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INTERPOLATION OF BESOV SPACES AND APPLICATIONS

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We define the analytic Besov spaces on a bounded symmetric domain associated with a rearrangement invariant space, give a description in terms of certain differential operators, prove an interpolation theorem and find their dual space; finally, as an application we formulate necessary and sufficient conditions in order to little Hankel operator H_f belongs to S_E, the Schatten ideal associated with a given rearrangement invariant sequence space E.

1. Introduction.

Let Ω be a bounded symmetric domain in C^n, in its standard Harish-Chandra realization; it is well-known (see e.g. [6]) that Ω is uniquely determined (up to biholomorphic mapping) by three analytic invariants (all of them positive integers): r, called the rank of Ω, a and b. The Bergman reproducing kernel of Ω is

$$K(z, w) = \frac{1}{h(z, w)^N}, \quad z, w \in \Omega$$

where $h(z, w)$ is a sum of homogeneous monomials in z and $\overline{w}$ and $N = a(r - 1) + b + 2$ and the Bergman projection P (the orthogonal projection of $L^2(dv)$ onto $L^2_a(dv)$) is given by the formula

$$Pf(z) = \int_\Omega \frac{f(w)}{h(z, w)^N}\, dv(w), \quad f \in L^2(dv), z \in \Omega$$

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where dv is the normalized volume measure on Ω. Let $\alpha > -1$ and let C_α be a positive constant such that the measure $dv_\alpha(z) = C_\alpha h(z, z)^\alpha \, dv(z)$ has total mass 1 on Ω. Let $L_a^2(dv_\alpha)$ be the closed subspace of all analytic functions in $L^2(dv_\alpha)$ and denote by P_α the corresponding Bergman projection

$$P_\alpha f(z) = C_\alpha \int_\Omega \frac{h(w, w)^\alpha f(w)}{h(z, w)^{N+\alpha}} \, dv(w), \qquad f \in L^2(dv_\alpha), z \in \Omega.$$

Standard arguments show that P_α reproduces functions in $L_a^1(dv_\alpha)$. Let us consider, as in [12], two radial differential operators:

$$D^{\alpha,\beta} f(z) = c_{\alpha,\beta} \lim_{r \to 1^-} \int_\Omega \frac{h(w, w)^\beta}{h(z, w)^{N+\alpha+\beta}} f(rw) \, dv(w), \qquad f \in H(\Omega), z \in \Omega$$

$$D_{\alpha,\beta} f(z) = c_{\alpha,\beta} \lim_{r \to 1^-} \int_\Omega \frac{h(w, w)^{\alpha+\beta}}{h(z, w)^{N+\beta}} f(rw) \, dv(w), \qquad f \in H(\Omega), z \in \Omega.$$

$D^{\alpha,\beta}$ and $D_{\alpha,\beta}$ are well defined (the limit always exists) and continuous on $H(\Omega)$ (when endowed with the topology of uniform convergence on compacts). When $c_{\alpha,\beta}$ are properly chosen, we also have the following representation formula

$$D^{\alpha,\beta} D_{\alpha,\beta} f(z) = f(z), \qquad f \in H(\Omega), \; z \in \Omega$$

(see Theorem 1 and Theorem 2 in [12]). For simplicity, we shall denote by D^α, the operator $D^{\alpha,0}$ and by D_α, the oprerator $D_{\alpha,0}$.

Let $E(d\lambda)$ a rearrangement invariant space over the measure space $(\Omega, d\lambda)$, where

$$d\lambda(z) = \frac{1}{h(z, z)^N} \, dv(z)$$

is the Möbius invariant measure on Ω; we define the analytic Besov spaces $B_E(\Omega)$ associated with $E(d\lambda)$ to be $B_E(\Omega) = PE(d\lambda)$, endowed with the quotient norm (see Definition 1). We shall prove that a holomorphic function f is in $B_E(\Omega)$ if and only if the function $h(z, z)^N D^N f(z) \in E(d\lambda)$ if and only if for any $\alpha > -1$ and for any real β, the function $h(z, z)^\alpha D^{\alpha,\beta} f(z) \in E(d\lambda)$ (Theorem 1 and Corollary 1); then we give an alternative description in terms of P_α ($\alpha > -1$): $B_E(\Omega) = P_\alpha E(d\lambda)$ (Theorem 2). When $E(d\lambda)$ has absolute continuous norm, we show that $B_E(\Omega)$ is a separable Banach space, with polynomials dense in it (Theorem 3) and whose dual is $B_{E'}(\Omega)$ (Theorem 5). Finally, we prove two interpolation theorem for pairs of Besov spaces (Theorem 5 and Proposition 2).

