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ON THE RADIAL SOLUTIONS OF A NONLINEAR MATUKUMA-TYPE EQUATION WITH DOUBLE SINGULAR TERMS

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This paper is concerned with the positive solutions of a Matukuma-type nonlinear equation with double singular terms,

$$\Delta_p u + |x|^{m_1} u^{\delta_1} + |x|^{m_2} u^{\delta_2} = 0, \ x \in \mathbb{R}^N,$$

where $p > 2$, $N \geq 1$, $\delta_2 > \delta_1 \geq 1$, $-p < m_2 < m_1 \leq 0$ and $-N < m_2 < m_1 \leq 0$.

Our objective is to generalize a Matukuma-type equation since its importance in both geometry and physics. In this context, we prove the existence of singular solutions and we present their explicit behavior near the origin.

1. Introduction

We are considering the next elliptic equation involving the p-Laplace operator

$$\Delta_p u + |x|^{m_1} u^{\delta_1} + |x|^{m_2} u^{\delta_2} = 0, \ \ x \in \mathbb{R}^N, \tag{1}$$

where $p > 2$, $N \geq 1$, $\delta_2 > \delta_1 \geq 1$, $-p < m_2 < m_1 \leq 0$ and $-N < m_2 < m_1 \leq 0$.

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We restrict our considerations to radial solutions of the previous equation. That is, we will study the subsequent equation

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + r^{m_1}u^{\delta_1} + r^{m_2}u^{\delta_2} = 0,\ r > 0, \tag{2}$$

where $p > 2$, $N \geq 1$, $\delta_2 > \delta_1 \geq 1$, $-p < m_2 < m_1 \leq 0$ and $-N < m_2 < m_1 \leq 0$.

The study of elliptic equations having singular coefficients has been the subject of recent research by many authors. In the case $p = 2$, Lai, Luo and Zhou [14], studied the next equation

$$u''(r) + \frac{N-1}{r}u'(r) + r^{m_1}u^{\delta_1} + r^{m_2}u^{\delta_2} = 0,\ \ r > 0, \tag{3}$$

where $N \geq 3$, $1 < \delta_1 < \delta_2$ and $-2 < m_2 < m_1 \leq 0$.
They gave the behavior near the origin and infinity of positive solutions. More precisely, they showed that if $\dfrac{N+m_1}{N-2} < \delta_1 < \delta_2$, $\delta_1 \neq \dfrac{N+2+2m_1}{N-2}$ and $\delta_2 \neq \dfrac{N+2+2m_2}{N-2}$, then $\lim\limits_{r\to 0} r^{\frac{m_2+2}{\delta_2-1}}u(r)$ and $\lim\limits_{r\to+\infty} r^{\frac{m_1+2}{\delta_1-1}}u(r)$ always exist.

If $m_1 = m_2$ and $\delta_1 = \delta_2$, rescaling, equation (3) is reduced to

$$u''(r) + \frac{N-1}{r}u'(r) + r^{m_1}u^{\delta_1} = 0. \tag{4}$$

Equation (4) has been extensively discussed in the literature, originating from the realms of both physics and mathematics, particularly in the field of astrophysics. It represents a generalization of Matukuma's equation, which was introduced in 1930 to describe the dynamics of globular clusters of stars [17], to delve deeper into this specific equation type, readers are encouraged to consult the following research studies [10, 13, 16, 18]. If $m_1 = 0$, the equation (4) is known in astrophysics as the Emden-Fowler equation, as shown in [11]. Moreover, in geometry, if $N \geq 3$ and $\delta_1 = \dfrac{N+2}{N-2}$, (4) is recognized as the conformal scalar curvature equation. The first existence results of equation (4) are due to Ni [19] in 1982, he proved that (4) has an infinite number of positive solutions, all of which are bounded by positive constants from below. Li and Ni [15] showed that, since u is a positive solution of (4), then $\lim\limits_{r\to+\infty} u(r)$ still exists. The behavior of positive solutions of (4) was described by Li [17]. Interesting results about this equation can be found in [1, 7, 8, 12, 19–21].

We observe that if $m_1 = m_2 = 0$, equation (3) reduces to:

$$u''(r) + \frac{N-1}{r}u'(r) + u^{\delta_1}(r) + u^{\delta_2}(r) = 0, \quad r > 0. \tag{5}$$

Equation (5) was recently investigated by Bamon, Flores, and Del Pino [2], under the assumption that the exponents δ_1 and δ_2 are subcritical and supercritical, respectively, i.e., $1 < \delta_1 < \frac{N+2}{N-2} < \delta_2$. They demonstrated that when δ_2 is fixed and δ_1 approaches the critical value $\frac{N+2}{N-2}$, equation (5) admits multiple radial solutions. Similarly, an analogous result holds when $\delta_1 > \frac{N}{N-2}$ and δ_2 approaches $\frac{N+2}{N-2}$. Furthermore, they proved that equation (5) has no solutions if δ_1 approaches $\frac{N}{N-2}$ while δ_2 remains fixed.

For the general case $p > 2$, Bouzelmate and Gmira [5, 6] established that equation (2) admits an explicit positive solution under the conditions $N > p$, $\delta > \frac{N(p-1)}{N-p}$, and either $m_1 = 0$ with $\delta_2 = 0$ or $m_2 = 0$ with $\delta_1 = 0$. Furthermore, they provided significant results regarding the existence, nonexistence, and asymptotic behavior of singular solutions to equation (2) near the origin, specifically those solutions satisfying $\lim_{r\to 0} u(r) = +\infty$.

Equation (2) has been studied in [9] with $\lim_{r\to 0} u(r) = a \in (0,+\infty)$. We established that $\lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r) = d$ for some $d \in \mathbb{R}$ and so we analyzed the problem

$$(P_a)\begin{cases} \left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + r^{m_1}u^{\delta_1} + r^{m_2}u^{\delta_2} = 0, \quad r > 0 \\ \lim_{r\to 0} u(r) = a, \ \lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r) = d, \end{cases}$$

where $p > 2$, $N \geq 1$, $\delta_2 > \delta_1 \geq 1$, $-p < m_2 < m_1 \leq 0$, $-N < m_2 < m_1 \leq 0$, $a > 0$ and $d \in \mathbb{R}$. In that work, we demonstrated the existence of a regular solution of problem (P_a), while avoiding the challenging case of singular solutions at the origin.

The primary objective of the current study is to extend the results obtained by Lai, Luo, and Zhou [14] to the case $p > 2$, with a particular focus on the more difficult scenario where a singularity at the origin is present. Specifically, we investigate the existence and asymptotic behavior of singular positive solutions near the origin for equation (2) under the condition $\lim_{r\to 0} u(r) = +\infty$. More precisely, we prove that for any solution u of (2) satisfying $\lim_{r\to 0} u(r) = +\infty$, the limit $\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) = 0$ holds when $N > p$. Consequently, our aim is to investigate a solution u defined on the interval $(0,+\infty)$, such that $u \in C^1(0,+\infty)$ and $|u'|^{p-2}u' \in C^1(0,+\infty)$ and satisfies

$$(P)\begin{cases} \left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + r^{m_1}u^{\delta_1} + r^{m_2}u^{\delta_2} = 0, \quad r > 0 \\ \lim_{r\to 0} u(r) = +\infty, \ \lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) = 0, \end{cases}$$

where $N > p > 2$, $\delta_2 > \delta_1 \geq 1$ and $-p < m_2 < m_1 \leq 0$.

