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APPROXIMATE CONTROLLABILITY OF IMPULSIVE INTEGRODIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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This paper considers the approximate controllability of mild solutions for impulsive semilinear integrodifferential equations with state-dependent delay in Hilbert spaces. We obtain our significant findings using Grimmer's resolvent operator theory and Schauder's fixed point theorem. We give an example at the end to ensure the compatibility of the results.

1. Introduction

This study investigates the existence of mild solutions and approximate controllability for the following impulsive integrodifferential equation with state-dependent delay

$$\begin{cases} \xi'(t) = A\xi(t) + \int_0^t B(t-s)\xi(s)ds + h\big(t, \xi_{\sigma(t,\xi_t)}\big) + \Gamma\nu(t), \\ \qquad\qquad t \in J = [0,b],\ t \neq \tau_i,\ i = 1,2,\ldots,m, \\ \Delta\xi|_{t=\tau_i} = I_i(\xi(\tau_i)),\ i = 1,\ldots,m, \\ \xi_0 = \psi \in \mathcal{B}, \end{cases} \tag{1}$$

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where A is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a Hilbert space $\mathbb{K}$, $\big(B(t)\big)_{t\geq 0}$ is a family of closed linear operators on $\mathbb{K}$ with domain $D(B(t)) \supset D(A)$, Γ is a bounded linear operator from $\mathbb{U}$ into $\mathbb{K}$, the control function v is given in $L^2(J,\mathbb{U})$, $h : J \times \mathcal{B} \to \mathbb{K}$ is a nonlinear function, where $\mathcal{B}$ is a phase space, which will be described later. The impulsive functions $I_i : \mathbb{K} \to \mathbb{K}$, for $i = 1,\dots,m$, and $\Delta\xi|_{t=\tau_i} = \xi(\tau_i^+) - \xi(\tau_i^-)$ with $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} = b$, for $i = 1,\dots,m$. The function $\xi_t : (-\infty,0] \to \mathbb{K}$, $\xi_t(\vartheta) = \xi(t+\vartheta)$, belongs to the phase space $\mathcal{B}$ and the function $\sigma : J \times \mathcal{B} \to (-\infty,b]$, is continuous. Integrodifferential equations can describe natural phenomena from many fields, such as electronics, fluid dynamics, biological models, and chemical kinetics. Classical differential equations can not describe those phenomena. That is why, in recent years, integrodifferential equations have attracted more and more attention from physicists, mathematicians, and engineers. According to the author of the citation, Volterra, the dynamics of certain types of elastic materials can be characterized by a partial integrodifferential response diffusion equation of the following form: In [41], the author makes the suggestion that the kinetics of certain classes of elastic materials can be represented by a partial integrodifferential response diffusion equation of the following form:

$$\frac{\partial u(\theta,t)}{\partial t} = \Delta u(\theta,t) + \int_0^t \phi(t,s)\Delta u(\theta,s)ds + \varphi(\theta,t), \quad (\theta,t) \in \mathbb{R}\times\mathbb{R}^+,$$

where ϕ and φ are appropriates functions. In their investigation of the electric displacement field in Maxwell Hopkinson dielectric, the authors in [9, 14] made use of the linear partial integrodifferential equation that is presented below:

$$\frac{\partial^2 u(\theta,t)}{\partial t^2} = \frac{1}{\eta\varepsilon}\Delta u(\theta,t) + \int_0^t \frac{1}{\eta\varepsilon}\psi(t-s)\Delta u(\theta,s)ds, \quad (\theta,t) \in \tilde{\Omega}\times[0,T),$$

for $T > 0$ and $\tilde{\Omega} \subset \mathbb{R}^3$, where $\eta,\varepsilon \in \mathbb{R}$ and ψ is a vector of scalar functions. The Rayleigh problem or the Stoke's first problem of viscoelasticity is given by the following integrodifferential equation:

$$\frac{\partial u(\theta,t)}{\partial t} = \int_0^t \Delta u(\theta,\tau)da(\tau) + h(\theta,t), \quad (\theta,t) \in [0,1]\times\mathbb{R}^+,$$

where $a : \mathbb{R}^+ \to \mathbb{R}$ is a function of bounded variation on each compact interval of $\mathbb{R}^+$ with $a(0) = 0$. This problem is a typical example of one-dimensional problems in viscoelasticity, like torsion of a rod, simple shearing motions, simple tension, etc, see [36]. A simple control system of integrodifferential equation is that of the electrical RLC circuit:

$$\frac{dL(t)}{dt} = -\frac{1}{RC}v(t) - \frac{1}{LC}\int_0^t L(t)dt + r(t), \quad t \in \mathbb{R}^+.$$

Here, $L(t)$ is the voltage in the electrical circuit and $r(t)$ is the current source which serves as the control function, see [10, Eq (2.2)]. In [31], In order to investigate the dynamics of several epidemiological systems, the authors in [31] made use of a few delayed integrodifferential equations, see [31, page 685, Eqs (11f)-(11g)]. An additional source of inspiration comes from biological sciences, physics, and other disciplines. These include areas such as elasticity, dynamics of populations, forecasting human populations, torsion of a wire, radiation transport, Bernoulies problems, oscillating magnetic field, mortality of equipment problems, and inverse problems of reaction-diffusion equations, etc, see for instance [9, 14, 30, 31, 36, 37, 41, 42] and the references therein.
Many authors have researched the qualitative properties of numerous integrodifferential equations in infinite dimensional spaces due to the importance of these properties in applications (see [16], [17], [18], and [19] for examples). These qualitative properties include existence, stability, optimal control, controllability, etc.
Because it is more successful than classical differential equations, the theory of impulsive differential equations is becoming an increasingly important research topic. As models, impulsive differential equations are suitable for studying the evolution of processes subject to abrupt changes in their states. The wide applications of these equations in fields such as mechanics, electrical engineering, medicine, biology, ecology, and other fields have recently attracted the attention of many scholars. Readers should consult the books written by [6] and [8], as well as the papers written by Chang[11] and Hernandez[24], along with the references listed within those published works.

Functional differential equation theory is now crucial to nonlinear analysis. Researchers have long modeled scientific processes using differential delay equations or functional differential equations. Many times, the delay is believed to be constant or integral, creating a distributed delay.We refer the reader to the books by Hale and Verduyn Lunel [23], Kolmanovskii and Myshkis [27], Smith [40], and Wu [44], and the references therein. Researchers have recently studied complex cases where delay depends on unknown functions. Researchers commonly refer to equations with state-dependent delay as such ([1, 2, 25]). Additionally, state-dependent delays are common and effective in applications, as demonstrated in ([3, 12]).

Controllability underpins mathematical control theory. We can differentiate between two controllability notions for infinite dimensional systems. There is exact and approximate controllability. Excact controllability indicates a system can reach a desired ultimate state in a finite time, whereas approximation controllability means it can steer into an arbitrarily small neighborhood of it. In infinite dimensional spaces, perfect controllability is frequently too strong and has limited applicability; (see [7, 39, 43]). Therefore, investigating the more abstract notion of controllability, specifically the concept of approximate controllability for nonlinear systems, is not only recommended but essential. In the past ten years, a great number of works have reported on the approximate controllability of the dynamical control system with delay via fixed point methods; for example, see([4, 5]). Muthukumar et al.[33] recently proved the approximate controllability for a class of impulsive neutral stochastic functional differential systems with state-dependent delay in Hilbert spaces by making use of semigroup theory, stochastic analysis techniques, and Sadovskii's fixed point theorem. In addition, the authors of the study in Zahra [32] investigated the approximate controllability for semilinear neutral integrodifferential systems with finite delays in Hilbert spaces. They did this by employing Sadovskii's fixed point theorem and the resolvent operator theory. Later, Fu and Zhang[20] used the fixed point method and semigroup theory to come up with enough conditions for approximate controllability for semilinear neutral functional differential systems with state-dependent delay. Ndambomve and al. [34] studied the approximate controllability for some nonlinear partial functional integrodifferential equations with infinite delay by making use of the measure of noncompactness and the Mönch fixed point theorem. A set of adequate criteria for approximate controllability of the semilinear impulsive functional differential system with nonlocal initial conditions was derived by authors very recently in [5]. These conditions were derived by invoking Schauder's fixed-point theorem.

However, to our knowledge, there has yet to be work on the approximate controllability for impulsive integrodifferential systems with state-dependent delay described by (1).

In this study, we want to fill in this gap by looking into how to roughly control impulsive integrodifferential equations with state-dependent delay. This is guided by the arguments that came before it.

The following is a summary of the major contributions that this work makes:

- A novel class of impulsive integrodifferential equations with state-dependent delay in Hilbert spaces has been proposed.
- Using Schauder's fixed point theorem and the theory of the resolvent operator in the sense of Grimmer, we develop a new set of adequate conditions that guarantee the existence of mild solutions for impulsive integrodifferential equations with state-dependent delay.
- An example is used to illustrate the main results.

