|\dot{u}_n\right|^{p-2}\dot{u}_n\cdot\dot{v}_n + V(t)\left|u_n\right|^{p-2}u_n\cdot v_n - \lambda\nabla W(t,u_n)\cdot v_n\right]dt\right| \\
&= \left|f'_\lambda(u_n)v_n\right| \le \left\|f'_\lambda(u_n)\right\|\|v_n\| = \left\|f'_\lambda(u_n)\right\|\|v\| \to 0,
\end{aligned}
$$

which implies (8). Since $(\widetilde{u}_n)$ is still bounded, up to a subsequence if necessary, there exists $u_\lambda \in E$ such that

$$
\begin{aligned}
&\widetilde{u}_n \rightharpoonup u_\lambda \ in\ E, \\
&\widetilde{u}_n \to u_\lambda \ in\ L^s_{Q,loc}(\mathbb{R})\ for\ s \in]p,\infty[, \\
&\widetilde{u}_n \to u_\lambda \ a.e.\ in\ \mathbb{R},
\end{aligned}
\tag{9}
$$

and $u_\lambda \neq 0$ by (7). We claim that for all compact $K \subset \mathbb{R}$, $\nabla W(t,\widetilde{u}_n) \to \nabla W(t,u_\lambda)$ in $L^{p'}_Q(K)$. Arguing indirectly, we may assume that there exist a constant $\varepsilon_0 > 0$ and a subsequence $(\widetilde{u}_{n_k})$ such that

$$
\int_K e^{Q(t)}\left|\nabla W(t,\widetilde{u}_{n_k}) - \nabla W(t,u_\lambda)\right|^{p'}dt \ge \varepsilon_0,\ \forall k \in \mathbb{N}. \tag{10}
$$

By (9), we can assume that

$$
\sum_{k=1}^{\infty}\|\widetilde{u}_{n_k} - u_\lambda\|_{L^p_Q(K)} < \infty \ and\ \sum_{k=1}^{\infty}\|\widetilde{u}_{n_k} - u_\lambda\|_{L^{p'(\mu-1)}_Q(K)} < \infty.
$$

Let $w(t) = \sum_{k=1}^{\infty}|\widetilde{u}_{n_k}(t) - u_\lambda(t)|$ for all $t \in K$. Then $w \in L^p_Q(K)\bigcap L^{p'(\mu-1)}_Q(K)$. By (5), there holds for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$

$$
\begin{aligned}
\left|\nabla W(t,\widetilde{u}_{n_k}) - \nabla W(t,u_\lambda)\right|^{p'} &\le 2^{p'-1}\left(\left|\nabla W(t,\widetilde{u}_{n_k})\right|^{p'} + \left|\nabla W(t,u_\lambda)\right|^{p'}\right) \\
&\le 2^{p'-1}\left[\left(\varepsilon\left|\widetilde{u}_{n_k}\right|^{p-1} + C_\varepsilon\left|\widetilde{u}_{n_k}\right|^{\mu-1}\right)^{p'}\right. \\
&\quad \left. + [\left(\varepsilon\left|u_\lambda\right|^{p-1} + C_\varepsilon\left|u_\lambda\right|^{\mu-1}\right)^{p'}\right] \\
&\le C_1\left[\left|\widetilde{u}_{n_k}\right|^p + \left|\widetilde{u}_{n_k}\right|^{p'(\mu-1)} + \left|u_\lambda\right|^p + \left|u_\lambda\right|^{p'(\mu-1)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le C_1\left[\left(\left|\widetilde{u}_{n_k} - u_\lambda\right| + \left|u_\lambda\right|\right)^{p'} + \left(\left|\widetilde{u}_{n_k} - u_\lambda\right|\right.\right. \\
&\quad \left.\left. + \left|u_\lambda\right|\right)^{p'(\mu-1)} + \left|u_\lambda\right|^p + \left|u_\lambda\right|^{p'(\mu-1)}\right] \\
&\le C_2\left[\left|w\right|^p + \left|w\right|^{p'(\mu-1)} + \left|u_\lambda\right|^p + \left|u_\lambda\right|^{p'(\mu-1)}\right]
\end{aligned}
$$

where C_1, C_2 are positive constants. Combining this with (9), Lebesgue's Dominated Convergence Theorem implies

$$\lim_{k\to\infty}\int_K e^{Q(t)}\,|\nabla W(t,\widetilde{u}_{n_k}) - \nabla W(t,u_\lambda)|^{p'}\,dt = 0$$