The structure of this paper is organized as follows. In Section 2, we present preliminary results related to singular solutions, which lay the foundation for the subsequent analysis. Section 3 focuses on the asymptotic behavior of singular solutions for problem (P) near the origin. In Section 4, we address the existence of singular solutions of problem (P). Finally, in Section 5, we summarize the findings and discuss potential perspectives for future research.

2. Preliminaries and Basic Results

In this section, we present some foundational results.

Proposition 2.1. *Assuming that $N > p$ and u is a solution to (2) under which $\lim_{r\to 0} u(r) = +\infty$. Then u is strictly decreasing on $(0,+\infty)$.*

Proof. Since $\lim_{r\to 0} u(r) = +\infty$ and u is continuous, there exists $\eta > 0$ sufficiently small such that $u(r) > 0$ for all $r \in (0,\eta)$.

Suppose, by contradiction, that u oscillates on $(0,\eta)$, and let r_0 be the first zero of u' in $(0,\eta)$. Substituting into equation (2), we obtain:

$$\left(|u'|^{p-2}u'\right)'(r) = -r^{m_1}u^{\delta_1}(r) - r^{m_2}u^{\delta_2}(r), \tag{6}$$

which implies that $\left(|u'|^{p-2}u'\right)'(r_0) < 0$ and thus $u'(r) < 0$ near $r = 0$.

To show that u is strictly decreasing on $(0,+\infty)$, assume, by contradiction, that there exists $r_1 > 0$ such that $u'(r_1) = 0$. This would imply

$$\left(|u'|^{p-2}u'\right)'(r_1) \geq 0.$$

However, this contradicts equation (2), which gives

$$\left(|u'|^{p-2}u'\right)'(r_1) < 0.$$

Thus, we conclude that $u'(r) < 0$ for all $r > 0$. □

Proposition 2.2. *Assuming that $N > p$ and $\delta_2 > \delta_1 > p-1$. Considering a solution u to (2) under which $\lim_{r\to 0} u(r) = +\infty$, it follows that for any $r > 0$, we have*

$$u(r) \leqslant M(N,p,\delta_2,m_2)\, r^{\frac{-(m_2+p)}{\delta_2+1-p}}, \tag{7}$$

where

$$M(N,p,\delta_2,m_2) = \left[(\delta_2+1-p)\left(\frac{1-2^{-m_2-N}}{m_2+N}\right)^{\frac{1}{p-1}}\left(\frac{1-2^{-\frac{m_2+p}{p-1}}}{m_2+p}\right)\right]^{\frac{-(p-1)}{\delta_2+1-p}}. \tag{8}$$

Proof. Employing (6), we derive the following inequality for all $r > 0$

$$\left(r^{N-1}|u'|^{p-2}u'\right)' < -r^{m_2+N-1}u^{\delta_2}(r). \tag{9}$$

Integrating (9) over the interval $\left(\frac{r}{2}, r\right)$ for any $r > 0$, we obtain

$$r^{N-1}|u'|^{p-2}u'(r) - \left(\frac{r}{2}\right)^{N-1}|u'|^{p-2}u'\left(\frac{r}{2}\right) < -\int_{\frac{r}{2}}^{r} s^{m_2+N-1}u^{\delta_2}(s)\,ds. \tag{10}$$

Given that u is strictly decreasing by Proposition 2.1, it follows that

$$r^{N-1}|u'|^{p-2}u'(r) < -\frac{1-2^{-m_2-N}}{m_2+N}r^{m_2+N}u^{\delta_2}(r). \tag{11}$$

Thus, for any $r > 0$, we find:

$$u'(r) < -\left(\frac{1-2^{-m_2-N}}{m_2+N}\right)^{\frac{1}{p-1}} r^{\frac{m_2+1}{p-1}} u^{\frac{\delta_2}{p-1}}(r). \tag{12}$$

Since u is strictly positive and $\delta_1 > p-1$, we deduce

$$\frac{p-1}{p-\delta_2-1}\left(u^{\frac{p-\delta_2-1}{p-1}}\right)' < -\left(\frac{1-2^{-m_2-N}}{m_2+N}\right)^{\frac{1}{p-1}} r^{\frac{m_2+1}{p-1}}. \tag{13}$$

Consequently

$$\left(u^{\frac{p-\delta_2-1}{p-1}}\right)' > \frac{\delta_2+1-p}{p-1}\left(\frac{1-2^{-m_2-N}}{m_2+N}\right)^{\frac{1}{p-1}} r^{\frac{m_2+1}{p-1}}. \tag{14}$$

Integrating inequality (14) over $\left(\frac{r}{2}, r\right)$ for $r > 0$, we conclude

$$u^{\frac{p-\delta_2-1}{p-1}}(r) > (\delta_2+1-p)\left(\frac{1-2^{-m_2-N}}{m_2+N}\right)^{\frac{1}{p-1}}\left(\frac{1-2^{-\frac{m_2+p}{p-1}}}{m_2+p}\right) r^{\frac{m_2+p}{p-1}}. \tag{15}$$

As a result, inequality (7) is verified. □

Proposition 2.3. *Assuming that $N > p$ and $\delta_2 > \delta_1 > p-1$. Let u be a solution of (2) with $\lim_{r\to 0} u(r) = +\infty$. Then*

$$\lim_{r\to 0} r^{(N-p)/(p-1)}u(r) = 0, \quad \lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0 \tag{16}$$

and

$$\frac{N-p}{p-1}u(r) + ru'(r) > 0 \ \text{ for small } r.$$

Before giving the proof, let us define for all real $\sigma \neq 0$ the following function

$$H_\sigma(r) = \sigma u(r) + ru'(r), \ r > 0. \tag{17}$$

It is evident that for each $r > 0$,

$$(r^\sigma u(r))' = r^{\sigma-1} H_\sigma(r). \tag{18}$$

Therefore, the analysis of the variation of $r^\sigma u(r)$ hinges on the sign of $H_\sigma(r)$. We obtain, by referring to equation (2) and for all $r > 0$ verifies $u'(r) \neq 0$, that

$$\begin{aligned}(p-1)|u'|^{p-2} H'_\sigma(r) = (p - N + \sigma(p-1))\,|u'|^{p-2} u'(r) - r^{m_1+1} u^{\delta_1} \\ - r^{m_2+1} u^{\delta_2}.\end{aligned} \tag{19}$$

If we have $H_\sigma(\rho) = 0$ for a certain $\rho > 0$, then we can deduce that

$$\begin{aligned}(p-1)|u'|^{p-2} H'_\sigma(r_0) = (N - p - \sigma(p-1))\,|\sigma|^{p-2}\sigma\, r_0^{1-p} |u|^{p-2} u(r_0) \\ - r_0^{m_1+1} u^{\delta_1}(r_0) - r_0^{m_2+1} u^{\delta_2}(r_0).\end{aligned} \tag{20}$$