This paper is structured as follows: In section 2, we give some fundamental definitions, assumptions and results needed to develop the approximate controllability of system (1). In section 3, we discuss the approximate controllability of system system (1) with the help of Schauder's fixed point theorem and the resolvent operator in the sense of Grimmer. An example is provided in the last section to illustrate the obtained results.

2. Preliminaries

In this section, some definitions, notations, and lemmas that are used throughout this work are stated. The norms in the state space $\mathbb{K}$ and control space $\mathbb{U}$ are denoted by $\|\cdot\|_{\mathbb{K}}$, and $\|\cdot\|_{\mathbb{U}}$, respectively. The inner product in $\mathbb{U}$ is represented by $(\cdot,\cdot)$. $\mathcal{L}(\mathbb{U};\mathbb{K})$ is the space of all bounded linear operators from $\mathbb{U}$ into $\mathbb{K}$, it is endowed with the norm $\|\cdot\|_{\mathcal{L}(\mathbb{U};\mathbb{K})}$ and we write $\mathcal{L}(\mathbb{K})$ when $\mathbb{U}=\mathbb{K}$. A function $\xi:[\rho,\theta]\to\mathbb{K}$ is called the normalised piecewise continuous function on the interval $[\rho,\theta]$, it is piecewise continuous on $[\rho,\theta]$ and left continuous on $(\rho,\theta]$. Let us detone by $\mathcal{PC}([\rho,\theta],\mathbb{K})$ the space of all normalised piecewise continuous function from $[\rho,\theta]$ into $\mathbb{K}$. $\mathcal{PC}([\rho,\theta],\mathbb{K})$ equipped with the norm $\|\xi\|_{\mathcal{PC}}=\sup_{s\in[\rho,\theta]}\|\xi(s)\|_{\mathbb{K}}$ is a Banach space. For convenience of notations, we use $\mathcal{PC}$ in place of $\mathcal{PC}(J,\mathbb{K})$.

2.1. Phase space

Now, we give the axiomatic definition of the phase space $\mathcal{B}$ introduced by Hino et al. in [26] and suitably modify to treat the impulsive evolution equations (cf. [35]). Specifically $\mathcal{B}$, is a linear space of all functions from $(-\infty,0]$ into $\mathbb{K}$ equipped with the norm $\|\cdot\|_{\mathcal{B}}$ and satisfying the following axioms:

(C_1) If $\xi:(-\infty,\rho+\theta)\to\mathbb{K}$, $\theta>0$, such that $\xi_\rho\in\mathcal{B}$ and $\xi|_{[\rho,\rho+\theta]}\in\mathcal{PC}([\rho,\rho+\theta];\mathbb{K})$, then for every $t\in[\rho,\rho+\theta]$, the following conditions are satisfied:

- **(a)** $\xi_t\in\mathcal{B}$.
- **(b)** $\|\xi_t\|_{\mathcal{B}}\leq F(t-\rho)\sup\{\|\xi(s)\|_{\mathbb{K}}:\rho\leq s\leq t\}+G(t-\rho)\|\xi_\rho\|_{\mathcal{B}}$, where F, $G:[0,\infty)\to[0,\infty)$ such that F is locally bounded, and both F, G are independent of $\xi(\cdot)$.

(C_2) The space $\mathcal{B}$ is complete.

Remark 2.1. *For any $\psi\in\mathcal{B}$, the function ψ_t, $t\leq 0$, defined as $\psi_t(\vartheta)=\psi(t+\vartheta)$, $\vartheta\in(-\infty,0]$. Thus, if the function $\xi(\cdot)$ fulfills the axiom (C_1) with $\xi_0=\psi$, then we may extend the mapping $t\mapsto\xi_t$ by setting $\xi_t=\psi_t$, $t\leq 0$, to the entire interval $(-\infty,b]$.*

Furthermore, for the function $\sigma : J \times \mathcal{B} \to (-\infty, b]$, we introduce the set

$$\mathcal{X}(\sigma^-) = \{\sigma(s,\phi) : \sigma(s,\phi) \leq 0, \text{ for } (s,\phi) \in J \times \mathcal{B}\},$$

and we give the following assumption on ψ_t. The function $t \mapsto \psi_t$ is well defined from $\mathcal{X}(\sigma^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function J^ψ : $\mathcal{X}(\sigma^-) \to (0,\infty)$ such that $\|\xi_t\|_{\mathcal{B}} \leq J^\psi(t)\|\psi\|_{\mathcal{B}}$ for every $t \in \mathcal{X}(\sigma^-)$.

Lemma 2.1. [38, Lemma 2.3] *Let $\xi : (-\infty, b] \to \mathbb{K}$, be a function such that $\xi_0 = \psi$ and $\xi|_J \in \mathcal{PC}$. Then,*

$$\|\xi_s\|_{\mathcal{B}} \leq F_1\|\psi\|_{\mathcal{B}} + F_2 \sup\{\|\xi(\mu)\|_{\mathbb{K}} : \mu \in [0, \max\{0,s\}]\}, \; s \in \mathcal{X}(\sigma^-) \cup J,$$

where $F_1 = \sup_{t \in \mathcal{X}(\sigma^-)} J^\psi(t) + \sup_{t \in J} G(t), \; F_2 = \sup_{t \in J} F(t).$

Example 2.1. *Phase space.*
Let $f : (-\infty, 0] \to \mathbb{R}^+$ be a Lebesgue function, which is locally bounded. Take $\mathcal{B} = \mathcal{PC}_\tau \times L^1_f(\mathbb{K})$ as the space of all functions $\psi : (-\infty, 0] \to \mathbb{K}$ such that $\psi|_{[-\tau,0]} \in \mathcal{PC}([-\tau,0];\mathbb{K})$, for some $\tau > 0$, ψ is Lebesgue measurable on $(-\infty,-\tau)$, and $f\|\psi(\cdot)\|_{\mathbb{K}}$ is Lebesgue integrable on $(-\infty,-\tau]$. The norm in $\mathcal{B}$ is defined as

$$\|\psi\|_{\mathcal{B}} := \int_{-\tau}^{0} \|\psi(\vartheta)\|_{\mathbb{K}} d\vartheta + \int_{-\infty}^{-\tau} f(\vartheta)\|\psi(\vartheta)\|_{\mathbb{K}} d\vartheta. \tag{2}$$

Furthermore, there exists a locally bounded function $H : (-\infty, 0] \to \mathbb{R}^+$ such that $f(t+\vartheta) \leq H(t)f(\vartheta)$, for all $t \leq 0$ and $\vartheta \in (-\infty,0) \setminus F_t$ where $F_t \subseteq (-\infty,0)$ is a set with a Lebesgue measure zero. A simple example of f is given by $f(\vartheta) = e^{s\vartheta}$, for some $s > 0$.

To verify that the space $\mathcal{B}$ is a phase space, first, we prove that it satisfies the axiom (C_1). Here, we choose $\rho = 0, \; \theta = b$. Let $\xi(\cdot)$ be a function such that $\xi_0 \in \mathcal{B}$ and $\xi|_J \in \mathcal{PC}(J;\mathbb{K})$. Remember that the function $\xi_t : (-\infty, 0] \to \mathbb{K}$ is described by

$$\xi_t(\vartheta) = \xi(t+\vartheta), \textit{ for each } t \in J.$$

For $t \in J$, we verify that $\xi_t \in \mathcal{B}$. It is easy to see that for each $t \in J$, the function $\xi_t|_{[-\tau,0]} \in \mathcal{PC}([-\tau,0];\mathbb{K})$, $\tau > 0$ and Lebesgue measurable. Now, for $t \in [0,\tau]$, *we estimate*

$$
\begin{aligned}
&\|\xi_t\|_{\mathcal{B}} \\
&= \int_{-\tau}^{0} \|\xi(t+\vartheta)\|_{\mathbb{K}} d\vartheta + \int_{-\infty}^{-\tau} f(\vartheta)\|\xi(t+\vartheta)\|_{\mathbb{K}} d\vartheta \\
&= \int_{-\tau}^{-t} \|\xi(t+\vartheta)\|_{\mathbb{K}} d\vartheta + \int_{-t}^{0} \|\xi(t+\vartheta)\|_{\mathbb{K}} d\vartheta + \int_{-\infty}^{-t-\tau} f(\vartheta)\|\xi(t+\vartheta)\|_{\mathbb{K}} d\vartheta \\
&\quad + \int_{-t-\tau}^{-\tau} f(\vartheta)\|\xi(t+\vartheta)\|_{\mathbb{K}} d\vartheta \\
&\leq \int_{-\tau}^{0} \|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta + \int_{0}^{t} \|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta + \int_{-\infty}^{-\tau} f(\vartheta - t)\|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta \\
&\quad + \int_{-\tau}^{0} f(\vartheta - t)\|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta \\
&\leq \int_{-\tau}^{0} \|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta + t \sup_{\vartheta\in[0,t]} \|\xi(\vartheta)\|_{\mathbb{K}} + H(-t)\int_{-\infty}^{-\tau} f(\vartheta)\|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta \\
&\quad + H(-t)\int_{-\tau}^{0} f(\vartheta)\|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta \\
&\leq \Big(1 + H(-t)\sup_{\vartheta\in[-\tau,0]} f(\vartheta)\Big)\int_{-\tau}^{0} \|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta + t\sup_{\vartheta\in[0,t]} \|\xi(\vartheta)\|_{\mathbb{K}} \\
&\quad H(-t)\int_{-\infty}^{-\tau} f(\vartheta)\|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta.
\end{aligned} \tag{3}
$$