The analytic Besov spaces $B_p(\Omega)$ over a bounded symmetric domain Ω associated with $L^p(d\lambda)$ were defined and studied in [11] and [12], while the classic case of the analytic Besov spaces B_p over the unit disk was studied in [13].

The importance of these spaces in operators theory arises from their relation with Hankel operators H_f that belongs to Schatten ideals S_p; the question of characterizing those symbols f such that the Hankel operator H_f belongs to the Schatten ideal S_p in the Bergman or Hardy space, has received considerable attention from many authors (see [2], [4], [7], [8], [9], [10], [11], [12] and the references therein). When the symbol f is a holomorphic function on the disk, it was shown in [7] that $H_f \in S_p$ if and only if $f \in B_p$, $1 \leq p \leq \infty$; in the theory of several complex variables, similar results were given in [2], [9] and [10]; where is proved that if $0 < p \leq 2n$, then $H_f \in S_p$ (on Hardy as well as on Bergman space) if and only if f is constant while if $2n < p \leq \infty$, $H_f \in S_p$ if and only if $f \in B_p$. The first who studied necessary and sufficient condition such that $H_f \in S_{p,q,}$ the Schatten ideal associated with Lorentz sequence spaces $l^{p,q}$ was Peller (see [8]); for this, he considered a new class of spaces, $\Gamma^{1/p}_{p,q}$ defined in terms of a maximal nontangential function defined on a annular domain and proved that $H_f \in S_{p,q}$ if and only $f \in \Gamma^{1/p}_{p,q}$.

In the last section, we investigate necessary and sufficient conditions in order to the Hankel operator on weighted Bergman spaces on a bounded symmetric domain $h^{(\alpha)}_f(g) = (I - P_\alpha)(\overline{f}g)$, $g \in L^2_a(dv_\alpha)$ belongs to Schatten ideals S_E associated with rearrangement invariant sequence spaces and show that if E is a given rearrangement invariant sequence space, $h^{(\alpha)}_f \in S_E$ if and only if $f \in B_E(\Omega)$ (Theorem 6 and Corollary 2). Our approach is completely different of those in [9] and the main point is the interpolation theorem of Besov spaces (Theorem 5).

2. Preliminaries.

We shall remind here some basic fact of interpolation theory that we shall use latter on; for notations, unexplained definition and details, the reader is referred to [3].

Let (X_0, X_1) be a compatible couple of Banach spaces, ρ a monotone Riesz-Fischer norm and $k(f, \ , X_0, X_1)$ be Peetre's little functional; then the space

$$(X_0, X_1)_\rho = \{f \in \overline{X_0 \bigcap X_1}^{X_0} + X_1 : \|f\|_{(X_0,X_1)_\rho} = \rho(k(f, \ , X_0, X_1)) < \infty\}$$

is a (monotone) interpolation space for the couple (X_0, X_1). This implies that for any linear (o quasilinear, when X_i, $i = 0, 1$ are Banach lattices) operator T that is bounded both on X_0 and on X_1, it follows that T is bounded on $(X_0, X_1)_\rho$, too. We say that (X_0, X_1) form a Calderon couple if all its interpolation spaces are monotone; in this case, it forms a Gagliardo couple, too, hence, by Theorem V 3.7 in [3], an intermediate space E for the couple (X_0, X_1) is an interpolation one, if and only if there exists a monotone Riesz-Fischer norm ρ such that $E = (X_0, X_1)_\rho$. In other words, all the interpolation spaces for a Calderon couple (X_0, X_1), are obtainable by using only the interpolation functor $(\,)_\rho$ defined above.

In this paper, we shall mainly deal with the Calderon couples (L^{p_0}, L^{p_1}) and (S_{p_0}, S_{p_1}), $1 \le p_0 \le p_1 \le \infty$ (see [5] and [1]), as well as with pairs of Besov spaces $(B_{p_0}(\Omega), B_{p_1}(\Omega))$.

Let (R, μ) denote either the measure space $(\Omega, d\lambda)$ or N endowed with the cardinal measure; let $\mathcal{M}(R, \mu)$ be the space of all μ measurable functions on R and ρ a monotone Riesz-Fischer norm which have Fatou's property; then the space

$$E = \{f \in \mathcal{M}(R, \mu) : \|f\|_E = \rho(f^*) < \infty\}$$

where f^* is the decreasing rearrangement of f, is called rearrangement invariant space (r.i.). E is said to have absolute continuous norm if for any $f \in E$ and for any sequence of measurable sets $E_n \downarrow \emptyset$, we have $\|f\chi_{E_n}\|_E \to 0$, as $n \to \infty$. If E is *r.i.*, then its associate,

$$E' = \{g \in \mathcal{M}(R, \mu) : \|g\|_{E'} = \sup\left\{\Big|\int_R fg\, d\mu\Big|,\ \|f\|_E \le 1\right\},$$

is a r.i., too. Hölder's inequality

$$\Big|\int_R fg\, d\mu\Big| \le \|f\|_E \|g\|_{E'}$$

is a consequence of the definition of E'; note that we shall always have $E'' = E$, for any r.i. E. If $g \in E'$, then $L_g(f) = \int_R fg\, d\mu$ defines a continuous functional on E; when E has absolute continuous norm, then the Banach dual E^* of E identifies with E', by mean of the canonic isomorphism $g \to L_g$.