Proof. Using 2.2 we have $r^{(m_2+p)/(\delta_2-p+1)}$ is bounded near the origin and since $\delta_2 > \dfrac{(m_2+N)(p-1)}{N-p}$ then $\dfrac{m_2+p}{\delta_2+1-p} < \dfrac{N-p}{p-1}$, hence we obtain $\lim\limits_{r\to 0} r^{(N-p)/(p-1)} u(r) = 0$. Now, by relation (6), we get that $\lim\limits_{r\to 0} r^{(N-1)/(p-1)} u'(r)$ exists and by the Hôspital's rule we have $\lim\limits_{r\to 0} r^{(N-1)/(p-1)} u'(r) = 0$ (because $\lim\limits_{r\to 0} u(r) = +\infty$ and $N > p$). Next, based on equation (19), we have $H'_{\frac{N-p}{p-1}}(r) < 0$ for small r, since $u > 0$ and $u' < 0$ near the origin. Consequently, $H_{\frac{N-p}{p-1}}(r) \neq 0$ for small r. Given that $\lim\limits_{r\to 0} r^{(N-p)/(p-1)} u(r) = 0$, it is necessarily the case that $H_{\frac{N-p}{p-1}}(r) > 0$ for small r, that is $\dfrac{N-p}{p-1} u(r) + ru'(r) > 0$ for small r. □

Thanks to the previous results, we are able to examine the problem (P).

Proposition 2.4. *Let $\delta_2 > \delta_1 > p-1$ and consider a solution u to problem (P). If $r^\sigma u(r)$ is bounded for small values of r and for some $\sigma > 0$, then $r^{\sigma+1} u'(r)$ is also bounded for small values of r.*

Proof. Using Proposition 2.3, we know that $H_{\frac{N-p}{p-1}}(r) > 0$ for sufficiently small r. Since u is strictly decreasing, it follows that:

$$r|u'(r)| < \frac{N-p}{p-1} u(r) \quad \text{for small } r.$$

Given that $r^\sigma u(r)$ is bounded for small r, it immediately follows that $r^{\sigma+1} u'(r)$ is also bounded as r approaches zero. □

We now adopt the logarithmic transformation introduced in [3, 6], which serves as a crucial tool in proving the subsequent theorems. For every $r > 0$, we define

$$z_\sigma(t) = r^\sigma u(r), \text{ where } t = -ln(r). \tag{21}$$

So z_σ verifies

$$Q'_\sigma(t) - \alpha_\sigma Q_\sigma(t) + e^{-\beta_\sigma t} z_\sigma^{\delta_1}(t) + e^{-\theta_\sigma t} z_\sigma^{\delta_2}(t) = 0, \tag{22}$$

with

$$l_\sigma(t) = z'_\sigma(t) + \sigma z_\sigma(t), \tag{23}$$

$$Q_\sigma(t) = |l_\sigma|^{p-2} l_\sigma(t), \tag{24}$$

$$\alpha_\sigma = \alpha_{\sigma,N,p} = N - p - \sigma(p-1), \tag{25}$$

$$\beta_\sigma = (m_1 + p) - \sigma(\delta_1 + 1 - p) \tag{26}$$

and

$$\theta_\sigma = (m_2 + p) - \sigma(\delta_2 + 1 - p). \tag{27}$$

We remark that

$$l_\sigma(t) = -r^{\sigma+1} u'(r). \tag{28}$$

Proposition 2.5. *Let* $\delta_2 > \delta_1 > \dfrac{(m_1+N)(p-1)}{N-p}$ *and consider a solution* u *to* (P). *If* $r^\sigma u(r)$ *with* $0 < \sigma \leq \dfrac{m_2+p}{\delta_2+1-p}$ *converges as* $r \to 0$, *then the function* $r^{\sigma+1}u'(r)$ *also converges as* $r \to 0$.

Proof. Using the transformation (21) with $0 < \sigma \leq \dfrac{m_2+p}{\delta_2+1-p}$, the function $z_\sigma(t) = r^\sigma u(r)$ is bounded for sufficiently small r. From Proposition 2.4, it follows that $l_\sigma(t) = r^{\sigma+1}u'(r)$ is bounded for sufficiently large t. Consequently, $Q_\sigma(t) = |l_\sigma(t)|^{p-2} l_\sigma(t)$ is also bounded when t is large.

Suppose, for contradiction, that there exist two sequences $\{k_j\}$ and $\{s_j\}$ such that both k_j and s_j approach $+\infty$ as $i \to +\infty$, where k_j corresponds to a local minimum and s_j to a local maximum of $Q_\sigma(t)$. Assume further that $k_j < s_j < k_{i+1}$ for all i, and that the sequences satisfy:

$$0 \leq \liminf_{t\to+\infty} Q_\sigma(t) = \lim_{i\to+\infty} Q_\sigma(k_j) < \lim_{i\to+\infty} Q_\sigma(s_j) = \limsup_{t\to+\infty} Q_\sigma(t) < +\infty, \tag{29}$$

and

$$Q'_\sigma(k_j) = Q'_\sigma(s_j) = 0. \tag{30}$$

From equation (22), we have

$$\begin{aligned} &-\alpha_\sigma Q_\sigma(k_j)+e^{-\theta_\sigma k_j}z_\sigma^{\delta_2}(k_j)+e^{-\beta_\sigma k_j}z_\sigma^{\delta_1}(k_j)=\\ &-\alpha_\sigma Q_\sigma(s_j)+e^{-\theta_\sigma s_j}z_\sigma^{\delta_2}(s_j)+e^{-\beta_\sigma s_j}z_\sigma^{\delta_1}(s_j). \end{aligned}$$

Since z_σ converges, and given that $\alpha_\sigma>0$ and $\beta_\sigma>0$ (due to $0<\sigma\leq \dfrac{m_2+p}{\delta_2+1-p}$, $\delta_2>\delta_1>\frac{(m_1+N)(p-1)}{N-p}$ and $m_2<m_1$), we deduce

$$\lim_{i\to+\infty}Q_\sigma(k_j)=\lim_{i\to+\infty}Q_\sigma(s_j). \tag{31}$$

This contradicts equation (29). Therefore, $Q_\sigma(t)$ converges as $t\to+\infty$. Consequently, $r^{\sigma+1}u'(r)$ converges as $r\to+\infty$ with $0<\sigma\leq\dfrac{m_2+p}{\delta_2+1-p}$. □

Proposition 2.6. *Assuming that Let $\delta_2>\delta_1>\frac{(m_1+N)(p-1)}{N-p}$. Suppose that u is a solution of (P) satisfying*

$$\lim_{r\to0}r^{\frac{m_2+p}{\delta_2+1-p}}u(r)=0.$$

Suppose there exists $0<\sigma_0<\frac{m_2+p}{\delta_2+1-p}$ so that

$$\lim_{r\to0}r^{\sigma_0}u(r)=+\infty,$$

then

$$H_{(m_2+p)/(\delta_2+1-p)}(r)>0 \quad \textit{and} \quad H_{\sigma_0}(r)<0 \quad \textit{for small } r.$$

Proof. Since

$$\lim_{r\to0}r^{\frac{m_2+p}{\delta_2+1-p}}u(r)=0 \quad \text{and} \quad \lim_{r\to0}r^{\sigma_0}u(r)=+\infty,$$

it suffices to show that $H_{(m_2+p)/(\delta_2+1-p)}(r)\neq0$ and $H_{\sigma_2}(r)\neq0$ for small r.