which contradicts (10). Hence the claim above is true. It follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}} e^{Q(t)}\Big(\nabla W(t,\widetilde{u}_n) - \nabla W(t,u_\lambda)\Big)\psi dt = 0,\ \forall\psi\in C_0^\infty(\mathbb{R},\mathbb{R}^N)$$

which implies that f'_λ is weakly sequentially continuous. Hence, by (8), we deduce

$$f'_\lambda(u_\lambda) = 0. \tag{11}$$

Now, by (W_4) and Fatou's lemma, one gets

$$\begin{aligned} c_\lambda &\geq \lim_{n\to\infty}\Big(f_\lambda(\widetilde{u}_n) - \frac{1}{p}f'_\lambda(\widetilde{u}_n)\widetilde{u}_n\Big) \\ &= \lim_{n\to\infty}\lambda\int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,\widetilde{u}_n)dt \\ &\geq \lambda\int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,u_\lambda)dt \\ &= f_\lambda(u_\lambda) - \frac{1}{p}f'_\lambda(u_\lambda)u_\lambda \\ &= f_\lambda(u_\lambda) \end{aligned}$$

The proof of Lemma 3.6 is completed. □

As a consequence of Lemmas 3.4,3.6, we have the following

Lemma 3.7. *Assume that (C_1), (W_1), (W_2) and (W_4) are satisfied. Then there exist $(\lambda_n)\subset[1,2]$ and $(u_n)\subset E\setminus\{0\}$ such that*

$$\lambda_n\to 1,\ f_{\lambda_n}(u_n)\leq c_{\lambda_n}\ \textit{and}\ f'_{\lambda_n}(u_n) = 0.$$

Lemma 3.8. *Under the assumptions of Theorem 3.1, the sequence (u_n) obtained in Lemma 3.7 is bounded.*

Proof. Suppose by contradiction that $\|u_n\|\to\infty$ as $n\to\infty$. Set $w_n = \frac{u_n}{\|u_n\|}$. Then $\|w_n\| = 1$, and by Lion's concentration compactness principle [22], either (w_n) is vanishing or it is nonvanishing. Hence the proof of the lemma will be completed if we show that (w_n) is neither vanishing nor nonvanishing. Assume that (w_n) is vanishing. Let $(s_n) \subset [0,1]$ be a sequence such that

$$f_{\lambda_n}(s_n u_n) = \max_{s\in[0,1]} f_{\lambda_n}(s u_n).$$

For any $M > 0$, let $v_n = \left(\frac{2\sqrt{M}}{\|u_n\|}\right) u_n = 2\sqrt{M} w_n$. Since (v_n) is vanishing and bounded, by Lemma 3.5 and (5), one has

$$\int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t, v_n) dt \to 0 \; as \; n \to \infty.$$

Now, for n large enough, $\frac{2\sqrt{M}}{\|u_n\|} \in]0,1[$, and by the definition of s_n, we deduce that

$$f_{\lambda_n}(s_n u_n) \geq f_{\lambda_n}(v_n) = 2M - \lambda_n \int_{\mathbb{R}} e^{Q(t)} W(t, v_n) dt \geq M,$$

which implies that

$$f_{\lambda_n}(s_n u_n) \to +\infty \; as \; n \to \infty. \tag{12}$$

Since $f_{\lambda_n}(0) = 0$ and $f_{\lambda_n}(u_n) \leq c_{\lambda_n} \leq c_1$, then $s_n \in]0,1[$ and

$$f'_{\lambda_n}(s_n u_n) s_n u_n = s_n \frac{d}{ds}\Big(f_{\lambda_n}(s u_n)\Big)_{|s=s_n} = 0. \tag{13}$$

Therefore, using (12) and (13), we deduce that

$$\begin{aligned} \int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t, s_n u_n) dt &= \frac{1}{\lambda_n}\Big(f_{\lambda_n}(s_n u_n) - \frac{1}{p} f'_{\lambda_n}(s_n u_n) s_n u_n\Big) \\ &= \frac{1}{\lambda_n} f_{\lambda_n}(s_n u_n) \to +\infty \; as \; n \to \infty. \end{aligned}$$

However, it follows from (W_4) and Lemma 3.7 that

$$\begin{aligned} \int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t, s_n u_n) dt &\leq \sigma \int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t, u_n) dt \\ &\leq \frac{\sigma}{\lambda_n}\Big[f_{\lambda_n}(u_n) - \frac{1}{p} f'_{\lambda_n}(u_n) u_n\Big] \\ &= \frac{\sigma}{\lambda_n} f_{\lambda_n}(u_n) \leq \frac{\sigma}{\lambda_n} c_{\lambda_n} \leq \sigma c_1, \; \forall n \in \mathbb{N}, \end{aligned}$$

yielding a contradiction.