Associate with any r.i. E, there are two (real) numbers $1 \le p_E \le q_E \le \infty$, called the Boyd indices of E. Boyd's interpolation theorem states that if T is an operator of weak type (p, p) and (q, q), then T is bounded on any r.i. E whose Boyd indices verify $1 \le p < p_E \le q_E < q \le \infty$ (see [3]).

Many important classes of r.i., such as the Lorentz spaces $L^{p,q}$ have absolute continuous norm if and only if have non trivial Boyd indices $1 < p_E =$

$q_E = p < \infty$; in this case $(L^{p,q})^* = L^{p',q'} = (L^{p,q})'$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$; a similar fact occurs with the Orlicz spaces L^{Φ}, that have absolutely continuous norm if and only if have non trivial inferior Boyd indices and if this condition is fulfilled, we have $(L^{\Phi})^* = (L^{\Phi})' = L^{\psi}$, where ψ is the Young's conjugate function of ϕ. Nevertheless, there exists examples of r.i. for which the two proprieties are completely independent. The following spaces, constructed by R. Sharpley (see[3]), furnish such an example.

Let E be an arbitrary r.i. and fix $1 \leq q \leq \infty$; let

$$\Lambda_q(E) = \Big\{ f \in \mathcal{M}(D, dA) : \|f\|_{\Lambda_q} = \Big(\int_0^{\infty} [f^{**}(t)\varphi_E(t)]^q \frac{dt}{t} \Big)^{1/q} < \infty \Big\},$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$ is the maximal function, f^* the decreasing rearrangement of f and φ_E the fundamental function of E and where, as usual, if $q = \infty$ we take sup instead of integral; then, if E has non trivial fundamental indices, $\Lambda_q(E)$ is a r.i. whose Boyd and fundamental indices coincide with the fundamental indices of E; further, $\Lambda_q(E)$ has absolute continuous norm if and only if $1 \leq q < \infty$ and if this conditions are satisfied, then $\Lambda_q(E)^* = \Lambda_q(E)' = \Lambda_{q'}(E')$ (for details, see e.g. [3], pg. 285). So, if we choose $q = \infty$ and E having non trivial fundamental indices, then $\Lambda_{\infty}(E)$ will have non trivial Boyd indices and its norm will not be absolute continuous. On the other hand, if we take $q < \infty$ and a r.i. E with trivial fundamental indices, then we get $\Lambda_{\infty}(E)$, a r.i. whose norm is absolute continuous but whose Boyd indices are trivial.

Notations. From now on, ρ will always denote a monotone Riesz-Fischer norm having Fatou property and $(\ )_\rho$ the corresponding interpolation functor; if $1 \leq p_0 \leq p_1 \leq \infty$ and if $E(d\mu) = ((L^{p_0}(d\mu), L^{p_1}(d\mu))_\rho$, then we shall simply designate by E the corresponding r.i. sequence space $E = (l^{p_0}, l^{p_1})_\rho$.

3. Analytic Besov spaces.

A very useful tool in what follows is a Forelli-Rudin type theorem for bounded symmetric domains.

Lemma 1. (see [6] or Lemma 2 in [11]). *Let Ω be a bounded symmetric domain, $t > -1$ and $c \in R$ such that $c > \frac{a(r-1)}{2}$; then*

$$I_{c,t}(z) = \int_{\Omega} \frac{h(w,w)^t}{|h(z,w)|^{N+t+c}}\, dv(w) \sim h(z,z)^{-c}$$

where, as usually, the notation $f \sim g$ means that there exists some constant $C > 0$ such that $1/Cg(z) \leq f(z) \leq Cg(z)$, $\forall z \in \Omega$.

Let us consider the following family of integral operators:

$$V_{\alpha,\beta} f(z) = h(z, z)^{\alpha} \int_{\Omega} \frac{h(w, w)^{\beta}}{h(z, w)^{N+\alpha+\beta}} f(w) dv(w), \quad z \in \Omega;$$

we shall denote, for simplicity, by V_α the operator $V_{\alpha,0}$.