Step 1. $H_{(m_2+p)/(\delta_2+1-p)}(r)\neq0$ for small r.
Suppose there exists a small r such that $H_{(m_2+p)/(\delta_2+1-p)}(r)=0$. Taking $\sigma=\frac{m_2+p}{\delta_2+1-p}$ in (20) and multiplying by $r^{\sigma_0(p-1)}$ we obtain

$$\begin{aligned} &(p-1)r^{(\sigma_0+1)(p-1)}|u'|^{p-2}H'_{(m_2+p)/(\delta_2+1-p)}(r)=\\ &r^{\sigma_0(p-1)}u^{p-1}\left(N-p-\frac{m_2+p}{\delta_2+1-p}(p-1)\right)\left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}\\ &-r^{\sigma_0(p-1)}u^{p-1}\left\{r^{m_1+p}u^{\delta_1+1-p}+r^{m_2+p}u^{\delta_2+1-p}\right\}. \end{aligned}$$

Since $\lim_{r\to 0} r^{\frac{m_2+p}{\delta_2+1-p}} u(r) = \lim_{r\to 0} r^{\frac{m_1+p}{\delta_1+1-p}} u(r) = 0$ and $\lim_{r\to 0} r^{\sigma_0} u(r) = +\infty$, it follows that

$$H'_{(m_2+p)/(\delta_2+1-p)}(r) > 0 \quad \text{for small } r.$$

That is $H_{(m_2+p)/(\delta_2+1-p)}(r) \neq 0$ for small r.
Step 2. $H_{\sigma_0}(r) \neq 0$ for small r.
In the same way as Step 1, assume there exists a small r such that $H_{\sigma_2}(r) = 0$. According to (20) with $\sigma = \sigma_0$, we have

$$(p-1)r^{(p-1)}|u'|^{p-2}H'_{\sigma_0}(r) = \\ u^{p-1}\left\{(N-p-\sigma_0(p-1))\,(\sigma_0)^{p-1} - r^{m_1+p}u^{\delta_1+1-p} - r^{m_2+p}u^{\delta_2+1-p}\right\}.$$

Multiplying this equality by $r^{\sigma_0(p-1)}$, we obtain

$$(p-1)r^{(\sigma_0+1)(p-1)}|u'|^{p-2}H'_{\sigma_0}(r) = \\ r^{\sigma_0(p-1)}u^{p-1}\left\{(N-p-\sigma_0(p-1))\,(\sigma_0)^{p-1} - r^{m_1+p}u^{\delta_1+1-p} - r^{m_2+p}u^{\delta_2+1-p}\right\}.$$

Taking into account our hypothesis and the fact that $0 < \sigma_0 < \dfrac{m_2+p}{\delta_2+1-p} < \dfrac{N-p}{p-1}$, we deduce that $H'_{\sigma_0}(r) > 0$. Consequently, we have $H_{\sigma_0}(r) \neq 0$ for small r. □

We now present the following lemma, which is a classical result due to Gidas and Spruck.

Lemma 2.7 ([12]). *Let F be a positive differentiable function satisfying the following conditions:*

i) $\int_{t_0}^{+\infty} F(t)\,dt < +\infty$ *for some* $t_0 > 0$,

ii) $F'(t)$ *is bounded for sufficiently large* t.

Then

$$\lim_{t\to+\infty} F(t) = 0.$$

3. Behavior of Singular Solution near the origin

This section explores the asymptotic behavior near the origin of singular solutions to the problem (P). To conduct this analysis, we utilize concepts and results found in the works [3, 4, 6].

Theorem 3.1. *Let* $\delta_2 > \delta_1 > \frac{(m_1+N)(p-1)}{N-p}$. *Suppose that* u *is a solution of* (P). *If* $\delta_2 \neq \frac{N(p-1)+p(m_2+1)}{N-p}$, *then* u *exhibits one of the following behaviors near the origin*
(i)

$$\lim_{r\to 0} r^{\frac{m_2+p}{\delta_2+1-p}} u(r) = \left(\left(N-p-\frac{m_2+p}{\delta_2+1-p}(p-1)\right)\left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}\right)^{\frac{1}{\delta_2+1-p}}. \tag{32}$$

(ii)

$$\lim_{r\to 0} r^{\sigma_0} u(r) = C, \text{ where } C > 0 \text{ and } 0 < \sigma_0 < \frac{m_2+p}{\delta_2+1-p}. \tag{33}$$

Proof. As a consequence of the transformation in (21) with $\sigma = \dfrac{m_2+p}{\delta_2+1-p}$, we have $z_\sigma(t) = r^{\frac{m_2+p}{\delta_2+1-p}} u(r)$, so the function z_σ satisfies the present equation

$$Q'_\sigma(t) - \alpha_\sigma Q_\sigma(t) + e^{-\beta_\sigma t} z_\sigma^{\delta_1}(t) + z_\sigma^{\delta_2}(t) = 0, \tag{34}$$

where

$$\alpha_\sigma = N-p-\frac{m_2+p}{\delta_2+1-p}(p-1) \tag{35}$$

and

$$\beta_\sigma = (m_1+p) - \frac{m_2+p}{\delta_2+1-p}(\delta_1+1-p). \tag{36}$$

Next we define the following energy function related to (34)

$$I_\sigma(t) = \frac{p-1}{p}|l_\sigma(t)|^p - \alpha_\sigma Q_\sigma(t) z_\sigma(t) \tag{37}$$

$$+ \frac{\delta_2}{\delta_2+1}\left(\alpha_\sigma - \frac{m_2+p}{\delta_2+1-p}\right)\alpha_\sigma^{\frac{1}{\delta_2}}|Q_\sigma(t)|^{\frac{\delta_2+1}{\delta_2}}$$

$$+ \frac{z_\sigma^{\delta_2+1}}{\delta_2+1}. \tag{38}$$

Since $z_\sigma(t)$, $l_\sigma(t)$, and $Q_\sigma(t)$ are bounded for sufficiently large t, it follows that $I_\sigma(t)$ is bounded as $t \to \infty$.