Assume that (w_n) is nonvanishing. Then, as in the proof of (10), by the translation invariance of system $(\mathcal{DV})$, one has $w_n \rightharpoonup w$ in E and $w_n(t) \to w(t)$ a.e. in $\mathbb{R}$ for some $w \in E \setminus \{0\}$. On the set $\Omega = \{t \in \mathbb{R}/w(t) \neq 0\}$, one has $|u_n(t)| \to +\infty$, and then by (W_3),

$$\frac{W(t, u_n)}{|u_n|^p} |w_n|^p \to +\infty \; as \; n \to \infty.$$

Therefore, taking into account $meas(\Omega) > 0$ and using Fatou's lemma, we obtain

$$\int_{\mathbb{R}} e^{Q(t)} \frac{W(t,u_n)}{\|u_n\|^p} dt \geq \int_{\Omega} e^{Q(t)} \frac{W(t,u_n}{|u_n|^p} |w_n|^p \, dt \to +\infty \; as \; n \to \infty.$$

On the other hand, since $0 \leq f_{\lambda_n}(u_n) \leq c_{\lambda_n} \leq c_1$, we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{Q(t)} \frac{W(t,u_n)}{\|u_n\|^p} dt = \frac{1}{p},$$

a contradiction. The proof of Lemma 3.8 is completed.

□

Now, we are in the position to complete the proof of Theorem 3.1. By Lemmas 3.7, 3.8 and (5), we have for any $v \in E$

$$f'(u_n)v = f'_{\lambda_n}(u_n)v + (\lambda_n - 1) \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u_n) \cdot v dt \to 0 \; as \; n \to \infty.$$

Hence $f'(u_n) \to 0$. Combining (5) and Lemma 3.8 yields

$$\begin{aligned} \|u_n\|^p &= \lambda_n \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u_n) \cdot u_n dt \\ &\leq \varepsilon \|u_n\|^p_{L^p_Q} + C_\varepsilon \|u_n\|^\mu_{L^\mu_Q} \\ &\leq \varepsilon \eta_p^p \|u_n\|^p + C_\varepsilon \eta_\mu^\mu \|u_n\|^\mu \end{aligned} \tag{14}$$

which implies that $\|u_n\| \geq d$, $\forall n \in \mathbb{N}$ for some constant $d > 0$. If (u_n) is vanishing, Lemma 2.3 and (14) imply that $u_n \to 0$, a contradiction. Hence (u_n) is nonvanishing. Proceeding as in the proof of Lemma 3.7, we can obtain a sequence $(y_n) \subset \mathbb{Z}$ such that if $\widetilde{u}_n(t) = u_n(t + y_n T)$, then $\widetilde{u}_n \to \widetilde{u}$ and $f'(\widetilde{u}) = 0$. Therefore system $(\mathcal{DV})$ possesses a nontrivial homoclinic solution.

Finally, we prove the existence of ground state homoclinic solution of $(\mathcal{DV})$. Set $K = \{u \in E \setminus \{0\} / f'(u) = 0\}$ and $m = \inf_K f(u)$. Using (W_4) and the fact $\widetilde{u} \in K$, we get $0 \leq m \leq f(\widetilde{u})$. By the definition of m, there exists a sequence $(v_n) \subset K$ such that $f(v_n) \to m$ as $n \to \infty$. Following the same procedures as in the proof of Lemma 3.8, we have (v_n) is bounded. Since $(v_n) \subset K$ and $f'(v_n) = 0$, similar to (14), we obtain $\|v_n\| \geq d_1 > 0$ for all n and (v_n) is nonvanishing. Hence, arguing as in (9)-(11), there exists $\widetilde{v} \in E \setminus \{0\}$ such that $f'(\widetilde{v}) = 0$ and $f(\widetilde{v}) \leq m$. Noting that $\widetilde{v} \in K$, one has $f(\widetilde{v}) \geq m$. Thus $f(\widetilde{v}) = m$. This ends the proof of Theorem 3.1.