Lemma 2. *With the notations above, if $\alpha > N - 1$ and β is an arbitrary real constant, then the operators V_α and $V_{N,\beta}$ are bounded on any $E(d\lambda)$.*

Proof. By interpolation, it will suffice to prove their boundedness on $L^1(d\lambda)$ and on $L^\infty(d\lambda)$. We shall give the proof for V_α, as the one for $V_{N,\beta}$ is similar.

If $f \in L^\infty(d\lambda)$; then, by Lemma 1

$$|V_\alpha f(z)| \le C h(z, z)^{\alpha} \int_{\Omega} \frac{dv(w)}{|h(z, w)|^{N+\alpha}}$$

so by Lemma 1, V_α is bounded on $L^\infty(d\lambda)$.

If $f \in L^1(d\lambda)$, then by Fubini's theorem

$$\int_{\Omega} |V_\alpha f(z)| \, d\lambda(z) \le \int_{\Omega} |f(w)| \int_{\Omega} \frac{h(z, z)^{\alpha-N}}{h(z, w)^{\alpha+N}} dv(z) dv(w) \sim$$

$$\sim \int_{\Omega} |f(w)| h(w, w)^{-N} \, dv(w),$$

where the last relation follows by Lemma 1, applied for $t = \alpha - N > -1$ and $c = \alpha$, taking in account that $\alpha > N - 1 = a(r - 1) = b + 1 > \frac{a(r-1)}{2}$, where the last inequality is verified in any bounded symmetric domain Ω. This proves that V_α is bounded on $L^1(d\lambda)$, too, ending the proof.

Definition 1. *Let $B_E(\Omega)$ be the set of all (analytic) functions on Ω of the form $f = P\varphi$, with $\varphi \in E(d\lambda)$; when normed with the natural quotient norm induced by the Bergman projection P, $\|f\|_{B_E(\Omega)} = \inf\{\|\varphi\|_{E(d\lambda)}, P\varphi = f\}$, it becomes a Banach space, called the Besov space associated with $E(d\lambda)$.*

We shall prove a first description of $B_E(\Omega)$ in term of derivatives.

Theorem 1. *The Besov space $B_E(\Omega)$ consists of all analytic functions f on Ω, integrable with respect to volume measure, such that the function $f_N(z) = h(z, z)^N D^N f(z)$ belongs to $E(d\lambda)$. Moreover, we have the equivalence of the norms: $\|f\|_{B_E(\Omega)} \sim \|f_N\|_{E(d\lambda)}$.*

Proof. Let $f \in B_E(\Omega)$ and let $\varphi \in E(d\lambda)$ such that $f = P\varphi$; then

$$f_N(z) = h(z,z)^N D^N (P\varphi)(z) = C_N h(z,z)^N \int_\Omega \frac{P\varphi(w) dv(w)}{h(z,w)^{2N}} =$$

$$C_N h(z,z)^N \int_\Omega \frac{1}{h(z,w)^{2N}} \int_\Omega \frac{\varphi(u)}{h(u,w)^N} dv(u) dv(w) =$$

$$C_N h(z,z)^N \int_\Omega \varphi(u) \int_\Omega \frac{1}{h(z,w)^{2N} h(u,w)^N} dv(w) dv(u) =$$

$$C_N h(z,z)^N \int_\Omega \frac{\varphi(u)}{h(z,u)^{2N}} dv(u) = C_N V_N \varphi(z) \in E(d\lambda)$$

since, by Lemma 2, the operator V_N is bounded on $E(d\lambda)$; this shows that $f_N \in E(d\lambda)$ and $\|f_N\|_{E(d\lambda)} \leq C\|\varphi\|_{B_E(\Omega)}$; passing at the infimum over all $\varphi \in E(d\lambda)$ such that $f = P\varphi$, we obtain $\|f_N\|_{E(d\lambda)} \leq C\|f\|_{B_E(\Omega)}$.

Now, if f is analytic and integrable in Ω and if $f_N \in E(d\lambda)$, we have the representation formula $f = D_N D^N f = P f_N$, hence $f \in B_E(\Omega)$ and $\|f\|_{B_E(\Omega)} \leq \|f_N\|_{E(d\lambda)}$, proving the set equality and, by the open mapping theorem, the equivalence of norms.

We can prove now, in a similar manner a description in terms of $D^{\alpha,\beta}$.