Additionally, we have the following expression for the derivative of $I_\sigma(t)$:

$$I'_\sigma(t) = \left(\frac{m_2+p}{\delta_2+1-p} - \alpha_\sigma\right) Y_\sigma(t) - e^{-\beta_\sigma t} z_\sigma^{\delta_1}(t) z'_\sigma(t) \tag{39}$$

$$- \left(\frac{m_2+p}{\delta_2+1-p} - \alpha_\sigma\right) e^{-\beta_\sigma t} z_\sigma^{\delta_1}(t)\left(\alpha_\sigma^{\frac{1}{\delta_2}}|Q_\sigma|^{\frac{1}{\delta_2}} - z_\sigma(t)\right),$$

where

$$Y_\sigma(t) = \left(z_\sigma(t) - \alpha_\sigma^{\frac{1}{\delta_2}} |Q_\sigma(t)|^{\frac{1}{\delta_2}}\right)\left(z_\sigma^{\delta_2} - \alpha_\sigma |Q_\sigma(t)|\right). \tag{40}$$

We will divide the remainder of the proof into three distinct steps.

Step 1. $I_\sigma(t)$ is convergent as $t \to +\infty$.
By integrating (39) over the interval (T,t) for sufficiently large T, we obtain:

$$\begin{aligned} I_\sigma(t) = I_\sigma(T) &+ \left(\frac{m_2+p}{\delta_2+1-p} - \alpha_\sigma\right) S_\sigma(t) \qquad (41)\\ &- \int_T^t e^{-\beta_\sigma s} z_\sigma^{\delta_1}(s) z'_\sigma(s)\, ds \\ &- \left(\frac{m_2+p}{\delta_2+1-p} - \alpha_\sigma\right) \int_T^t e^{-\beta_\sigma s} z_\sigma^{\delta_1}(s) \left(\alpha_\sigma^{\frac{1}{\delta_2}} |Q_\sigma(s)|^{\frac{1}{\delta_2}} - z_\sigma(s)\right) ds, \qquad (42) \end{aligned}$$

where

$$S_\sigma(t) = \int_T^t Y_\sigma(s)\, ds. \tag{43}$$

Since the function $s \to s^{\delta_2}$ is increasing, $Y_\sigma(t)$ is positive, and therefore $S_\sigma(t)$ is positive and increasing. We now demonstrate that $S_\sigma(t)$ is bounded as $t \to \infty$. Given that $\delta_2 \neq \frac{N(p-1)+p(m_2+1)}{N-p}$, we have $\alpha_\sigma - \frac{m_2+p}{\delta_2+1-p} \neq 0$, and thus

$$\begin{aligned} S_\sigma(t) = &\frac{1}{\frac{m_2+p}{\delta_2+1-p} - \alpha_\sigma} \left(I_\sigma(t) - I_\sigma(T)\right) + \\ &\frac{1}{\frac{m_2+p}{\delta_2+1-p} - \alpha_\sigma} \left(\frac{e^{-\beta_\sigma t}}{\delta_1+1} z_\sigma^{\delta_1+1} - \frac{e^{-\beta_\sigma T}}{\delta_1+1} z_\sigma^{\delta_1+1} + \frac{\beta_\sigma}{\delta_1+1} \int_T^t e^{-\beta_\sigma s} z_\sigma^{\delta_1+1}(s) ds\right) \\ &+ \int_T^t e^{-\beta_\sigma s} v^{\delta_1} \left(\alpha_\sigma^{\frac{1}{\delta_2}} |Q_\sigma(s)|^{\frac{1}{\delta_2}} - z_\sigma(s)\right) ds. \end{aligned} \tag{44}$$

Recall that $z_\sigma(t)$, $Q_\sigma(t)$, and $I_\sigma(t)$ are bounded as $t \to \infty$, and $\theta_\sigma > 0$. Hence, $S_\sigma(t)$ is bounded as t becomes large. Consequently, $S_\sigma(t)$ is convergent as $t \to +\infty$. Therefore, $I_\sigma(t)$ converges as $t \to +\infty$.

Step 2. $\lim_{t \to +\infty} Q'_\sigma(t) = 0.$

Note that for any $1 < \iota \leq 2$, there exists $C_\iota > 0$ such that

$$\left(|\kappa_1|^{\delta-2}\kappa_1 - |\kappa_2|^{\delta-2}\kappa_2\right)(\kappa_1-\kappa_2) \geq C_\iota(\kappa_1-\kappa_2)^2\left(|\kappa_1|+|\kappa_2|\right)^{\delta-2} \tag{45}$$

for any $\kappa_1, \kappa_2 \in \mathbb{R}$ such that $|\kappa_1|+|\kappa_2|>0$. In particular, for $\iota = \frac{\delta_2+1}{\delta_2}$, there exists $C_\iota > 0$ such that

$$\left(z_\sigma(t) - \alpha_\sigma^{\frac{1}{\delta_2}}|Q_\sigma(t)|^{\frac{1}{\delta_2}}\right)\left(z_\sigma^{\delta_2}(t) - \alpha_\sigma|Q_\sigma(t)|\right) \geq C_\iota\left(z_\sigma^{\delta_2}(t) - \alpha_\sigma|Q_\sigma(t)|\right)^2 \times \left(z_\sigma^{\delta_2}(t) + \alpha_\sigma|Q_\sigma(t)|\right)^{\frac{1}{\delta_2}-1}. \tag{46}$$

Knowing that $Q_\sigma(t)$ is strictly negative for large t, we use (34) to obtain

$$\left(z_\sigma^{\delta_2} + \alpha_\sigma|Q_\sigma(t)|\right)^{1-\frac{1}{\delta_2}} Y_\sigma(t) \geq C_\iota\left(Q'_\sigma(t) + e^{-\beta_\sigma t}z_\sigma^{\delta_1}(t)\right)^2.$$

Using the fact that $z_\sigma(t)$ and $Q_\sigma(t)$ are bounded for large t and that $\delta_2 > 1$, there exists $C'_\iota > 0$ such that for large t,

$$\left(Q'_\sigma(t) + e^{-\beta_\sigma t}z_\sigma^{\delta_1}(t)\right)^2 \leq C'_\iota Y_\sigma(t).$$

This leads to the inequality

$$\int_T^t \left(Q'_\sigma(s) + e^{-\beta_\sigma s}z_\sigma^{\delta_1}(s)\right)^2 ds \leq C'_\iota S_\sigma(t).$$

Consequently, we obtain

$$\int_T^t Q'_\sigma(s)^2 ds \leq C'_\iota S_\sigma(t) - 2\int_T^t e^{-\beta_\sigma s}Q'_\sigma(s)z_\sigma^{\delta_1}(s)ds.$$

Since $S_\sigma(t)$, $z_\sigma(t)$, and $Q'_\sigma(t)$ are bounded for sufficiently large t and $\beta_\sigma > 0$, it follows that

$$\int_T^t Q'_\sigma(s)^2 ds$$

is bounded. Furthermore, since $\int_T^t Q'_\sigma(s)^2 ds$ is increasing, we conclude that

$$\int_T^{+\infty} Q'_\sigma(t)^2 dt < +\infty.$$

On the other hand, by differentiating equation (34), we get

$$Q''_\sigma(t) - \alpha_\sigma Q'_\sigma(t) + \delta_2 z_\sigma^{\delta_2 - 1} z'_\sigma(t) - \theta_\sigma e^{-\beta_\sigma t} z_\sigma^{\delta_1}(t) + \delta_1 e^{-\beta_\sigma t} z_\sigma^{\delta_1 - 1}(t) z'_\sigma(t) = 0. \tag{47}$$

Since $Q'_\sigma(t)$, $z'_\sigma(t)$, and $z_\sigma(t)$ are bounded for large t, it follows that $Q''_\sigma(t)$ is bounded for sufficiently large t. Thus, by Lemma (2.7), we conclude that

$$\lim_{t\to+\infty} Q'_\sigma(t) = 0.$$

Step 3. $z_\sigma(t)$ converges as $t \to +\infty$.