4. Asymptotically quadratic growth

In this Section, we are concerned with the existence of homoclinic solution for the differentrial system $(\mathcal{DV})$ when $W(t,x)$ is periodic in t and asymtotically quadratic with respect to the second variable. More precisely, we assume that $W(t,x)$ is of the form

$$W(t,x) = \frac{1}{p}S\,|x|^p + V(t,x)$$

where S is a positive constant and we take the following assumptions:

$$(W_5) \qquad W(t,x) \geq 0,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,$$

(W_6) There exist positive constants c, r and $\mu > p$ such that

$$|\nabla W(t,x)| \leq c\,|x|^{\mu-1},\ \forall t \in \mathbb{R},\ |x| \leq r,$$

$$(W_7) \qquad V(t,x) = o(|x|^{p-1})\ as\ |x| \to \infty,$$

(W_8) $\widetilde{W}(t,x) = \frac{1}{p}\nabla W(t,x)\cdot x - W(t,x) \geq 0$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, and there are positive constants $a, b > 0$, $R > r$ and $\alpha \in]1, p[$ such that

$$\widetilde{W}(t,x) \geq \begin{cases} a\,|x|^{\mu},\ \forall t \in \mathbb{R},\ |x| \leq r, \\ b\,|x|^{\alpha},\ \forall t \in \mathbb{R},\ |x| \geq R. \end{cases}$$

Theorem 4.1. *Assume that (C_1), $(W_5) - (W_8)$ and the following condition*

$$(C_2) \qquad \inf_{u\in E, u\neq 0} \frac{\int_{\mathbb{R}} e^{Q(t)}[|\dot{u}(t)|^p + V(t)\,|u(t)|^p]dt}{\int_{\mathbb{R}} |u(t)|^p\,dt} < S$$

are satisfied. Then the differentrial system $(\mathcal{DV})$ possesses at least one homoclinic solution.

Remark 4.2. Let

$$W(t,x) = \begin{cases} \left(\frac{1}{p}S - a(t)\right)|x|^p\ if\ |x| \leq 1, \\ \frac{1}{p}S\,|x|^p - a(t)\,|x|^{\alpha}\ if\ |x| \geq 1, \end{cases}$$

where $a \in C(\mathbb{R},\mathbb{R})$ is periodic in t, $0 < \inf_{t\in\mathbb{R}} a(t) \leq \sup_{t\in\mathbb{R}} a(t) < \frac{S}{p}$ and $1 < \alpha < p$. It is easy to check that the above function W satisfies conditions $(W_5) - (W_8)$.

4.1. Proof of Theorem 4.1

Consider the functional f defined on the space E introduced in Section 2 by

$$f(u) = \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}\left[|\dot{u}(t)|^p + V(t)\,|u(t)|^p\right]dt - \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))dt$$
$$= \frac{1}{p}\,\|u\|^p - \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))dt.$$

It is well known that f is continuously differentiable on E and for all $u, v \in E$, we have

$$f'(u)v = \int_{\mathbb{R}} e^{Q(t)}\left[|\dot{u}(t)|^{p-2}\,u(t)\cdot\dot{v}(t) + V(t)\,|u(t)|^{p-2}\,u(t)\cdot v(t)\right]dt$$
$$- \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u(t))\cdot v(t)dt.$$

Moreover, critical points of f are classical solutions of $(\mathcal{DV})$ satisfying $\dot{u}(t)\to 0$ as $|t|\to\infty$.
In the following, we will reason by successive lemmas.

Lemma 4.3. *Assume that (C_1), (W_6) and (W_7) are satisfied. Then for any bounded vanishing sequence $(u_n)\in E$, we have*

$$\lim_{n\to\infty}\int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt = 0.$$

Proof. By (W_6) and (W_7), for every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$|\nabla W(t,x)| \le \varepsilon\,|x|^{p-1} + C_\varepsilon\,|x|^{\mu-1},\ \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N. \tag{15}$$

Since $W(t,0) = 0$, we deduce

$$|W(t,x)| \le \frac{\varepsilon}{p}\,|x|^p + \frac{C_\varepsilon}{\mu}\,|x|^\mu,\ \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N. \tag{16}$$

Let $(u_n)\subset E$ be a bounded vanishing sequence. Then Lemma 2.3 implies that $u_n\to 0$ in $L^s_Q(\mathbb{R})$ for all $s\in]p,\infty[$. Combining this with (15) and (16) yields

$$\left|\int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u_n(t))\cdot u_n(t)dt\right| \le \varepsilon\,\|u_n\|^p_{L^p_Q} + C_\varepsilon\,\|u_n\|^\mu_{L^\mu_Q} \to 0 \; as\; n\to\infty$$

and

$$\int_{\mathbb{R}} e^{Q(t)}W(t,u_n(t)) \le \frac{\varepsilon}{p}\,\|u_n\|^p_{L^p_Q} + \frac{C_\varepsilon}{\mu}\,\|u_n\|^\mu_{L^\mu_Q} \to 0 \; as\; n\to\infty.$$

Hence $\int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt \to 0$ as $n\to\infty$ and the proof of Lemma 4.3 is completed. □