Corollary 1. *Let Ω be a bounded symmetric domain, $\alpha > N-1$ and $E(d\lambda)$ a r.i.; let $f \in H(\Omega)$; then $f \in B_E(\Omega)$ if and only if for any real number β, the function $h(z,z)^\alpha D^{\alpha,\beta} f(z)$ belongs to $E(d\lambda)$.*

Proof. If $f \in B_E(\Omega)$, then there exists a function $\varphi \in E(d\lambda)$ such that

$$f(z) = \int_\Omega \frac{\varphi(w)}{h(z,w)^N} dv(w), \quad z \in \Omega;$$

since the integral above is uniformly convergent on compact sets and since the operator $D^{\alpha,\beta}$ is continuous on $H(\Omega)$, we may derivate inside the integral sign and obtain

$$D^{\alpha,\beta} f(z) = \int_\Omega D_z^{\alpha,\beta}[h(z,w)^{-N}] \varphi(w)\, dv(w), \quad z \in \Omega.$$

By the proof of Theorem 1 in [12], there exists a constant $C > 0$ such that

$$h(z,z)^\alpha |D^{\alpha,\beta} f(z)| \leq C h(z,z)^\alpha \int_\Omega \frac{|\varphi(w)|}{|h(z,w)|^{N+\alpha}} dv(w) = C\tilde{V}_\alpha \varphi(z), \quad z \in \Omega,$$

where the last equality serves as definition of the quasilinear operator $\tilde{V}_\alpha$. Proceeding as in the proof of Lemma 2, we can verify that $\tilde{V}_\alpha$ is bounded on any $E(d\lambda)$, hence, the relation above, together with the fact that $E(d\lambda)$ is an order ideal, imply that $h(z,z)^\alpha D^{\alpha,\beta} f(z) \in E(d\lambda)$.

In order to prove the converse, assume that

$$\varphi(z) = h(z,z)^\alpha D^{\alpha,\beta} f(z) \in E(d\lambda);$$

then, proceeding as in the proof of Theorem 4 of [12], we obtain

$$h(z,z)^N |D^N f(z)| \leq C h(z,z)^N \left| \int_\Omega \frac{h^\beta(w,w)}{h(z,w)^{2N+\beta}} \varphi(w)\, dv(w) \right| =$$

$$= |V_{N,\beta}\varphi(z)| \in E(d\lambda),$$

since, by Lemma 2, the operator $V_{N,\beta}$ is bounded on $E(d\Lambda)$.

Finally, we shall prove a description of Besov spaces in terms of the family of Bergman type projections P_β, defined in Section 1.

Theorem 2. *Let $\beta > -1$; then $B_p(\Omega) = P_\beta E(d\lambda)$.*

Proof. If $f \in B_E(\Omega)$, then by Corollary 1, $h(z,z)^N D^{N,\beta} f(z) \in E(d\lambda)$. Using the definition of P_β, reproducing formulas and Fubini's Theorem we obtain

$$P_\beta[h(z,z)^N D^{N,\beta} f(z)] = D_{N,\beta} D^{N,\beta} f(z) = f(z),$$

so $B_E(\Omega) \subseteq P_\beta(E(d\lambda))$.

Now, if $f = P_\beta\varphi$, with $\varphi \in E(d\lambda)$, then by Theorem 1 in [12]

$$h(z,z)^N D^{N,\beta} f(z) = h(z,z)^N \int_\Omega h(w,w)^\beta D^{N,\beta}[h(z,w)^{-(N+\beta)}] \varphi(w)\, dv(w) =$$

$$= h(z,z)^N \int_\Omega \frac{h(w,w)^\beta}{h(z,w)^{2N+\beta}} \varphi(w)\, dv(w) = V_{N,\beta}\varphi(z) \in E(d\lambda),$$

since $V_{N,\beta}$ is bounded on $E(d\lambda)$. This proves the inclusion $P_\beta E(d\lambda) \subseteq B_E(\Omega)$, ending the proof.

We shall discuss now the separability of Besov spaces. We begin by making a simple remark.

Proposition 1. *Any analytic Besov space $B_E(\Omega)$ contain all the analytic functions on $\overline{\Omega}$; in particular, $B_E(\Omega)$ contain all the polynomials.*

Proof. Let $f \in H(\overline{\Omega})$; then there exists $r > 1$ such that $f \in H(\overline{\Omega_r})$, where $\Omega_r = \{rz, z \in \Omega\}$; by the proof of Theorem 7 in [11], it follows that $D^N f \in H(\Omega_r) \supset H(\overline{\Omega})$; this fact implies, as one can easily check, that the function $h(z,z)^N D^N f(z) \in L^1(d\lambda) \cap L^\infty(d\lambda) \hookrightarrow E(d\lambda)$, hence, by Theorem 1, $f \in B_E(\Omega)$.