Since $z_\sigma(t)$ is bounded, $\lim\limits_{t\to+\infty} Q'_\sigma(t) = 0$, and $\beta_\sigma > 0$, we get from (34) that

$$\lim_{t\to+\infty} \left(-\alpha_\sigma Q_\sigma(t) + z_\sigma^{\delta_2}(t) \right) = 0. \tag{48}$$

Now, assume for the sake of contradiction that $z_\sigma(t)$ oscillates for large t. Then, there exist two sequences $\{\eta_j\}$ and $\{\zeta_j\}$, both tending to $+\infty$ as $j \to +\infty$, such that η_j and ζ_j are the local minimum and maximum of $v(t)$, respectively, satisfying $\eta_j < \zeta_j < \eta_{j+1}$. Moreover, we have the following relations:

$$0 \le \lim_{j\to+\infty} z_\sigma(\eta_j) = \liminf_{t\to+\infty} z_\sigma(t) = \rho_1 < \lim_{j\to+\infty} z_\sigma(\zeta_j) = \limsup_{t\to+\infty} z_\sigma(t) = \rho_2 < +\infty.$$

Since $z'_\sigma(\eta_j) = z'_\sigma(\zeta_j) = 0$, we have

$$l_\sigma(\eta_j) = \frac{m_2 + p}{\delta_2 + 1 - p} z_\sigma(\eta_j) \quad \text{and} \quad l_\sigma(\zeta_j) = \frac{m_2 + p}{\delta_2 + 1 - p} z_\sigma(\zeta_j).$$

This implies

$$Q_\sigma(\eta_j) = \left(\frac{m_2 + p}{\delta_2 + 1 - p} \right)^{p-1} z_\sigma^{p-1}(\eta_j) \quad \text{and}$$
$$Q_\sigma(\zeta_j) = \left(\frac{m_2 + p}{\delta_2 + 1 - p} \right)^{p-1} z_\sigma^{p-1}(\zeta_j).$$

Combining with relation (48) and letting $i \to +\infty$, we get

$$\rho_1^{p-1} \left(\rho_1^{\delta_2 + 1 - p} - \left(N - p - \frac{m_2 + p}{\delta_2 + 1 - p}(p-1) \right) \left(\frac{m_2 + p}{\delta_2 + 1 - p} \right)^{p-1} \right) = 0,$$

and

$$\rho_2^{p-1}\left(\rho_2^{\delta_2+1-p}-\left(N-p-\frac{m_2+p}{\delta_2+1-p}(p-1)\right)\left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}\right)=0.$$

Since $\rho_1<\rho_2$, it follows that $\rho_1=0$, and

$$\rho_2=\left(\left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}\left(N-p-\frac{m_2+p}{\delta_2+1-p}(p-1)\right)\right)^{\frac{1}{\delta_2+1-p}}.$$

Using (37), we obtain

$$\lim_{j\to+\infty} I_\sigma(\eta_j)=0,$$

and

$$\lim_{j\to+\infty} I_\sigma(\zeta_j)=\frac{-(m_2+p)}{p(\delta_2+1)}\left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}\rho_2^p<0.$$

This is a contradiction, as $I_\sigma(t)$ converges as $t\to+\infty$. Hence, $z_\sigma(t)$ converges as $t\to+\infty$. Now, using Proposition 2.5, we have that $l_\sigma(t)$ converges, and by (23), the limit of $z'_\sigma(t)$ must be zero as $t\to+\infty$.

Let

$$\lim_{r\to0} r^{\frac{m_2+p}{\delta_2+1-p}}u(r)=a_1,$$

then

$$\lim_{t\to+\infty} l_\sigma(t)=\frac{(m_2+p)}{\delta_2+1-p}a_1,$$

and by relation (24), we have

$$\lim_{t\to+\infty} Q_\sigma(t)=\left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}a_1^{p-1}.$$

Since both $z_\sigma(t)$ and $Q_\sigma(t)$ converge, from equation (34), we see that $Q'_\sigma(t)$ must converge to 0. By letting $t\to+\infty$ in (34), we obtain

$$a_1^{p-1}\left(a_1^{\delta_2+1-p}-\left(N-p-\frac{m_2+p}{\delta_2+1-p}(p-1)\right)\left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}\right)=0. \tag{49}$$

Thus, $a_1 = 0$ or

$$a_1 = \left(\left(N - p - \frac{m_2+p}{\delta_2+1-p}(p-1) \right) \left(\frac{m_2+p}{\delta_2+1-p} \right)^{p-1} \right)^{\frac{1}{\delta_2+1-p}}.$$

Step 4. If $a_1 = 0$ we show that $\lim_{r\to 0} r^{\sigma_0} u(r) = C > 0$, where $0 < \sigma_0 < \dfrac{m_2+p}{\delta_2+1-p}$. It is sufficient to prove that $r^{\sigma_0}u(r)$ is bounded for small r, that is, $z_{\sigma_0}(t)$ is bounded for small t.

By contradiction, suppose that

$$\lim_{t\to+\infty} z_{\sigma_0}(t) = +\infty.$$

Since $\sigma_0 < \dfrac{m_2+p}{\delta_2+1-p}$, we use the Proposition 2.6.

We have $H_{\sigma_0}(r) < 0$ and $H_{(m_2+p)/(\delta_2+1-p)}(r) > 0$ for small r.

Hence, using the fact that $u'(r) < 0$ for small r, we get

$$\sigma_0 < \frac{r|u'|}{u} < \frac{m_2+p}{\delta_2+1-p}.$$

Then, using the change (21), we have as $t \to +\infty$

$$(\sigma_0)^{p-1} < Q_{\sigma_0}(t) z_{\sigma_0}^{1-p}(t) < \left(\frac{m_2+p}{\delta_2+1-p} \right)^{p-1}. \tag{50}$$

Multiplying equation (22) (for $\sigma = \sigma_0$) by $z_{\sigma_0}^{1-p}(t)$, we get

$$\left(Q_{\sigma_0}(t) z_{\sigma_0}^{1-p}(t) \right)' + (p-1)(l_{\sigma_0}(t))^p z_{\sigma_0}^{-p}(t) - (N-p) Q_{\sigma_0}(t) z_{\sigma_0}^{1-p}(t) + G_{\sigma_0}(t) = 0, \tag{51}$$

where

$$G_{\sigma_0}(t) = e^{-\beta_{\sigma_0} t} z_{\sigma_0}^{\delta_1+1-p}(t) + e^{-\theta_{\sigma_0} t} z_{\sigma_0}^{\delta_2+1-p}(t).$$

Since

$$\lim_{t\to+\infty} e^{-\beta_{\sigma_0} t} z_{\sigma_0}^{\delta_1+1-p}(t) = \lim_{t\to+\infty} e^{-\theta_{\sigma_0} t} z_{\sigma_0}^{\delta_2+1-p}(t) = 0,$$

we deduce that $G_{\sigma_0}(t) \to 0$ as $t \to +\infty$.