In a similar manner, for $t > \tau$, *we get*

$$
\begin{aligned}
&\|\xi_t\|_{\mathcal{B}} \\
&\quad \leq \Big(1 + H(-t)\sup_{\vartheta\in[-\tau,0]} f(\vartheta)\Big)\int_{-\tau}^{0} \|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta \\
&\qquad + \Big(t + \int_{-t}^{-\tau} f(\vartheta) d\vartheta\Big)\sup_{\vartheta\in[0,t]} \|\xi(\vartheta)\|_{\mathbb{K}} + H(-t)\int_{-\infty}^{-\tau} f(\vartheta)\|\xi(\vartheta)\|_{\mathbb{K}} d\vartheta.
\end{aligned} \tag{4}
$$

From (3) *and* (4), *we deduce that the axiom* (C_1) *holds with*

$$
F(t) = \begin{cases} t, & \textit{for} \quad 0 \leq t \leq \tau \\ t + \int_{-t}^{-\tau} f(\vartheta) d\vartheta, & \textit{for}\,, \quad \tau < t, \end{cases}
$$

and

$$
G(t) = \max\Big\{1 + H(-t)\sup_{\vartheta\in[-\tau,0]} f(\vartheta), H(-t)\Big\}.
$$

Finally, the completeness of the space $\mathcal{B}$, under the norm $\|\cdot\|_{\mathcal{B}}$ described in (2) *follows similarly to Theorem 1.3.1 [26] and hence the axiom* (C_2) *is satisfied. In particular, we take* $\tau = 0$ *and in this case, we consider the following condition:*
(A1) The function $\phi \in \mathcal{PC}_0 \times L^1_f(\mathbb{K})$ *and* $K := \sup_{\vartheta\in(-\infty,0]} \dfrac{|\mu(-\vartheta)|}{f(\vartheta)}$.

2.2. Resolvent operators in Banach spaces

In this part, we recall some basic results about the operators for the following linear homogeneous equation

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds \quad \text{for } t \geq 0 \\ x(0) = x_0 \in \mathbb{X}, \end{cases} \tag{5}$$

where A and $B(t)$ are closed operators on Banach space $\mathbb{X}$. Let $\mathbb{Y}$ be the Banach space formed from $D(A)$ with the graph norm

$$\|x\|_{\mathbb{Y}} = \|Ax\| + \|x\|, \text{ for } x \in D(A).$$

In the sequel, we suppose that A and $\Big(B(t)\Big)_{t\geq 0}$ satisfy the following conditions:

- (P_1) A is the infinitesimal generator of C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on $\mathbb{X}$.
- (P_2) $(B(t))_{t\geq 0}$ is a family of closed linear operator on $\mathbb{X}$ such that $B(t)$ is continuous when regarded as linear map from $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ into $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and the map $t \to B(t)x$ is measurable for all $x \in \mathbb{X}$ and $t \geq 0$, and belongs to $\mathbb{W}^{1,1}(\mathbb{R}^+, \mathbb{X})$. Moreover there exists an integrable function $\gamma : [0, +\infty) \longrightarrow \mathbb{R}^+$ such as

$$\left\| \frac{d}{dt} B(t)x \right\| \leq \gamma(t) \|x\|_{\mathbb{Y}}, \quad x \in \mathbb{Y}, t \geq 0.$$

Definition 2.2. *[22]* We call resolvent operator for Eq. (5), a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{X})$ for $t \geq 0$, verifying the following properties:

- **(i)** $R(0) = I$ (identity operator on $\mathbb{X}$) and $\|R(t)\| \leq Me^{\omega t}$ for some constants M and ω.
- **(ii)** For each $x \in \mathbb{X}$, $R(t)x$ is strongly continuous for $t \geq 0$.
- **(iii)** For $x \in \mathbb{Y}$, $R(\cdot)x \in C^1([0, +\infty), \mathbb{X}) \cap C([0, +\infty), \mathbb{Y})$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \quad \text{for } t \geq 0. \end{aligned}$$

Remark 2.2. *In general, the resolvent operator* $(R(t))_{t\geq 0}$ *for Eq.* (5)*, does not satisfy the semigroup property, namely,*

$$R(t+s) \neq R(t)R(s) \text{ for some } t,s>0.$$

Theorem 2.3. [22] *Suppose that* (P_1) *and* (P_2) *hold. Then, Eq.* (5) *has a unique resolvent operator* $(R(t))_{t\geq 0}$.

Lemma 2.4. [29] *Let* (P_1) *and* (P_2) *be satisfied. Then, for all* $t>0$ *there exists a constant* L *such that*

$$\|R(t+\rho)-R(\rho)R(t)\|_{\mathcal{L}(\mathbb{X})} \leq L\rho \text{ for } 0\leq \rho \leq t \leq b.$$

Theorem 2.5. [15] *Assume that* (P_1) *and* (P_2) *hold. The resolvent operator* $(R(t))_{t\geq 0}$ *is compact for* $t>0$ *if only if the semigroup* $(T(t))_{t\geq 0}$ *is compact for* $t>0$.

Let us define the operators

$$\begin{cases} C_0^b := \int_0^b R(b-t)\Gamma\Gamma^* R^*(b-t)dt, \\ G(\alpha, C_0^b) := (\alpha I + C_0^b)^{-1}, \alpha>0, \end{cases} \tag{6}$$

where $R^*(t)$ and Γ^* denote the adjoints of the operators $R(t)$ and Γ, respectively.

Definition 2.6. A function $\xi(\cdot;\psi,u):(-\infty,b]\to\mathbb{K}$ is called a mild solution of Eq. (1), if it satisfies the following:

(i) $\xi(t)=\psi(t)\in\mathcal{B},\ t\in(-\infty,0]$,

(ii) $\Delta\xi|_{t=\tau_i}=I_i(\xi(\tau_i)),\ i=1,\dots,m$,

(iii) $x(\cdot)|_J\in\mathcal{PC}$ and the following integral equation is verified:

$$\xi(t)=R(t)\psi(0)+\int_0^t R(t-s)h\big(s,\xi_{\sigma(s,\xi_s)}\big)ds+\int_0^t R(t-s)\Gamma\nu(s)ds + \sum_{0<\tau_i<t} R(t-\tau_i)I_i(\xi(\tau_i)) \tag{7}$$

Next, we present the concept of approximate controllability.

Definition 2.7. The system (1) is approximately controllable on J, if for any initial function $\psi\in\mathcal{PC}$, the set $\mathcal{R}(b,\psi)$ is dense in $\mathbb{K}$, i.e.:

$$\overline{\mathcal{R}(b,\psi)}=\mathbb{K},$$

where $\mathcal{R}(b,\psi)=\{\xi(b,\psi,u),\ u(\cdot)\in L^2(J,\mathbb{U})\}$.

To obtain the approximate controllability results for Eq. (1), we need the following hypotheses.

(A_0) $\alpha G(\alpha, C_0^b)(g) \to 0$ as $\alpha \downarrow 0$ in the strong topology.

(A_1) $(R(t))_{t\geq 0}$ is compact for $t > 0$. Let $N = \sup_{t\in[0,b]} \|R(t)\|$.

(A_2) The linear operator $\Gamma : \mathbb{U} \to \mathbb{K}$ is bounded with $\|\Gamma\|_{\mathcal{L}(\mathbb{U};\mathbb{K})} = N_\Gamma$.

(A_3) **(i)** Let $\xi : (-\infty, b] \to \mathbb{K}$ be such that $\xi_0 = \psi$ and $\xi|_J \in \mathcal{PC}$. The function $t \mapsto h(t, \xi_t)$ is strongly measurable on J and $t \mapsto h(s, \xi_t)$ is continuous on $\mathcal{X}(\sigma^-) \cup J$, for every $s \in J$. Moreover, for each $t \in J$, the function $h(t,\cdot) : \mathcal{B} \to \mathbb{K}$ is continuous.