In the following, we define on E the family of functionals

$$f_\lambda(u) = A(u) - \lambda B(u),\ \lambda \in [1,2]$$

where

$$A(u) = \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}\Big[|\dot{u}(t)|^p + V(t)\,|u(t)|^p\Big]dt$$

and

$$B(u) = \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt.$$

Lemma 4.4. *Assume that (C_1), (C_2) and (W_7) are satisfied, then there exists $v_0 \in E\setminus\{0\}$ such that $f_1(v_0) = f(v_0) < 0$.*

Proof. By (C_2), we can choose a nonnegative function $\varphi \in E$ such that

$$\int_{\mathbb{R}} e^{Q(t)}\,|\varphi(t)|^p\,dt = 1 \text{ and } \int_{\mathbb{R}} e^{Q(t)}\Big[|\dot{\varphi}(t)|^p + V(t)\,|\varphi(t)|^p\Big]dt < S.$$

Assumption (W_7) implies that for all $t \in \mathbb{R}$ with $\varphi(t) \neq 0$

$$\lim_{s\to\infty}\frac{W(t,s\varphi(t))}{s^p} = \lim_{s\to\infty}\frac{W(t,s\varphi(t))}{|s\varphi(t)|^p}\,|\varphi(t)|^p = \frac{1}{p}S\,|\varphi(t)|^p,$$

which together with (C_2) and Fatou's lemma implies

$$\begin{aligned}
\lim_{s\to\infty}\frac{f(s\varphi)}{s^p} &= \frac{1}{p}\int_{\mathbb{R}}\Big[|\dot{\varphi}(t)|^p + V(t)\,|\varphi(t)|^p\Big]dt - \lim_{s\to\infty}\int_{\mathbb{R}} e^{Q(t)}\frac{W(t,s\varphi(t))}{s^p}dt\\
&< \frac{S}{p} - \int_{\mathbb{R}} e^{Q(t)}\lim_{s\to\infty}\frac{W(t,s\varphi(t))}{s^p}dt\\
&\leq \frac{S}{p} - \int_{\mathbb{R}} e^{Q(t)}\frac{S}{p}\,|\varphi(t)|^p\,dt = 0.
\end{aligned}$$

Consequently, there exists a positive constant s_0 large enough such that the element $v_0 = s_0\varphi$ satisfies $v_0 \neq 0$ and $f(v_0) < 0$. The proof of Lemma 4.4 is completed.

□

Now, let

$$c_\lambda = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} f_\lambda(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1],E)/\gamma(0) = 0, \gamma(1) = v_0\}.$$

Lemma 4.5. *Assume that* $(W_6)-(W_8)$ *are satisfied. Then for any sequence* $(u_n)\subset E$ *satisfying*

$$0<\lim_{n\to\infty} f_\lambda(u_n)\le c_\lambda \text{ and } f'_\lambda(u_n)\to 0 \text{ as } n\to\infty,$$

there exists a subsequence (u_{n_k}) *such that*

$$u_{n_k}\rightharpoonup u_\lambda\neq 0 \text{ with } f_\lambda(u_\lambda)\le c_\lambda \text{ and } f'_\lambda(u_\lambda)=0.$$

Proof. Note that

$$\int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt=\frac{1}{\lambda}\left[f_\lambda(u_n)-\frac{1}{p}f'_\lambda(u_n)u_n\right]\to\frac{1}{\lambda}\lim_{n\to\infty}f_\lambda(u_n)>0.$$

Since (u_n) is bounded, then Lemma 4.3 implies that (u_n) does not vanish, i.e., there exist positive constants $r,\delta>0$ and a sequence $(s_n)\subset\mathbb{R}$ such that

$$\lim_{n\to\infty}\int_{I_r(s_n)} e^{Q(t)}\,|u_n|^p\,dt\ge\delta, \tag{17}$$

where $I_r(s_n)=[s_n-r,s_n+r]$. From the boundedness of (u_n), we can assume, after passing to a subsequence, that $u_n\rightharpoonup u_\lambda$ in E and $u_n\to u_\lambda$ in $L^p_{Q,loc}(\mathbb{R})$, which with (17) implies that $u_\lambda\neq 0$. By the weakly sequentially continuity of f_λ and the fact $f'_\lambda(u_n)\to 0$ as $n\to\infty$, we obtain

$$f'_\lambda(u_\lambda)v=\lim_{n\to\infty}f'_\lambda(u_n)v=0,\ \forall v\in E.$$

Hence $f'_\lambda(u_\lambda)=0$. Combining (W_8) with Fatou's lemma yields

$$\begin{aligned}
c_\lambda &\ge \lim_{n\to\infty}f_\lambda(u_n)=\lim_{n\to\infty}\left[f_\lambda(u_n)-\frac{1}{p}f'_\lambda(u_n)u_n\right]\\
&=\lim_{n\to\infty}\lambda\int_{\mathbb{R}}e^{Q(t)}\widetilde{W}(t,u_n(t))dt\ge\lambda\int_{\mathbb{R}}e^{Q(t)}\widetilde{W}(t,u_\lambda(t))dt\\
&=f_\lambda(u_\lambda)-\frac{1}{p}f'_\lambda(u_\lambda)u_\lambda=f_\lambda(u_\lambda).
\end{aligned}$$