Theorem 3. *If $E(d\lambda)$ has absolute continuous norm, then the Besov space $B_E(\Omega)$ is a separable Banach space, with polynomials dense in it.*

Proof. Let $f \in B_E(\Omega)$; there exists $\varphi \in E(d\lambda)$ such that $f = P\varphi$; consider $\Omega_n \uparrow \Omega$ a compact exhaustion of Ω and let $\varphi_n = \varphi\chi_{\Omega_n}$, $n \geq 1$; then $\varphi_n \in E(d\lambda)$, as $E(d\lambda)$ has absolute continuous norm. Let $\varepsilon > o$ and choose N_ε such that $\|\varphi - \varphi_{N_\varepsilon}\|_{E(d\Lambda)} < \varepsilon/2$ denote, for simplicity of notations by $g = P\varphi_{N_\varepsilon}$; then $\|f - g\|_{B_E(\Omega} < \varepsilon/2$. Since φ_n were chosen to be compactly supported in Ω, we can easily see that the function $g \in H(\overline{\Omega})$; so there exists $r > 1$ such that $g \in H(\overline{\Omega_r})$, where Ω_r is defined as above and the function $h(z,z)^N D^N g(z) \in L^1(d\lambda) \cap L^\infty(d\lambda) \hookrightarrow E(d\lambda)$; at this point, if we proceed as in the proof of Theorem 7 in [11], we obtain

$$\|h(z,z)^N D^N g(z)\|_{E(d\lambda)} \leq \|h(z,z)^N D^N g(z)\|_{L^1(d\lambda)\cap L^\infty(d\lambda)} \leq$$
$$\leq \sup\{|D^N g(\frac{z}{r})|,\ z \in \Omega\}.$$

Since the function $D^N g$ is holomorphic in Ω, its homogeneous expansion converges uniformly and absolutely on Ω_r; so, if we replace g by its remainder in its homogeneous expansion in the above estimation, we find a polynomial $p \in B_E(\Omega)$ that approximates g in the norm of $B_E(\Omega)$ with an error $< \varepsilon/2$; putting all this together, we obtain $\|f - p\|_{B_E(\Omega)} \leq \|f - g\|_{B_E(\Omega)} + \|g - p\|_{B_E(\Omega)} < \varepsilon$ ending the proof.

Lemma 3. (see [11], Lemma 16). *Let $E(d\lambda)$ be a r.i. with absolute continuous norm. Then, for all $f \in E(d\lambda)$ and $g \in E'(d\lambda)$, the following proprieties hold:*

1) $V_\alpha^2 f = V_\alpha f$

2) $V_\alpha P_\alpha f = V_\alpha f$

3) $P_\alpha V_\alpha f = P_\alpha f$

4) $\int_\Omega V_\alpha f(z)\overline{g(z)}\, d\lambda(z) = \int_\Omega f(z)\overline{V_\alpha g(z)}\, d\lambda(z).$

Proof. First observe that the operators V_α, V_α^2 and $V_\alpha P_\alpha$ are all bounded on $E(d\lambda)$. Since, by hypothesis, $E(d\lambda)$ has absolute continuous norm, it will coincide with the closure of all bounded functions that are compactly supported on Ω. This means that it will suffice to check the proprieties 1-4 only for such functions f. But this requires straightforward computations that involves the reproducing proprieties of P_α, D_α and Fubini's theorem.

Theorem 4. *If $E(d\lambda)$ has absolute continuous norm, then with respect to the pairing*

$$< f, g >_\alpha = \int_\Omega V_\alpha f(z)\overline{V_\alpha g(z)}\, d\lambda(z),$$

we have $B_E^*(\Omega) = B_{E'(\Omega)}$.

Proof. Let $g \in B_{E'(\Omega)}$ and consider the functional

$$L_g(f) = \int_\Omega V_\alpha f(z)\overline{V_\alpha g(z)}\, d\lambda(z);$$

then clearly L_g is a bounded linear functional on $E(d\lambda)$, as, by Holder's inequality, we have

$$|L_g(f)| \le \|V_\alpha f\|_{E(d\lambda)} \|V_\alpha g\|_{E'(d\lambda)} = \|f\|_{B_E(\Omega)} \|g\|_{B_{E'}(\Omega)};$$

this prove that $B_{E'}(\Omega) \hookrightarrow B_E(\Omega)$, continuously.

Let now F be an arbitrary bounded functional on $B_E(\Omega)$; we have to prove that there exists a function $g \in E'(d\lambda)$ such that $F = L_g$. As V_α maps $B_E(\Omega)$ boundedly to $E(d\lambda)$, we may extend the functional $F \circ V_\alpha^{-1} : V_\alpha(B_E(\Omega)) \to C$, by using Hahn-Banach theorem, up to a bounded linear functional on $E(d\lambda)$, denoted by $\widetilde{F \circ V_\alpha^{-1}}$. Since $E(d\lambda)$ has absolute continuous norm, its dual $E(d\lambda)^*$ is canonically isomorphic with its associate space $E'(d\lambda)$, so there exists a function $\psi \in E'(d\lambda)$, such that

$$\widetilde{F \circ V_\alpha^{-1}}(h) = \int_\Omega h(z)\overline{\psi(z)} d\lambda(z), \quad \forall h \in E(d\lambda);$$

consequently,

$$F(f) = \widetilde{F \circ V_\alpha^{-1}}(V_\alpha f) = \int_\Omega V_\alpha f(z)\overline{\psi(z)}\, d\lambda(z), \quad \forall f \in B_E(\Omega).$$