(ii) For each positive integer a, there exists a function $m_a(\cdot) \in L^1(J;\mathbb{R}^+)$ such that

$$\sup_{\|\psi\|_{\mathcal{B}}\leq a} \|h(t,\psi)\|_{\mathbb{K}} \leq m_a(t), \quad \text{for a.e. } t \in J \text{ and } \psi \in \mathcal{B},$$

and

$$\liminf_{a\to\infty} \int_0^b \frac{m_a(t)}{a} dt = \theta < \infty.$$

(A_4) For $i = 1, \ldots, m$, the impulses $I_i : \mathbb{K} \to \mathbb{K}$ are completely continuous. Moreover, we assume that there exist constants ω_i such that $\|I_i(\xi)\|_{\mathbb{K}} \leq \omega_i$ for all $\xi \in \mathbb{K}$, $i = 1, \ldots, m$.

Note that the assumption (A_0) is equivalent to the fact that the linear control system:

$$\begin{cases} \xi'(t) = A\xi(t) + \int_0^t B(t-s)\xi(s)ds + \Gamma\nu(t), \quad t \in J, \\ \qquad\qquad\qquad\qquad \text{a.e} \quad (t,\xi) \in [0,b] \times \Omega, \\ \xi(0) = \psi(0) \in \mathcal{B}, \end{cases} \tag{8}$$

corresponding to Eq. (1) is approximately controllable on J.

Theorem 2.8. [13] *The following statements are equivalent:*

(a) *The linear control system* (8) *is approximately controllable on* J.

(b) $\Gamma^* R^*(t)\xi = 0$ *for all* $t \in J$, $\Rightarrow$ $\xi = 0$.

(c) *The assumption* (A_0) *is satisfied.*

We end this section with Schauder's fixed point theorem, which we will use in the following to prove our results.

Theorem 2.9 (Schauder's fixed point theorem)**.** *[21] Let $\mathbb{F}$ be a nonempty closed convex bounded subset of a Banach space $\mathbb{X}$. Then every continuous compact mapping $\mathcal{S}:\mathbb{F}\to\mathbb{F}$ has a fixed point.*

3. Approximate controlability of the nonlinear system (1)

In this section, we establish the approximate controllability of the system (1). To this end, we first show that, for every $\alpha>0$ and $\xi_b\in\mathbb{K}$, the system (1) has at least one mild solution using the following control

$$v_\alpha=\Gamma^*R^*(b-t)G(\alpha,C_0^b)\vartheta(\xi(\cdot)),\quad t\in J, \tag{9}$$

where

$$\vartheta(\xi(\cdot))=\xi_b-R(b)\psi(0)-\int_0^b R(b-s)h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds-\sum_{i=1}^m R(b-\tau_i)I_i(\bar{\xi}(\tau_i)), \tag{10}$$

and $\bar{\xi}:(-\infty,b]\to\mathbb{K}$ such that $\bar{\xi}_0=\psi$ and $\bar{\xi}=\xi$ on J.

Theorem 3.1. *Assume that assumptions (P_1)-(P_2), (A_0)-(A_4) are satisfied. Then, for every $\alpha>0$ and for fixed $\xi_b\in\mathbb{K}$, the system* (1) *using the control* (9) *has at least one mild solution on J, provided that*

$$NF_2\theta\big(1+\frac{N_\Gamma^2N^2b}{\alpha}\big)\leq 1. \tag{11}$$

Proof. Let $\mathcal{H}=\{\xi\in\mathcal{PC}:\xi(0)=\psi(0)\}$ be the space endowed with the norm $\|\cdot\|_{\mathcal{PC}}$. Let q be a positive constant. We now consider the set $\mathcal{H}_q=\{\xi\in\mathcal{H}:\|\xi\|_{\mathcal{PC}}\leq q\}$.

For $0<\alpha<b$, we define the operator $\mathcal{W}_\alpha:\mathcal{H}\to\mathcal{H}$ as

$$\begin{aligned}(\mathcal{W}_\alpha\xi)(t)&=R(t)\psi(0)+\int_0^t R(t-s)h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds+\int_0^t R(t-s)\Gamma v_\alpha(s)ds\\&\quad+\sum_{0<\tau_i<t}R(t-\tau_i)I_i(\bar{\xi}(\tau_i)).\end{aligned} \tag{12}$$

It is clear from the definition of $\mathcal{W}_\alpha$, for $\alpha>0$, that the fixed point of $\mathcal{W}_\alpha$, is a mild solution of the system (1). Next, we will prove that $\mathcal{W}_\alpha$ has a fixed point by applying Schauder's fixed point theorem. The proof is split in several steps:
Step 1: $\mathcal{W}_\alpha(\mathcal{H}_q)\subset\mathcal{H}_q$, for some q. Indeed, suppose that our claim is false. Then for any $\alpha>0$ and for all $q>0$, there exists $\xi^q(\cdot)\in\mathcal{W}_\alpha$, such that

$\|(\mathcal{W}_\alpha \xi^q)(t)\|_{\mathbb{K}} > q$, for some $t \in J$, where t may depend upon q. First, by using the expression defined in (9), and (A_1), (A_2) and (A_3), we estimate

$$
\begin{aligned}
&\|\nu_\alpha(t)\|_{\mathbb{Y}} \\
&\quad = \|\Gamma^* R^*(b-t) G(\alpha, C_0^b)\vartheta(\xi(\cdot))\|_{\mathbb{Y}} \\
&\quad \le \frac{1}{\alpha}\|\Gamma^*\|_{\mathcal{L}(\mathbb{K}^*;\mathbb{U})}\|R^*(b-t)\|_{\mathcal{L}(\mathbb{K}^*)}\|G(\alpha, C_0^b)\vartheta(\xi(\cdot))\|_{\mathbb{K}^*} \\
&\quad \le \frac{N_\Gamma N}{\alpha}\Bigg(\|\xi_b\|_{\mathbb{K}} + \|R(b)\|_{\mathcal{L}(\mathbb{K})}\|\psi(0)\|_{\mathbb{K}} \\
&\qquad + \int_0^b \|R(b-s)\|_{\mathcal{L}(\mathbb{K})}\|h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\|_{\mathbb{K}} ds \\
&\qquad + \sum_{i=1}^m \|R(b-\tau_i)\|_{\mathcal{L}(\mathbb{K})}\|I_i(\bar{\xi}(\tau_i))\|_{\mathbb{K}}\Bigg) \\
&\quad \le \frac{N_\Gamma N}{\alpha}\Bigg(\|\xi_b\|_{\mathbb{K}} + N\|\psi(0)\|_{\mathbb{K}} \\
&\qquad + N\int_0^b \|h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\|_{\mathbb{K}} ds + N\sum_{i=1}^m \|I_i(\bar{\xi}(\tau_i))\|_{\mathbb{K}}\Bigg) \\
&\quad \le \frac{N_\Gamma N}{\alpha}\Bigg(\|\xi_b\|_{\mathbb{K}} + N\|\psi(0)\|_{\mathbb{K}} + N\int_0^b \|h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\|_{\mathbb{K}} ds + N\sum_{i=1}^m \omega_i\Bigg) < +\infty,
\end{aligned} \tag{13}
$$

which implies that $\|\nu_\alpha(t)\|_{\mathbb{U}}$ is bounded for all $t \in J$.