The proof of Lemma 4.5 is completed. □

Lemma 4.6. *Assume that* $(W_5)-(W_7)$ *are satisfied. Then for any* $\lambda\in[1,2]$, *there exists a sequence* $(v_n)\subset E$ *such that*

$$(v_n) \text{ is bounded},\ f_\lambda(v_n)\to c_\lambda \text{ and } f'_\lambda(v_n)\to 0. \tag{18}$$

Proof. For the $v_0 \in E$ obtained in Lemma 4.4, we have $f(v_0) < 0$. It follows from (W_5) that $f_\lambda(v_0) \leq f(v_0) < 0$, for all $\lambda \in [1,2]$. By (16) and (4), we get

$$\int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt \leq \frac{\varepsilon}{p} \eta_p^p \|u\|^p + \frac{C_\varepsilon}{\mu} \eta_\mu^\mu \|u\|^\mu, \ \forall u \in E.$$

Since ε is arbitrary, then

$$\int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt = o(\|u\|^p) \text{ as } u \to 0.$$

Hence, there exists a constant $0 < r_0 < \|v_0\|$ such that

$$\int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt \leq \frac{1}{2p} \|u\|^p, \ \forall \|u\| \leq r_0.$$

For all $\gamma \in \Gamma$, there is $s_\gamma \in [0,1]$ such that $\left\|\gamma(s_\gamma)\right\| = r_0$ and

$$\begin{aligned} \max_{s\in[0,1]} f_\lambda(\gamma(s)) &= f_\lambda(\gamma(s_\gamma)) \\ &= \frac{1}{p} \left\|\gamma(s_\gamma)\right\|^p - \int_{\mathbb{R}} e^{Q(t)} W(t,\gamma(s_\gamma))dt \\ &\geq \frac{1}{2p} \left\|\gamma(s_\gamma)\right\|^p = \frac{r_0^p}{2p} \end{aligned}$$

which implies that

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} f_\lambda(\gamma(s)) \geq \frac{r_0^p}{2p} > 0, \ \forall \lambda \in [1,2]$$

and

$$c_\lambda > \max\{f_\lambda(0), f_\lambda(v_0)\}.$$

Hence, the family $(f_\lambda)_{\lambda\in[1,2]}$ satisfies the hypotheses of Lemma 2.1, which completes the proof of Lemma 4.6. □

Combining Lemmas 4.5 and 4.6, we deduce that there exist a sequence $(\lambda_n) \subset [1,2]$ converging to 1 and a sequence $(u_n) \subset E$ satisfying

$$u_n \neq 0, \ f_{\lambda_n}(u_n) \leq c_{\lambda_n} \text{ and } f'_{\lambda_n}(u_n) = 0. \tag{19}$$

Since

$$\frac{1}{p} \|u_n\|^p - \lambda_n \int_{\mathbb{R}} e^{Q(t)} W(t,u_n(t))dt \leq c_{\lambda_n}$$

and

$$\|u_n\|^p = \lambda_n \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u_n(t)) \cdot u_n dt,$$

we deduce that
$$\int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt \le \frac{c_{\lambda_n}}{\lambda_n},\ \forall n\in\mathbb{N}.$$
It is clear that $(\frac{c_{\lambda_n}}{\lambda_n})$ is decreasing and bounded by c_1, which implies that
$$\int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt \le c_1,\ \forall n\in\mathbb{N}.$$

Lemma 4.7. *Assume that (C_1) and $(W_6)-(W_8)$ are satisfied. Then the sequence obtained in (19) is bounded.*