Let

$$\chi_{\sigma_0}(t) = Q_{\sigma_0}(t) z_{\sigma_0}^{1-p}(t). \tag{52}$$

Then estimation (50) yields that for $t \to +\infty$

$$(\sigma_0)^{p-1} < \chi_{\sigma_0}(t) < \left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}. \tag{53}$$

Furthermore, we have by (51)

$$(\chi_{\sigma_0})'(t) + (p-1)(\chi_{\sigma_0}(t))^{p/(p-1)} - (N-p)\chi_{\sigma_0}(t) + G_{\sigma_0}(t) = 0, \tag{54}$$

Let

$$\omega(s) = s^{p/(p-1)} - \frac{N-p}{p-1}s, \quad \text{for } s \geq 0.$$

Since $\chi_{\sigma_0}(t) > 0$ for t large, then

$$-\chi'_{\sigma_0}(t) = (p-1)\omega(\chi_{\sigma_0}(t)) + G_{\sigma_0}(t).$$

Using the study of the function ω, then there exists a constant $\mathcal{K}_0 > 0$ so that $\omega(s) < -\mathcal{K}_0$ for $(\sigma_0)^{p-1} < s < \left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1} < \left(\frac{N-p}{p-1}\right)^{p-1}$.
As $\chi_{\sigma_0}(t)$ satisfies (52) when t tends to $+\infty$ and $\lim_{t\to+\infty} G_{\sigma_0}(t) = 0$. Then from (54), there exists $\mathcal{K}_1 > 0$ so that

$$\chi'_{\sigma_0}(t) > \mathcal{K}_1, \qquad \text{for large } t. \tag{55}$$

Integrating (55) on (t_0, t) for large t_0, we get $\lim_{t\to+\infty} \chi_{\sigma_0}(t) = +\infty$. Which gives a contradiction with the fact that $\chi_{\sigma_0}(t)$ is bounded for large t. Which implies that $z_{\sigma_0}(t)$ is bounded for large t and as follows $z_{\sigma_0}(t)$ converges as $t \to +\infty$. That is, there exists $C > 0$ so that $\lim_{r\to 0} r^{\sigma_0}u(r) = C$, with $0 < \sigma_0 < \frac{m_2+p}{\delta_2+1-p}$. The proof is complete. □

Currently, we describe the behavior of the derivative of u in the vicinity of the origin.

Theorem 3.2. *Let $\delta_2 > \delta_1 > \frac{(m_1+N)(p-1)}{N-p}$. Suppose that u is a solution of (P). If $\delta_2 \neq \frac{N(p-1)+p(m_2+1)}{N-p}$, then the derivative of u, denoted as $u'(r)$, exhibits one of the following behaviors near $r = 0$:*

(i)

$$\lim_{r\to 0} r^{\frac{m_2+p}{\delta_2+1-p}+1}u'(r) = -\frac{m_2+p}{\delta_2+1-p}\times$$
$$\left(\left(N-p-\frac{m_2+p}{\delta_1+1-p}(p-1)\right)\left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}\right)^{\frac{1}{\delta_2+1-p}}.$$

(ii)

$$\lim_{r\to 0} r^{\sigma_0+1}u'(r) = -\sigma_0 C, \quad \textit{with } 0<\sigma_0<\frac{m_2+p}{\delta_2+1-p} \textit{ and } C>0.$$

Proof. Since $z_\sigma(t)=r^\sigma u(r)$ converges as $t\to+\infty$ for $\sigma=\frac{m_2+p}{\delta_2+1-p}$ or $\sigma=\sigma_0$, by Proposition 2.5, the function $l_\sigma(t)=r^{\sigma+1}u'(r)$ also converges as $t\to+\infty$. Consequently, we obtain that:

$$\lim_{t\to+\infty} z'_\sigma(t)=0.$$

Therefore, the following limits hold

$$\lim_{t\to+\infty} l_\sigma(t)=\frac{m_2+p}{\delta_2+1-p}\times$$
$$\left(\left(N-p-\frac{m_2+p}{\delta_2+1-p}(p-1)\right)\left(\frac{m_2+p}{\delta_2+1-p}\right)^{p-1}\right)^{\frac{1}{\delta_2+1-p}}$$

or

$$\lim_{t\to+\infty} l_\sigma(t)=\sigma_0 C.$$

This completes the proof. □

We now present the behavior of the singular solution u for problem (P) in the critical case where $\delta_2=\dfrac{N(p-1)+p(m_2+1)}{N-p}$.

Theorem 3.3. *Assume that* $\delta_2=\dfrac{N(p-1)+p(m_2+1)}{N-p}$. *Let u be a solution of problem* (P). *Then we have one of the following behaviors*

$i)$ $\lim_{r\to 0} r^{\frac{m_2+p}{\delta_2+1-p}}u(r)=0$
or
$$\lim_{r\to 0} r^{\frac{m_2+p}{\delta_2+1-p}}u(r)=\left(\left(N-p-\tfrac{m_2+p}{\delta_2+1-p}(p-1)\right)\left(\tfrac{m_2+p}{\delta_2+1-p}\right)^{p-1}\right)^{\frac{1}{\delta_2+1-p}}.$$

$ii)$ $r^{\frac{m_2+p}{\delta_2+1-p}}u(r)$ *oscillates near* 0 *and satisfies:*

$$0\le \liminf_{r\to 0} r^{\frac{m_2+p}{\delta_2+1-p}}u(r)=\rho_1<\left(\frac{N-p}{p}\right)^{\frac{p}{\delta_2+1-p}}$$
$$<\limsup_{r\to 0} r^{\frac{m_2+p}{\delta_2+1-p}}u(r)=\rho_2\le\left(\left(\frac{\delta_2+1}{p}\right)\left(\frac{N-p}{p}\right)^p\right)^{\frac{1}{\delta_2+1-p}}. \quad (56)$$

Moreover, if $\rho_1 = 0$*, then*

$$\rho_2 = \left(\left(\frac{\delta_2+1}{p}\right)\left(\frac{N-p}{p}\right)^p\right)^{\frac{1}{\delta_2+1-p}}.$$

Proof. Since $z_\sigma(t)$ remains bounded for $\sigma = \dfrac{m_2+p}{\delta_2+1-p}$ from Proposition 2.2, there are two potential cases to examine:

• If $z_\sigma(t)$ converges as $t \to +\infty$, then Proposition 2.5 ensures that $l_\sigma(t)$ also converges as $t \to +\infty$. Consequently, $z'_\sigma(t) \to 0$ as $t \to +\infty$. Using (34) and (49), this directly leads to the limits stated in the theorem.