Let us define $\|\nu_\alpha\|_\infty = \sup_{t\in J}\|\nu_\alpha(t)\|_{\mathbb{U}}$. Using (C_1)(b) and (A_3)(ii), we obtain

$$
\begin{aligned}
&q < \|(\mathcal{W}_\alpha \xi^q)(t)\|_{\mathbb{K}} \\
&\quad = \Big\| R(t)\psi(0) + \int_0^t R(t-s) h\big(s, \bar{\xi}^q_{\sigma(s,\bar{\xi}^q_s)}\big) ds + \int_0^t R(t-s)\Gamma \nu_\alpha(s) ds \\
&\qquad + \sum_{0<\tau_i<t} R(t-\tau_i) I_i(\bar{\xi}^q(\tau_i))\Big\|_{\mathbb{K}} \\
&\quad \le \|R(t)\|_{\mathcal{L}(\mathbb{K})}\|\psi(0)\|_{\mathbb{K}} + \int_0^t \|R(t-s)\|_{\mathcal{L}(\mathbb{K})}\|h\big(s, \bar{\xi}^q_{\sigma(s,\bar{\xi}^q_s)}\big)\|_{\mathbb{K}} ds \\
&\qquad + \int_0^t \|R(t-s)\|_{\mathcal{L}(\mathbb{K})}\|\Gamma\nu_\alpha(s)\|_{\mathbb{K}} ds + \sum_{i=1}^m \|R(t-\tau_i)\|_{\mathcal{L}(\mathbb{K})}\|I_i(\bar{\xi}^a(\tau_i))\|_{\mathbb{K}} \\
&\quad \le N\|\psi(0)\|_{\mathbb{K}} + N\int_0^t \|h\big(s, \bar{\xi}^q_{\sigma(s,\bar{\xi}^q_s)}\big)\|_{\mathbb{K}} ds + NN_\Gamma b\|\nu_\alpha\|_\infty + N\sum_{i=1}^m \omega_i.
\end{aligned} \tag{14}
$$

On the other hand, for any $\xi \in \mathcal{H}_q$, it follows from Lemma 2.1 that

$$\|\bar{\xi}^q_{\sigma(s,\bar{\xi}^q_s)}\|_{\mathcal{B}} \leq F_1\|\psi\|_{\mathcal{B}} + F_2 q = \bar{q}. \tag{15}$$

Then, using (13),(14), (15) and (A_0), we have

$$\begin{aligned} q &< \|(\mathcal{W}_\alpha \xi^q)(t)\|_{\mathbb{K}} \\ &\leq N\|\psi(0)\|_{\mathbb{K}} + N\int_0^t m_{\bar{q}}(s)ds + NN_\Gamma b\|v_\alpha\|_\infty + N\sum_{i=1}^m \omega_i \\ &\leq N\|\psi(0)\|_{\mathbb{K}} + N\int_0^b m_{\bar{q}}(s)ds + \frac{N^2N_\Gamma^2 b}{\alpha}\Big(\|\xi_b\|_{\mathbb{K}} + N\|\psi(0)\|_{\mathbb{K}} \\ &\quad + N\int_0^b m_{\bar{q}}(s)ds + N\sum_{i=1}^m \omega_i\Big) + N\sum_{i=1}^m \omega_i \\ &\leq M + N\big(1 + \frac{N^2N_\Gamma^2 b}{\alpha}\big)\int_0^b m_{\bar{q}}(s)ds, \end{aligned} \tag{16}$$

where $M = N\|\psi(0)\|_{\mathbb{K}} + \dfrac{N^2N_\Gamma^2 b}{\alpha}\big(\|\xi_b\|_{\mathbb{K}} + N\|\psi(0)\|_{\mathbb{K}} + N\sum_{i=1}^m \omega_i\big) + N\sum_{i=1}^m \omega_i$.
From (A_3), one can see that

$$\liminf_{q\to\infty}\int_0^b \frac{m_{\bar{q}}(t)}{q}dt = \liminf_{\bar{q}\to\infty}\int_0^b \frac{m_{\bar{q}}(t)}{\bar{q}}dt.\frac{\bar{q}}{q} = F_2\theta.$$

Dividing by q in the inequality (16) and letting $q \to \infty$, we obtain that

$$N\big(1 + \frac{N^2N_\Gamma^2 b}{\alpha}\big)F_2\theta > 1, \tag{17}$$

which contradicts (11). Hence $\mathcal{W}_\alpha(\mathcal{H}_q) \subset \mathcal{H}_q$.
Step 2: We prove that $\mathcal{W}_\alpha$ is a continuous operator.
Let $\{\xi^n\}_{n=1}^\infty \subseteq \mathcal{H}_q$ such that $\xi^n \to \xi$ in $\mathcal{H}_q$, that is,

$$\lim_{n\to\infty}\|\xi^n - \xi\|_{\mathcal{PC}} = 0.$$

Now, by virtue of Lemma 2.1, we deduce that

$$\|\bar{\xi}^n_s - \bar{\xi}_s\|_{\mathcal{B}} \leq F_2 \sup_{\eta\in[0,b]}\|\bar{\xi}^n_s(\eta) - \bar{\xi}_s(\eta)\|_{\mathbb{K}} = F_2\|\xi^n - \xi\|_{\mathcal{PC}} \to 0, \text{ as } n \to \infty,$$

for all $s \in \mathcal{X}(\sigma^-) \cup J$. Since $\sigma(s,\bar{\xi}^n_s) \in \mathcal{X}(\sigma^-) \cup J$, then

$$\|\bar{\xi}^n_{\sigma(s,\bar{\xi}^n_s)} - \bar{\xi}_{\sigma(s,\bar{\xi}^n_s)}\|_{\mathcal{B}} \to 0 \text{ as } n \to \infty, \text{ for all } s \in J.$$

Hence, by assumption (A_3) and the above convergence, we infer that

$$\begin{aligned}
&\left\|h(s,\bar{\xi}^n_{\sigma(s,\bar{\xi}^n_s)})-h(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)})\right\|_{\mathbb{K}}\\
&\leq \left\|h(s,\bar{\xi}^n_{\sigma(s,\bar{\xi}^n_s)})-h(s,\bar{\xi}_{\sigma(s,\bar{\xi}^n_s)})\right\|_{\mathbb{K}}+\left\|h(s,\bar{\xi}_{\sigma(s,\bar{\xi}^n_s)})-h(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)})\right\|_{\mathbb{K}}\\
&\to 0 \text{ as } n\to\infty, \text{ uniformly for all } s\in J.
\end{aligned}\tag{18}$$

Now, we estimate

$$\begin{aligned}
&\|\vartheta(\xi^n(\cdot))-\vartheta(\xi(\cdot))\|_{\mathbb{K}}\\
&\leq \left\|\int_0^b R(b-s)\big(h(s,\bar{\xi}^n_{\sigma(s,\bar{\xi}^n_s)})-h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)})\big)ds\right\|_{\mathbb{K}}\\
&\quad+\left\|\sum_{i=1}^m R(b-\tau_i)\big(I_i(\bar{\xi}^n(\tau_i))-I_i(\bar{\xi}(\tau_i))\big)\right\|_{\mathbb{K}}\\
&\leq N\int_0^b \left\|h(s,\bar{\xi}^n_{\sigma(s,\bar{\xi}^n_s)})-h(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)})\right\|_{\mathbb{K}}\\
&\quad+N\sum_{i=1}^m \left\|I_i(\bar{\xi}^n(\tau_i))-I_i(\bar{\xi}(\tau_i))\right\|_{\mathbb{K}}.
\end{aligned}$$

According to the convergence (18) together with Lebesgue's dominated convergence Theorem and assumption (A_4), we devire that

$$\|\vartheta(\xi^n(\cdot))-\vartheta(\xi(\cdot))\|_{\mathbb{K}}\to 0 \text{ as } n\to\infty. \tag{19}$$

Using (19), we obtain that

$$\begin{aligned}
&\|G(\alpha,C_0^b)\vartheta(\xi^n(\cdot))-G(\alpha,C_0^b)\vartheta(\xi(\cdot))\|_{\mathbb{K}}\\
&=\frac{1}{\alpha}\|\alpha G(\alpha,C_0^b)(\vartheta(\xi^n(\cdot))-\vartheta(\xi(\cdot)))\|_{\mathbb{K}}\\
&\leq\frac{1}{\alpha}\|\vartheta(\xi^n(\cdot))-\vartheta(\xi(\cdot))\|_{\mathbb{K}}\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}\tag{20}$$

Hence, $G(\alpha,C_0^b)\vartheta(\xi^n(\cdot))\to G(\alpha,C_0^b)\vartheta(\xi(\cdot))$ in $\mathbb{K}$ as $n\to\infty$. Remember that operator B is compact if and only if its adjoint B^* is compact. Since the operator $R(t)$ is compact for all $t\in J$, then the operator $R^*(t)$ is compact for all $t\in J$. Therefore, by using (20) and compactness of the operator $R^*(\cdot)$, we have

$$\begin{aligned}
&\|v^n_\alpha(t)-v_\alpha(t)\|_{\mathbb{U}}\\
&=\|\Gamma^*R^*(b-t)\big(G(\alpha,C_0^b)\vartheta(\xi^n(\cdot))-G(\alpha,C_0^b)\vartheta(\xi(\cdot))\big)\|_{\mathbb{U}}\\
&\leq\|\Gamma^*\|_{\mathcal{L}(\mathbb{K}^*,\mathbb{U})}\|R^*(b-t)\|\|\big(G(\alpha,C_0^b)\vartheta(\xi^n(\cdot))-G(\alpha,C_0^b)\vartheta(\xi(\cdot))\big)\|_{\mathbb{K}^*}\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}\tag{21}$$