By Lemma 3

$$\int_\Omega V_\alpha f(z)\overline{\psi(z)}\, d\lambda(z) = \int_\Omega V_\alpha^2 f(z)\overline{\psi(z)}\, d\lambda(z) =$$

$$= \int_\Omega V_\alpha f(z)\overline{V_\alpha \psi(z)}\, d\lambda(z), \quad f \in B_E(\Omega).$$

So if we take $g = V_\alpha \psi$, then we obtain $g \in B_{E'}(\Omega)$ and $F(f) = L_g(f)$, $f \in B_E(\Omega)$.

We shall prove now an interpolation theorems for Besov spaces, which shall be the key tool for the study of Schatten ideals of little Hankel operators.

Theorem 5. *If ρ be a monotone Riesz - Fischer norm,* $1 \leq p_0 < p_1 \leq \infty$ *and* $E(d\lambda) = (L^{p_0}(d\lambda), L_{p_1}(d\lambda))_\rho$, *then* $(B_{p_0}(\Omega), B_{p_1}(\Omega))_\rho = B_E(\Omega)$.

Proof. Consider again the linear operator $V_N f(z) = h(z,z)^N D^N f(z)$; since V_N is bounded from $B_{p_i}(\Omega)$ to $L^{p_i}(\Omega)$, $i = 0, 1$ by interpolation, V_N maps boundedly $(B_{p_0}(\Omega), B_{p_1}(\Omega))_\rho$ to $E(d\lambda)$; hence for all $f \in (B_{p_0}(\Omega), B_{p_1}(\Omega))_\rho$, we have

$$\|f\|_{B_E(\Omega)} = \|h^N(z,z) D^N f(z)\|_{E(d\lambda)} \leq \|f\|_{(B_{p_0}(\Omega), B_{p_1}(\Omega))_\rho},$$

proving the continuous inclusion $(B_{p_0}(\Omega), B_{p_1}(\Omega))_\rho \hookrightarrow B_E(\Omega)$.

Now assume that $f \in B_E(\Omega)$ and let $f = P\varphi$, where $\varphi \in E(d\lambda)$; we want to proof that $f \in (B_{p_0}(\Omega), B_{p_1}(\Omega))_\rho$.

By the definition of the Besov spaces, the Bergman projection maps boundedly $L^{p_i}(d\lambda)$ onto $B_{p_i}(\Omega)$, $i = 0, 1$ hence, by interpolation, we deduce that P maps boundedly $E(d\lambda)$ to $(B_{p_0}(\Omega), B_{p_1}(\Omega))_\rho$. So, if $f \in B_E(\Omega)$ and $\varphi \in E(d\lambda)$ is such that $P\varphi = f$, then $\|f\|_{(B_{p_0}(\Omega), B_{P_1}(\Omega))_\rho} \leq \|\varphi\|_{E(d\lambda)}$; finally, we get $\|f\|_{(B_{p_0}(\Omega), B_{P_1}(\Omega))_\rho} \leq \|f\|_{B_E(\Omega)}$, proving that $B_E(\Omega) \hookrightarrow (B_{p_0}(\Omega), B_{P_1}(\Omega))_\rho$ continuously.

We may prove a Boyd type theorem for Besov spaces. We say that a quasilinear operator T is of weak type (p, q) if T is bounded from $B_{p,1}$ to $B_{p,\infty}$, where we denoted by $B_{p,q}$ the space $B_{L^{p,q}(d\lambda)}(\Omega)$.

Proposition 1. *Let* $E(d\lambda)$ *be a r.i. over* $(\Omega, d\lambda)$, *and let* p *and* q *be such that* $1 \leq p < p_E \leq q_E < q < \infty$, *where* p_E *and* q_E *are the Boyd indices of* $E(d\lambda)$. *Then any operator* T *that is of weak type* (p, q) *and* (q, q) *is bounded on* $B_E(\Omega)$.