• If $z_\sigma(t)$ oscillates when t is large. Thus, there exists two sequences $\{\eta_j\}$ and $\{\zeta_j\}$ satisfying $\eta_j < \zeta_j < \eta_{j+1}$ and

$$\rho_1 = \lim_{j\to+\infty} z_\sigma(\eta_j) = \liminf_{t\to+\infty} z_\sigma(t) < \lim_{j\to+\infty} z_\sigma(\zeta_j) = \limsup_{t\to+\infty} z_\sigma(t) = \rho_2.$$

By equation (34) and the fact that $\alpha_\sigma = \dfrac{m_2+p}{\delta_2+1-p} = \dfrac{N-p}{p}$, we have:

$$0 \le Q'_\sigma(\eta_j) = \frac{N-p}{p} Q_\sigma(\eta_j) - z_\sigma^{\delta_2}(\eta_j) - e^{-\beta_\sigma \eta_j} z_\sigma^{\delta_1}(\eta_j)$$

and

$$0 \ge Q'_\sigma(\zeta_j) = \frac{N-p}{p} Q_\sigma(\zeta_j) - z_\sigma^{\delta_2}(\zeta_j) - e^{-\beta_\sigma \zeta_j} v^{\delta_1}(\zeta_j).$$

Taking the limit as $j \to +\infty$ in the above relations, we obtain

$$\rho_1 \le \left(\frac{N-p}{p}\right)^{\frac{p}{\delta_2+1-p}} \le \rho_2. \tag{57}$$

On the other hand, using relation (41), we deduce

$$I_\sigma(t) = I_\sigma(T) - \frac{e^{-\beta_\sigma t}}{\delta_2+1} z_\sigma^{\delta_2+1} + \frac{e^{-\beta_\sigma T}}{\delta_2+1} z_\sigma^{\delta_2+1} - \frac{\beta_\sigma}{\delta_2+1}\int_T^t e^{-\beta_\sigma s} z_\sigma^{\delta_2+1}(s)\,ds.$$

Since z_σ is bounded and $\beta_\sigma > 0$, it follows that $I_\sigma(t)$ converges as $t \to +\infty$. Therefore, by relation (37), we have

$$\lim_{t\to+\infty} I_\sigma(t) = \omega_1(\rho_1) = \omega_1(\rho_2), \tag{58}$$

where

$$\omega_1(s) = \frac{s^{\delta_2+1}}{\delta_2+1} - \frac{1}{p}\left(\frac{N-p}{p}\right)^p s^p, \quad s \ge 0.$$

A straightforward analysis of the function ω_1 reveals

$$\omega_1(0)=\omega_1\left(\left(\left(\frac{\delta_2+1}{p}\right)\left(\frac{N-p}{p}\right)^p\right)^{\frac{1}{\delta_2+1-p}}\right)=0,$$

$$\omega_1'(0)=\omega_1'\left(\left(\frac{N-p}{p}\right)^{\frac{p}{\delta_2+1-p}}\right)=0,$$

$$\omega_1'(s)<0 \quad \text{for} \quad 0<s<\left(\frac{N-p}{p}\right)^{\frac{p}{\delta_2+1-p}},$$

$$\omega_1'(s)>0 \quad \text{for} \quad s>\left(\frac{N-p}{p}\right)^{\frac{p}{\delta_2+1-p}}.$$

Hence, combining the study of the function ω_1 with relations (57) and (58) we get relation (56) and if $\rho_1=0$, then $\rho_2=\left(\left(\frac{\delta_2+1}{p}\right)\left(\frac{N-p}{p}\right)^p\right)^{\frac{1}{\delta_2+1-p}}$. This completes the proof. □

4. Existence of singular solutions

In this section, we focus on establishing the existence of singular solutions to problem (P). Our principal finding is derived from the work presented in [9].

Theorem 4.1. *Assume that $\delta_2>\delta_1>\frac{(m_1+N)(p-1)}{N-p}$. Then there exists a singular solution of problem (P).*

Proof. We have previously demonstrated in [9] that the problem (P_a) has a maximal solution defined on $(0,R_{\max})$ for some $R_{max}>0$. By applying the maximum principle, it follows that $a\mapsto u_a$ is monotonically increasing. Furthermore, from Proposition 2.2, we know that $u_a(r)\le M(N,p,\delta_2,m_2)r^{-(m_2+p)/(\delta_2+1-p)}$, where the constant $M(N,p,\delta_2,m_2)$ is explicitly given by (8). As a result, u_a converges as $a\to+\infty$ to a solution u, which satisfies the problem (P) on the maximal interval $(0,R_{\max})$. Next, we show that $R_{max}=+\infty$. Assume for contradiction that $R_{\max}<+\infty$. We define the following energy function

$$A(r)=\frac{p-1}{p}|u'|^p+\frac{r^{m_1}}{\delta_1+1}u^{\delta_1+1}+\frac{r^{m_2}}{\delta_2+1}u^{\delta_2+1}. \tag{59}$$

Using equation (2), we compute the derivative of $A(r)$:

$$A'(r)=-\frac{N-1}{r}|u'|^p+\frac{m_1}{\delta_1+1}r^{m_1-1}u^{\delta_1+1}+\frac{m_2}{\delta_2+1}r^{m_2-1}u^{\delta_2+1}. \tag{60}$$

Since $\lim_{r\to R_{\max}} |u(r)| = \lim_{r\to R_{\max}} |u'(r)| = +\infty$ and given that $N \geq 1$ and $m_2 < m_1 \leq 0$, we conclude that

$$\lim_{r\to R_{\max}} A'(r) = -\infty \quad \text{and} \quad \lim_{r\to R_{\max}} A(r) = +\infty,$$

which is a contradiction. Therefore $R_{\max} = +\infty$. The proof is achieved. □

5. Conclusion and Perspective

In this paper we have studied a nonlinear elliptic Matukuma-type equation near the origin. We have established existence results for singular positive solutions and examined their behavior near zero. More precisely, the singular solution u exhibits one of the following behaviors:

i)

$$u(r) \underset{0}{\sim} \left(\left(N - p - \frac{m_2+p}{\delta_2+1-p}(p-1) \right) \left(\frac{m_2+p}{\delta_2+1-p} \right)^{p-1} \right)^{\frac{1}{\delta_2+1-p}} r^{\frac{-(m_2+p)}{\delta_2+1-p}}.$$

ii)

$$u(r) \underset{0}{\sim} C r^{-\sigma_0}, \quad \text{where } C \text{ and } \sigma_0 \text{ are strictly positive constants.}$$

Also, we demonstrate that the derivative of u has one of the present behaviors

i)

$$u'(r) \underset{0}{\sim} -\frac{m_2+p}{\delta_2+1-p} \times \left(\left(N - p - \frac{m_2+p}{\delta_1+1-p}(p-1) \right) \left(\frac{m_2+p}{\delta_2+1-p} \right)^{p-1} \right)^{\frac{1}{\delta_2+1-p}} r^{-\frac{m_2+p}{\delta_2+1-p}-1}.$$

ii)

$$u'(r) \underset{0}{\sim} -\sigma_0 C r^{-(\sigma_0+1)}, \quad \text{where } C \text{ and } \sigma_0 \text{ are strictly positive constants.}$$

The uniqueness of singular solutions remains an open question and will be addressed in future research.

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