Consequently, by (18),(21) and assumption (A_4), we obtain

$$
\begin{aligned}
&\|(\mathcal{W}_\alpha \xi^n)(t) - (\mathcal{W}_\alpha \xi)(t)\|_{\mathbb{K}} \\
&\quad \le \left\| \int_0^t R(t-s)\big(h(s, \bar{\xi}^n_{\sigma(s,\bar{\xi}^n_s)}) - h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\big) ds \right\|_{\mathbb{K}} \\
&\qquad + \left\| \int_0^t R(t-s)\Gamma\big(\nu^n_\alpha(s) - \nu_\alpha(s)\big) ds \right\|_{\mathbb{K}} \\
&\qquad + \Big\| \sum_{0<\tau_i<t} R(t-\tau_i)\big(I_i(\bar{\xi}^n(\tau_i)) - I_i(\bar{\xi}(\tau_i))\big) \Big\|_{\mathbb{K}} \\
&\quad \le \int_0^t \left\| R(t-s)\big(h(s, \bar{\xi}^n_{\sigma(s,\bar{\xi}^n_s)}) - h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\big) ds \right\|_{\mathbb{K}} + N N_\Gamma b \|\nu^n_\alpha - \nu_\alpha\|_\infty \\
&\qquad + N \sum_{i=1}^m \big\| I_i(\bar{\xi}^n(\tau_i)) - I_i(\bar{\xi}(\tau_i)) \big\|_{\mathbb{K}} \to 0 \text{ as } n \to \infty \text{ uniformly for } t \in J,
\end{aligned}
\tag{22}
$$

which implies that $\mathcal{W}_\alpha$ is continuous.

Step 3: $\mathcal{W}_\alpha$ is a compact operator. To prove this, first we show that $\mathcal{W}_\alpha(\mathcal{H}_q)$ is equicontinuous. For $0 \leq t_1 \leq t_2 \leq b$ and $\xi \in \mathcal{H}_q$, we evaluate

$$
\begin{aligned}
&\|(\mathcal{W}_\alpha \xi)(t_2) - (\mathcal{W}_\alpha \xi)(t_1)\|_{\mathbb{K}} \\
&\leq \|R(t_2)\psi(0) - R(t_1)\psi(0)\|_{\mathbb{K}} + \left\| \int_0^{t_1} \big(R(t_2-s) - R(t_1-s)\big) h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big) ds \right\|_{\mathbb{K}} \\
&\quad + \left\| \int_0^{t_1} \big(R(t_2-s) - R(t_1-s)\big)\Gamma \nu_\alpha(s) ds \right\|_{\mathbb{K}} + \left\| \int_{t_1}^{t_2} R(t_2-s) h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big) \right\|_{\mathbb{K}} \\
&\quad + \left\| \int_{t_1}^{t_2} R(t_2-s)\Gamma \nu_\alpha(s) \right\|_{\mathbb{K}} + \left\| \sum_{0<\tau_i<t_1} \big(R(t_2-\tau_i) - R(t_1-\tau_i)\big) I_i(\bar{\xi}(\tau_i)) \right\|_{\mathbb{K}} \\
&\quad + \left\| \sum_{t_1 \leq \tau_i < t_2} R(t_2-\tau_i) I_i(\bar{\xi}(\tau_i)) \right\|_{\mathbb{K}} \\
&\leq \|R(t_2)\psi(0) - R(t_1)\psi(0)\|_{\mathbb{K}} + \int_0^{t_1} \|R(t_2-s) - R(t_1-s)\| \|h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\|_{\mathbb{K}} ds \\
&\quad + \int_0^{t_1} \|R(t_2-s) - R(t_1-s)\| \|\Gamma \nu_\alpha(s)\|_{\mathbb{K}} ds \\
&\quad + + N \int_{t_1}^{t_2} \|h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\|_{\mathbb{K}} ds + N \int_{t_1}^{t_2} \|\Gamma \nu_\alpha(s)\|_{\mathbb{K}} ds \\
&\quad + \sum_{0<\tau_i<t_1} \|R(t_2-\tau_i) - R(t_1-\tau_i)\|_{\mathbb{K}} \|I_i(\bar{\xi}(\tau_i))\|_{\mathbb{K}} + N \sum_{t_1 \leq \tau_i < t_2} \|I_i(\bar{\xi}(\tau_i))\|_{\mathbb{K}} \\
&\leq \|R(t_2)\psi(0) - R(t_1)\psi(0)\|_{\mathbb{K}} + N N_\Gamma \|\nu_\alpha\|_\infty (t_2 - t_1) + N \int_{t_1}^{t_2} m_{\bar{q}}(s) ds \\
&\quad + \int_0^{t_1} \|R(t_2-s) - R(t_1-s)\| m_{\bar{q}}(s) ds + N_\Gamma \|\nu_\alpha\|_\infty \int_0^{t_1} \|R(t_2-s) - R(t_1-s)\| ds \\
&\quad + \sum_{0<\tau_i<t_1} \|R(t_2-\tau_i) - R(t_1-\tau_i)\|_{\mathbb{K}} \omega_i + N \sum_{t_1 \leq \tau_i < t_2} \omega_i. \qquad (23)
\end{aligned}
$$

By the continuity of $(R(t))_{t \geq 0}$ in the operator-norm topology and the dominated convergence Theorem, we conclude that the right-hand side of (23) tends to zero uniformly for $\xi \in \mathcal{H}_q$, as $t_1 \to t_2$. As a consequence, $\mathcal{W}_\alpha(\mathcal{H}_q)$ is equicontinuous.

Next, we show that for each $\alpha > 0$, the operator $\mathcal{W}_\alpha$ maps $\mathcal{H}_q$ into a relatively compact subset of $\mathcal{H}_q$. For this purpose, we prove that for every $t \in J$, the set $F(t) = \{(\mathcal{W}_\alpha \xi)(t) : \in \mathcal{H}_q\}$ is precompact in $\mathbb{K}$. By (A_1) and (A_5), we know that the operators $R(t)$ and I_i, $i = 1, \ldots, m$ are compact. Therefore, in order to prove the compactness of $F(t)$, for each $t \in J$, we have to prove that the operator $(\Phi_\alpha \xi)(t) := \int_0^t R(t-s) h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big) ds + \int_0^t R(t-s)\Gamma \nu_\alpha(s) ds$ is compact. To this end, we demonstrate that the following set

$$
F_1(t) = \left\{ (\Phi_\alpha \xi)(t) := \int_0^t h\big(s, \bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big) ds + \int_0^t R(t-s)\Gamma \nu_\alpha(s) ds : \in \mathcal{H}_q \right\}
$$

is relatively compact in $\mathbb{K}$. Let $t \in J$ be fixed and $\rho \in (0,1)$. For $\xi \in \mathcal{H}_q$, we define the following operators

$$
\begin{aligned}
(\Phi_\alpha^\rho \xi)(t) &= R(\rho)\int_0^{t-\rho} R(t-s-\rho)h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds \\
&\quad + R(\rho)\int_0^{t-\rho} R(t-s-\rho)\Gamma \nu_\alpha(s)ds \\
(\widehat{\Phi_\alpha^\rho \xi})(t) &= \int_0^{t-\rho} R(t-s)h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds + \int_0^{t-\rho} R(t-s)\Gamma \nu_\alpha(s)ds.
\end{aligned}
$$

The set $\{(\Phi_\alpha^\rho \xi)(t) : \xi \in \mathcal{H}_q\}$ is relatively compact thanks to the compactness of the operator $R(\rho)$ in $\mathbb{K}$, for every $\rho \in (0,1)$.
On the other hand, by Lemma 2.4 and Hölder's inequality, for all $\xi \in \mathcal{H}_q$, we have

$$
\begin{aligned}
&\|(\Phi_\alpha^\rho \xi)(t) - (\widehat{\Phi_\alpha^\rho \xi})(t)\| \\
&= \Big\| R(\rho)\int_0^{t-\rho} R(t-s-\rho)h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds \\
&\quad + R(\rho)\int_0^{t-\rho} R(t-s-\rho)\Gamma \nu_\alpha(s)ds \\
&\quad - \int_0^{t-\rho} R(t-s)h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds - \int_0^{t-\rho} R(t-s)\Gamma \nu_\alpha(s)ds \Big\| \\
&= \Big\| \int_0^{t-\rho} [R(\rho)R(t-s-\rho) - R(t-s)]h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds \\
&\quad + \int_0^{t-\rho} [R(\rho)R(t-s-\rho) - R(t-s)]\Gamma \nu_\alpha(s)ds \Big\| \\
&\le \int_0^{t-\rho} \|R(\rho)R(t-s-\rho) - R(t-s)\| \|h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\| \\
&\quad + \int_0^{t-\rho} \|R(\rho)R(t-s-\rho) - R(t-s)\| \|\Gamma\| \|\nu_\alpha(s)\| ds \\
&\le L\rho \int_0^{t-\rho} \|h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\| ds + L\rho N_\Gamma \int_0^{t-\rho} \|\nu_\alpha(s)\| \to 0 \text{ as } \rho \to 0^+.
\end{aligned}
$$