Proof. Using (W_7) and (W_8) respectively, we can find a positive constant C_1 such that
$$\int_{\{t\in\mathbb{R}/|u_n(t)|\ge r\}} e^{Q(t)}\left|\nabla W(t,u_n(t))\right||u_n|\,dt \le C_1\int_{\{t\in\mathbb{R}/|u_n(t)|\ge r\}} e^{Q(t)}\left|u_n(t)\right|^p dt \tag{20}$$
and
$$\begin{aligned}&\int_{\{t\in\mathbb{R}/|u_n(t)|\ge r\}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt\\ &=\int_{\{t\in\mathbb{R}/r\le|u_n(t)|\le R\}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt+\int_{\{t\in\mathbb{R}/|u_n(t)|\ge R\}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt\\ &\ge\frac{1}{R^\alpha}\inf_{\{t\in\mathbb{R},r\le|x|\le R\}}\widetilde{W}(t,x)\int_{\{t\in\mathbb{R}/r\le|u_n(t)|\le R\}} e^{Q(t)}\left|u_n(t)\right|^\alpha dt\\ &+b\int_{\{t\in\mathbb{R}/|u_n(t)|\ge R\}} e^{Q(t)}\left|u_n(t)\right|^\alpha dt\\ &\ge C_2\int_{\{t\in\mathbb{R}/|u_n(t)|\ge r\}} e^{Q(t)}\left|u_n(t)\right|^\alpha dt\end{aligned} \tag{21}$$
where $C_2=\inf\left\{\frac{1}{R^\alpha}\inf_{\{t\in\mathbb{R},r\le|x|\le R\}}\widetilde{W}(t,x),b\right\}$. By (18), we have for a positive constant C_3
$$\frac{f_{\lambda_n}(u_n)-\frac{1}{p}f'_{\lambda_n}(u_n)u_n}{\lambda_n}\le C_3,$$
which with (W_8) and (21) implies
$$\begin{aligned}C_3\ge\frac{f_{\lambda_n}(u_n)-\frac{1}{p}f'_{\lambda_n}(u_n)u_n}{\lambda_n}&=\int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt\\ &=\int_{\{t\in\mathbb{R}/|u_n(t)|\le r\}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt+\int_{\{t\in\mathbb{R}/|u_n(t)|\ge r\}} e^{Q(t)}\widetilde{W}(t,u_n(t))dt\\ &\ge a\int_{\{t\in\mathbb{R}/|u_n(t)|\le r\}} e^{Q(t)}\left|u_n\right|^\mu dt+C_2\int_{\{t\in\mathbb{R}/|u_n(t)|\ge r\}} e^{Q(t)}\left|u_n\right|^\alpha dt.\end{aligned} \tag{22}$$

Take $s \in]0, \frac{\alpha}{p}[$, then Hölder's inequality, (22) and (4) imply

$$\begin{aligned}
&\int_{\{t\in\mathbb{R}/|u_n(t)|\geq r\}} e^{Q(t)}|u_n|^p\,dt \\
&= \int_{\{t\in\mathbb{R}/|u_n(t)|\geq r\}} e^{Q(t)}|u_n|^{ps}|u_n|^{p(1-s)}\,dt \\
&\leq \Big(\int_{\{t\in\mathbb{R}/|u_n(t)|\geq r\}} e^{Q(t)}|u_n|^{\alpha}\,dt\Big)^{\frac{ps}{\alpha}}\Big(\int_{\{t\in\mathbb{R}/|u_n(t)|\geq r\}} e^{Q(t)}|u_n|^{\frac{p\alpha(1-s)}{\alpha-ps}}\,dt\Big)^{\frac{\alpha-ps}{\alpha}} \\
&\leq C_4\|u_n\|^{p(1-s)},
\end{aligned}$$

where $C_4 = \left(\frac{C_3}{C_2}\right)^{\frac{ps}{\alpha}}\eta^{p(1-s)}_{\frac{p\alpha(1-s)}{\alpha-s}}$ and $\frac{p\alpha(1-s)}{\alpha-s} \geq p$. Now, since $f'_{\lambda_n}(u_n)u_n = 0$, then (W_6), (20), (22) and (4) imply

$$\begin{aligned}
\|u_n\|^p &= \lambda_n\int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u_n(t))\cdot u_n(t)dt \\
&\leq 2\int_{\{t\in\mathbb{R}/|u_n(t)|\leq r\}} e^{Q(t)}\nabla W(t,u_n(t))\cdot u_n(t)dt \\
&+2\int_{\{t\in\mathbb{R}/|u_n(t)|\geq r\}} e^{Q(t)}\nabla W(t,u_n(t))\cdot u_n(t)dt \\
&\leq 2c\int_{\{t\in\mathbb{R}/|u_n(t)|\leq r\}} e^{Q(t)}|u_n(t)|^{\mu}\,dt + 2C_1\int_{\{t\in\mathbb{R}/|u_n(t)|\geq r\}} e^{Q(t)}|u_n(t)|^p\,dt \\
&\leq 2c\frac{C_3}{a} + 2C_1C_4\|u_n\|^{p(1-s)}
\end{aligned} \tag{23}$$

where $p(1-s) < p$. Hence (23) implies that (u_n) is bounded and the proof of Lemma 4.7 is completed. □