Proof. By Theorem 1, the linear operator $V_N f(z) = h(z,z)^N D^N f(z)$ is bounded from $B_E(\Omega)$ to $E(d\Lambda)$, so the quasilinear operator $V_N \circ T \circ P$ is of (classic) weak type (p, p) and (q, q) so, according to Boyd's theorem (see [3], chap III), it is bounded on $E(d\lambda)$, too and there exists a constant $C > 0$ such that $\|V_N \circ T \circ P\varphi\|_{E(d\lambda)} \leq C\|\varphi\|_{E(d\lambda)}$ $\varphi \in E(d\lambda)$. Now fix $f \in B_E(\Omega)$; then there exists $\varphi \in E(d\lambda)$ such that $f = P\varphi$, so $\|V_N \circ Tf\|_{E(d\lambda)} \leq C\|\varphi\|_{E(d\lambda)}$; passing at the infimum over all $\varphi \in E(d\Lambda)$ such that $f = P\varphi$, we get $\|V_N \circ Tf\|_{E(d\lambda)} \leq C\|f\|_{B_E(\Omega)}$, for all $f \in B_E(\Omega)$. Since the function Tf is analytic, the latter relation implies that $Tf \in B_E(\Omega)$ and $\|Tf\|_{B_E(\Omega)} \leq C\|f\|$, for all $f \in B_E(\Omega)$, proving that T is bounded on $B_E(\Omega)$.

4. Schatten ideals of Hankel operators.

Let E be a monotone Riesz-Fischer sequences space and denote by

$$S_E = \{T : L_a^2(dv_\alpha) \to \overline{L_a^2(dv_\alpha)},\ T \text{ compact}:$$

$$\|T\|_{S_e} = \|\{s_n(T)\}_{n\in N}\|_E < \infty\}$$

the Schatten ideal associated with the r.i. sequence space E where

$$s_n(T) = \inf\{\|T - R\|,\ \text{rank}R \le n\}$$

is the n^{th} singular number of the compact operator T; $\{s_n\}_{n\in N}$ form a decreasing sequence, that coincides with the decreasing rearrangement of the eigenvalues of the compact and positive operator $(TT^*)^{1/2}$; when $E = l^p$, we shall denote by S_p the classic Schatten ideal Slp. The interpolation proprieties of the Calderon couple $S_{p_0}, S_{p_1})$ were investigated in [1]. We shall remind this useful result.

Lemma 4. (see [1], Theorem 2). *If ρ is a monotone Riesz-Fischer norm, $1 \le p_0 \le p_1 \le \infty$, then $(S_{p_0}, S_{p_1})_\rho = S_{(lp_0,\ lp_1)_\rho}$.*

Now we shall reformulate a result which is implicitly proved in [11].

Lemma 5. (see [11], Lemma 21). *With the notations above, if $1 \le p_0 \le p_1 \le \infty$, and if $f \in E(d\lambda)$, then the little Hankel operator $h_f \in S_E$.*

Proof. Just use Lemma 2 in [11] and Lemma 4.

At this point, if we proceed as in the proof of Theorem 22 in [11] and use the previous lemma, we can easily deduce the following.

Lemma 6. *Let $E(d\lambda)$ be a r.i. with absolute continuous norm; then for all $f \in E(d\lambda)$ and $g \in E'(d\lambda)$, the operator $h_{\overline{f}} h_{\overline{g}}$ is in the trace class S_1 and*

$$Tr(h_{\overline{f}} h_{\overline{g}}) = \int_\Omega V_\alpha f(z)\overline{V_\alpha g(z)}\, d\lambda(z).$$

Finally, we may prove

Theorem 6. *Let $\alpha > -1$, $f \in L^2(\Omega, dv_\alpha)$ and $E(d\lambda)$ a r.i. with absolute continuous norm; then we have $h_{\overline{f}} \in S_E$ if and only if $V_\alpha f \in E(d\lambda)$.*

Proof. By [11], Lemma 20, we have $h_{\overline{V_\alpha f}} = h_{\overline{f}}$, for all $f \in L^2(\Omega, dv_\alpha)$, hence, since by hypothesis $V_\alpha f \in E(d\lambda)$, by Lemma 5, it follows that $h_{\overline{f}} \in S_E$.

For the converse, if $f \in L^2(\Omega, dv_\alpha)$ and $h_{\overline{f}} \in S_E$, then using Lemma 6, the boundedness from $E'(d\lambda)$ to $S_{E'}$ of the operator $g \to h_{\overline{g}}$ and a duality argument, we obtain

$$\|V_\alpha f\|_{E(d\lambda)} = \sup\left\{\left|\int_\Omega V_\alpha f(w)\overline{g(w)}\,d\lambda(w)\right|, \quad \|g\|_{E'(d\lambda)} \leq 1\right\} =$$

$$\sup\{Tr(h_{\overline{f}} h_{\overline{g}}), \quad \|g\|_{E'(d\lambda)} \leq 1\} < \infty.$$

Corollary 2. *Under the same hypothesis of Theorem 6, if f is holomorphic on Ω, then we have $h_{\overline{f}} \in S_E$ if and only if $f \in B_E$.*

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