By the total bounbedness principle, the set $\{(\widehat{\Phi_\alpha^\rho \xi})(t) : \xi \in \mathcal{H}_q\}$ is relatively compact. Using the same technique, we obtain

$$\begin{aligned}
&\|(\Phi_\alpha\xi)(t)-(\widehat{\Phi_\alpha^\rho\xi})(t)\| \\
=&\quad \Big\|\int_0^t R(t-s)h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds+\int_0^t R(t-s)\Gamma\nu_\alpha(s)ds \\
&-\int_0^{t-\rho} R(t-s)h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds-\int_0^{t-\rho} R(t-s)\Gamma\nu_\alpha(s)ds\Big\| \\
=&\quad \Big\|\int_{t-\rho}^t R(t-s)h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)ds+\int_{t-\rho}^t R(t-s)\Gamma\nu_\alpha(s)ds\Big\| \\
\le& N\int_{t-\rho}^t \|h\big(s,\bar{\xi}_{\sigma(s,\bar{\xi}_s)}\big)\|ds+N\int_{t-\rho}^t \|\Gamma\|\|\nu_\alpha(s)\|ds \\
\le& N\int_{t-\rho}^t m_{\bar{q}}(s)ds+\rho N N_\Gamma\|\nu\|_\infty \to 0 \text{ as } \rho\to 0^+,
\end{aligned}$$

and there are precompact sets arbitrarily close to the set $\{(\Phi_\alpha\xi)(t) : \xi\in\mathcal{H}_q\}$. Hence, the set $\{(\Phi_\alpha\xi)(t) : \xi\in\mathcal{H}_q\}$ is relatively compact in $\mathbb{X}$. Finally, $F(t)$ is relatively compact in $\in\mathcal{H}_q$. By Arzelá-Ascoli Theorem, we conclude that the operator $\mathcal{W}_\alpha$ is compact. Thus, by using Schauder's fixed point theorem, we see that $\mathcal{W}_\alpha$ has a fixed point in $\mathcal{H}_q$, which is a mild solution of system (1). □

To establish the approximate controllability of system (1), we assume the following assumption on $h(\cdot,\cdot)$.
(A_5) $h : J\times\mathcal{B}\to\mathbb{K}$ satisfies the assumption (A_3)-(i) and uniformly bounded, that is, there exists a constant $\bar{M}>0$ such that

$$\|h(t,\psi)\|_{\mathbb{K}}\le\bar{M}, \text{ for all } (t,\psi)\in J\times\mathcal{B}.$$

Theorem 3.2. *Let assumptions (P_1)-(P_2), (A_0)-(A_2) and (A_4)-(A_5) hold. Then the system* (1) *is approximately controllable on J.*

Proof. By Theorem 3.1, we know that for every $\alpha>0$ and $\xi_b\in\mathbb{K}$, there exists a mild solution $\xi^\alpha(\cdot)$ such that

$$\begin{aligned}
\xi^\alpha(t)=R(t)\psi(0)+\int_0^t R(t-s)h\big(s,\bar{\xi^\alpha}_{\sigma(s,\bar{\xi^\alpha}_s)}\big)ds+\int_0^t R(t-s)\Gamma\nu^\alpha(s)ds \\
+\sum_{0<\tau_i<t} R(t-\tau_i)I_i(\bar{\xi^\alpha}(\tau_i))\ t\in J,
\end{aligned}$$

with the control

$$\nu^\alpha(t)=\Gamma^* R^*(b-t)G(\alpha,C_0^b)l(\xi^\alpha(\cdot)), \tag{24}$$

where

$$l(\xi^{\alpha}(\cdot)) = \xi_b - R(b)\psi(0) - \int_0^b R(b-s)h\big(s, \bar{\xi}^{\alpha}_{\sigma(s,\bar{\xi}^{\alpha}_s)}\big)ds$$
$$- \sum_{i=1}^{m} R(t-\tau_i)I_i(\bar{\xi^{\alpha}}(\tau_i)).$$

Using (24), one can see that

$$\xi^{\alpha}(t) = \xi_b - \alpha G(\alpha, C_0^b)l(\xi^{\alpha}(\cdot)). \tag{25}$$

Applying the assumption (A_5), we get

$$\int_0^b \left\| h\big(\cdot, \bar{\xi}^{\alpha}_{\sigma(s,\bar{\xi}^{\alpha}_s)}\big) \right\|_{\mathbb{K}}^2 \leq \bar{M}^2 b.$$

That is, the sequence $\left\{ h\big(\cdot, \bar{\xi}^{\alpha}_{\sigma(s,\bar{\xi}^{\alpha}_s)}\big) \,:\, \alpha > 0 \right\}$ is bounded in $L^2(J;\mathbb{K})$.

Therefore, there exists a subsequence relabeled as, such that $\left\{ h\big(\cdot, \bar{\xi}^{\alpha}_{\sigma(s,\bar{\xi}^{\alpha}_s)}\big) \,:\, \alpha > 0 \right\}$ such that

$$h\big(\cdot, \bar{\xi}^{\alpha}_{\sigma(s,\bar{\xi}^{\alpha}_s)}\big) \rightharpoonup h(\cdot).$$

In addition, by (A_4), we have

$$\left\| I_i(\bar{\xi^{\alpha}}(\tau_i)) \right\|_{\mathbb{K}} \leq \omega_i,$$

for each $i = 1, \ldots, m$. Hence, the sequence $\left\{ I_i(\bar{\xi^{\alpha}}(\tau_i)) \,:\, \alpha > 0 \right\}$ is bounded in $\mathbb{K}$. Once again using the same argument, we can find subsequences relabeled as $\left\{ I_i(\bar{\xi^{\alpha}}(\tau_i)) \,:\, \alpha > 0 \right\}$, which are weakly convergent to the pointwise weak limit $\delta_i \in \mathbb{K}$, for each $i = 1, \ldots, m$. Let us now define

$$\mu := \xi_b - R(b)\psi(0) - \int_0^b R(b-s)h(s)ds - \sum_{i=1}^{m} R(t-\tau_i)\delta_i.$$

Then, we have

$$\begin{aligned} \|l(\xi^{\alpha}(\cdot)) - \mu\|_{\mathbb{K}} \leq & \left\| \int_0^b R(b-s)\Big[h\big(\cdot, \bar{\xi^{\alpha}}_{\sigma(s,\bar{\xi^{\alpha}}_s)}\big) - h(s)\Big] \right\|_{\mathbb{K}} \\ & + \sum_{i=1}^{m} \left\| R(t-\tau_i)\Big(I_i(\bar{\xi^{\alpha}}(\tau_i)) - \delta_i\Big) \right\|_{\mathbb{K}} \\ & \to 0 \text{ as } \alpha \to 0^+. \end{aligned} \tag{26}$$

In the right-hand side of (26), the first term tends to zero thanks to the compactness of the operator $(Sh)(\cdot) = \int_0^{\cdot} R(\cdot - s)h(s)ds : L^2(J;\mathbb{K}) \to C(J;\mathbb{K})$ (see [28, Lemma 3.2, Chapter 3]), and the second term converges to zero using the compactness of the resolvent operator $(R(t))_{t\geq 0}$.
Next, we compute $\|\xi^\alpha(b) - \xi_b\|_{\mathbb{K}}$ as

$$\begin{aligned}\|\xi^\alpha(b) - \xi_b\|_{\mathbb{K}} &= \|\alpha G(\alpha, C_0^b) l(\xi^\alpha(\cdot))\|_{\mathbb{K}} \\ &\leq \|\alpha G(\alpha, C_0^b)\nu\|_{\mathbb{K}} + \|\alpha G(\alpha, C_0^b)(l(\xi^\alpha(\cdot)) - \nu)\|_{\mathbb{K}}\end{aligned} \tag{27}$$

According to (26) and assumption (A_0), we get

$$\|\xi^\alpha(b) - \xi_b\|_{\mathbb{K}} \to 0 \text{ as } \alpha \to 0^+,$$

which gives the approximate controllability of system (1). □

Remark 3.1. *Theorem 3.2 assumes that the operator A generates a compact semigroup and, consequently, the associated linear control system* (1) *is not exactly controllable. As a result, Theorem 3.2 has no analogue for the concept of exact controllability.*

4. Example

In this section, we provide an example to illustrate the obtained results. Consider the following equation:

$$\begin{cases} \dfrac{\partial}{\partial t}x(t,y) = \dfrac{\partial^2}{\partial t^2}x(t,y) + \zeta(t,y) + \displaystyle\int_0^t \omega(t-s)\dfrac{\partial^2}{\partial y^2}x(s,y)ds \\ \qquad\qquad + e^{-t}\displaystyle\int_{-\infty}^t e^{-2(s-t)}\dfrac{x(s-\sigma_1(t)\sigma_2(\|x(t)\|),y)}{50}ds \\ \qquad\qquad y \in [0,\pi],\ t \in J = [0,b],\ t \neq \tau_i,\ i = 1,\dots,m, \\ \Delta x(t,y) = \displaystyle\int_0^\pi \rho_i(\tau,y)\dfrac{(x(t_i,\tau))^2}{1+(x(t_i,\tau))^2}d\tau, \quad i = 1,2,3\ y \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0, \quad t \in J, \\ x(\varsigma,y) = \phi(\varsigma,y),\ y \in [0,\pi],\ \varsigma \leq 0, \end{cases} \tag{28}$$

where $0 < t_1 < t_2 < t_3 < b$ are real constants, $\rho_i \in C([0,\pi] \times [0,\pi],\mathbb{R}),\ i = 1,2,3$, the function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded and C^1 function with a derivative $|\omega'(t)| \leq \omega(t)$ for all $t \geq 0$, the functions $\sigma_k : [0,\infty) \to [0,\infty)$, for $k = 1,2$, are continuous. Also the function $\zeta : J \times [0,\pi] \to [0,\pi]$ is continuous in t, and the function ϕ belongs to the phase space which will specify later.