Now, we are in position to prove Theorem 4.1. Let (u_n) be the bounded sequence obtained in (19). By taking a subsequence if necessary, we can assume that $u_n \rightharpoonup u$ and $u_n \to u$ a.e. on $\mathbb{R}$. Using (19), we get for all $v \in E$

$$\lim_{n\to\infty} f'(u_n)v = \lim_{n\to\infty}\Big[f'_{\lambda_n}(u_n)v + (\lambda_n - 1)\int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u_n(t))\cdot v(t)dt\Big] = 0.$$

We distingsh two cases.
First case: $\limsup_{n\to\infty} f_{\lambda_n}(u_n) > 0$. In this case, the result follows from Lemma 4.5.
Second case: $\limsup_{n\to\infty} f_{\lambda_n}(u_n) \leq 0$. Let $(s_n) \subset [0,1]$ be such that

$$f_{\lambda_n}(s_nu_n) = \max_{s\in[0,1]} f_{\lambda_n}(su_n),$$

we denote by (v_n) the sequence defined by $v_n = s_n u_n$. It is clear that (v_n) is bounded. Using (4) and (15), we get for all $n \in \mathbb{N}$ and $u \in E$

$$\begin{aligned}
f'_{\lambda_n}(u)u &= \|u\|^p - \lambda_n \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u(t)) \cdot u(t)dt \\
&\geq \|u\|^p - 2 \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u(t)) \cdot u(t)dt \\
&\geq \|u\|^p - 2\varepsilon \eta_p^p \|u\|^p - 2C_\varepsilon \eta_\mu^\mu \|u\|^\mu .
\end{aligned}$$

Take $\varepsilon = \frac{1}{4\eta_\mu^\mu}$, we obtain

$$f'_{\lambda_n}(u)u \geq \frac{1}{2}\left[1 - 4C_\varepsilon \eta_\mu^\mu \|u\|^{\mu-p}\right] \|u\|^p .$$

Let $r_1 = \left(8C_\varepsilon \eta_\mu^\mu\right)^{-\frac{1}{\mu-2}}$, then we have

$$f'_{\lambda_n}(u)u \geq \frac{1}{4} \|u\|^p ,\ \forall u \in B(0,r_1). \tag{24}$$

Similarly, using (4) and (16), we can find a positive constant r_2 such that

$$f_{\lambda_n}(u) \geq \frac{1}{4p} \|u\|^p ,\ \forall u \in B(0,r_2).$$

Combining (24) with the fact $f'_{\lambda_n}(u_n) = 0$ yields

$$\|u_n\| \geq \theta,\ \forall n \in \mathbb{N}$$

where $\theta = \inf(r_1,r_2)$. Let $0 < \xi < 1$, then for all $n \in \mathbb{N}$, $\bar{s}_n = \xi \frac{\theta}{\|u_n\|} \in]0,1[$. Note that by (4.22)

$$f_{\lambda_n}(s_n u_n) \geq f_{\lambda_n}(\bar{s}_n u_n) \geq \frac{1}{4p} \bar{s}_n^p \|u_n\|^p \geq \frac{1}{4p}(\xi\theta)^p, \tag{25}$$

which with $f_{\lambda_n}(0) = 0$ implies that $s_n > 0$. Moreover, we have

$$\begin{aligned}
\limsup_{n\to\infty} f(u_n) &= \limsup_{n\to\infty} \left[f_{\lambda_n}(u_n) + (\lambda_n - 1) \int_{\mathbb{R}} e^{Q(t)} W(t,u_n(t))dt \right] \\
&= \limsup_{n\to\infty} f_{\lambda_n}(u_n) \leq 0,
\end{aligned}$$

which with (25) implies $s_n < 1$. Hence $s_n \in]0,1[$ and then $f'_{\lambda_n}(v_n)v_n = 0$ for all $n \in \mathbb{N}$ and

$$\lambda_n \int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t,v_n(t))dt = f_{\lambda_n}(v_n) - \frac{1}{p} f'_{\lambda_n}(v_n)v_n = f_{\lambda_n}(v_n).$$

Consequently, we deduce from (25)

$$\limsup_{n\to\infty}\int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,v_n(t))dt = \limsup_{n\to\infty} f_{\lambda_n}(v_n) > 0.$$

Since (v_n) is bounded, it follows from Lemma 4.3 that (v_n) does not vanish, so (u_n) does not vanish. By going to a subsequence if necessary, Lemma 4.5 implies that $v_n \rightharpoonup v \neq 0$ with $f'(v) = 0$ and the proof of Theorem 4.1 is finished.

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