Step 1: Resolvent operator.
Let $\mathbb{K} = L^2([0,\pi];\mathbb{R})$, and $\mathbb{U} = L^2([0,\pi];\mathbb{R})$. We define the operator A induced on $\mathbb{K}$ as follows:

$$\begin{cases} D(A) = H^2(0,\pi) \cap H_0^1(0,\pi), \\ A\theta = \theta''. \end{cases} \tag{29}$$

It is well know that A generates C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ in $\mathbb{K}$, which is compact. Moreover, the operator A can be expressed as

$$A\theta = \sum_{n=1}^{\infty} n^2 \langle \theta, \lambda_n \rangle, \quad \theta \in D(A),$$

where $\lambda_n(y) = \sqrt{\frac{2}{\pi}} \sin(ny),\ n = 1,2,\ldots$ is a orthonormal set of eigenvectors in A and

$$T(t)\theta = \sum_{n=1}^{\infty} e^{-n^2} \langle \theta, \lambda_n \rangle \lambda_n, \quad \theta \in \mathbb{K}.$$

Next, we define the bounded linear operator $\Gamma : L^2([0,\pi],\mathbb{R}) \to \mathbb{K}$ by

$$\Gamma(\nu(t))(y) := \nu(t)(y) = \zeta(t,y),\ t \in J,\ y \in [0,\pi].$$

Moreover since ω a bounded and C^1 function such that ω' is a bounded and uniformly continuous, (P_2) is satisfied. Therefore, (P_1) and (P_2) hold and hence, by Theorem 2.3, system (28) has a unique resolvent operator $(R(t))_{t\geq 0}$. Additionaly, in virtue of Theorem 2.5, $(R(t))_{t\geq 0}$ is also compact for $t > 0$, and then (A_1) is satisfied.
Step 2: Phase space.
For the phase space, we take $\mathcal{B} = \mathcal{PC}_0 \times L_f^1(\mathbb{K})$ *which is described in Section 2 with* $r = 0$*. Let* $f(\tau) = e^{2\tau}$*, for* $\tau < 0$*, then*

$$p = \int_{-\infty}^{0} f(\tau) d\tau = \frac{1}{2},$$

and define the norm

$$\|\varphi\|_{\mathcal{B}} = \int_{-\infty}^{0} f(\tau) \sup_{\varsigma \in [\tau,0]} \|\varphi(\varsigma)\|_{\mathbb{K}} d\tau.$$

Step 3: Abstract formulation and approximate controllability. Let us define

$$\xi(t)(y) := x(t,y),\ \text{for } t \in J \text{ and } y \in [0,\pi],$$

and the bounded linear operator $\Gamma : L^2([0,\pi],\mathbb{R}) \to \mathbb{K}$ by

$$\Gamma(\nu(t))(y) := \nu(t)(y) = \zeta(t,y),\ t \in J,\ y \in [0,\pi].$$

The function $\psi : (-\infty,0] \to \mathbb{K}$, is defined as

$$\psi(t)(y)) = \phi(t,y),\ y \in [0,\pi].$$

Next, we define the function $h : J \times \mathcal{B} \to \mathbb{K}$ such that

$$h(t,\phi)y := e^{-t}\int_{-\infty}^{0} e^{2\tau}\frac{\phi}{25}d\tau \text{ for } y \in [0,\pi].$$

Also, we define the state-dependent function $\sigma : J \times \mathcal{B} \to \mathbb{K}$ as

$$\sigma(t,\phi) := t - \sigma_1(t)\sigma_2(\|\phi(0)\|_{\mathbb{K}}).$$

Now, we define the impulses $I_i : \mathbb{K} \to \mathbb{K}$ by

$$I_i(\xi(\tau_i))y := I_i(x(\tau_i,y)),$$

for $i = 1,2,3$ and $y \in [0,\pi]$. Take

$$I_i(x(\tau_i,y)) = \int_0^{\pi} \eta_i(y,s)\frac{\xi^2(\tau_i,s)}{1+\xi^2(\tau_i,s)}ds,$$

where $\eta_i \in \mathcal{C}([0,\pi]\times[0,\pi];\mathbb{R}),\ y \in [0,\pi]$.

Now, according to the above definitions, the system (28) is written in the abstract form

$$\begin{cases} \xi'(t) = A\xi(t) + \int_0^t B(t-s)\xi(s)ds + h\big(t,\xi_{\sigma(t,\xi_t)}\big) + \Gamma\nu(t), \\ \qquad\qquad t \in J = [0,b],\ t \neq \tau_i,\ i = 1,2,\ldots,n, \\ \Delta\xi|_{t=\tau_i} = I_i(\xi(\tau_i)),\ i = 1,\ldots,m, \\ \xi_0 = \psi \in \mathcal{B}, \end{cases} \tag{30}$$

Step 4: Verification of hypotheses

It is obvious that h is a continuous and bounded function. In addition, we obtain, for $\|\varphi\| \le q$, that

$$\begin{aligned} \|h(t,\varphi)y\| &\le \Big(\int_0^{\pi}\Big(e^{-t}\int_{-\infty}^{0} e^{2s}\frac{\varphi}{25}ds\Big)^2 dy\Big)^{1/2} \\ &\le \Big(\int_0^{\pi}\Big(\frac{1}{50}e^{-t}\int_{-\infty}^{0} e^{2s}\sup\|\varphi\|ds\Big)^2 dy\Big)^{1/2} \\ &\le \frac{\sqrt{\pi}}{50}e^{-t}\|\varphi\|_{\mathcal{B}} \le \frac{\sqrt{\pi}}{50}e^{-t}q. \end{aligned}$$

We take $m_q(t) = \dfrac{\sqrt{\pi}}{50}e^{-t}q$ and in this case $\theta = \dfrac{\sqrt{\pi}}{50}(1-e^{-1})$. Thus, (A_2) is satisfied.

Clearly, the impulses $I_i, i = 1,2,3$ are completely continuous and satisfy assumption (A_3). Moreover, an adequate choice of $\alpha > 0$ allows us to verify (11). Therefore, all assumptions of Theorem 3.1 are satisfied and then system (28) has at least a mild solution.

Finally, it remains to check that the corresponding linear system of (28) is approximately controllable. In order to do this, let $\Gamma^* R^*(b-s)\xi^* = 0$ for any $\xi^* \in \mathbb{K}^*$. Since $\Gamma = I$ in the system (28), then we have

$$\Gamma^* R^*(b-s)\xi^* = 0 \Rightarrow R^*(b-s)\xi^* = 0 \Rightarrow \xi^* = 0.$$

Consequently, in accordance with Theorem 2.8, the linear system corresponding linear system of (28) is approximately controllable and the condition (A_0) is satisfied. Thus, by Theorem 3.2, the semilinear system (28) is approximately controllable.

5. Conclusion

In this paper, we have discussed the approximate controllability for a class of impulsive integrodifferential equations with state-dependent delay in Hilbert spaces. We fisrt proved the existence of mild solutions for the impulsive integrodifferential equations with state-dependent delay by applying Schauder's fixed point theorem and the resolvent operators theory in Grimmer's sense. Then, we established the approximate controllability of the considered system under some conditions by assuming that the corresponding linear system is approximately controllable. Finally, an example has been provided to illustrate the effectiveness of our findings.

However, there are many way to extend this work. Thus, for the future research, we propose the analysis of approximate controllability for neutral impulsive integrodifferential stochastic equations with state-dependentdelay driven by a Rosenblatt process which is an interesting topic.

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