LE MATEMATICHE
Vol. LXXIX (2024) – Issue II, pp. 249–266
doi: 10.4418/2024.79.2.1

ON THE INITIAL VALUE PROBLEM WITH ALMOST PERIODIC LINEAR PART IN A BANACH SPACE

P. KOUMANTOS

In this article we present strong global solutions for the initial value problem in a reflexive Banach space, when the linear part of the corresponding differential evolution equation is Bohl-Bohr or Stepanov almost periodic function, and the displayed operator is infinitesimal generator of (C_0)-semigroup. As an application, we consider a problem from magnetohydrodynamics.

1. Introduction

Let $(\mathrm{B}, \|\cdot\|_{\mathrm{B}})$ be a reflexive Banach space and the semilinear differential evolution equation:

$$\dot{x}(t) + Sx(t) = F(t, x(t)), \quad t \in \mathbb{R}^+, \tag{1}$$

where the symbol of the dot is for $\frac{d}{dt}$, $-S$ is the infinitesimal generator of a semigroup $\left\{e^{-tS} : t \geq 0\right\}$ of operators of class (C_0), $F : \mathbb{R}^+ \times \mathrm{B} \to \mathrm{B}$ is a nonlinear function, and let the initial data $x(0) = x_0 \in \mathrm{B}$.

Also, we assume that there exist real positive constants c, a such that:

$$\left\|e^{-tS}\right\| \leq ce^{-at}, \quad t \in \mathbb{R}^+. \tag{2}$$

Received on December 4, 2023

AMS 2010 Subject Classification: 34K30; 34G20; 47D06; 35B10
Keywords: evolution equations, (C_0)-semigroups, strong solutions, almost periodic functions

We present strong global solutions for the above evolution equation by continuous perturbations of Carathéodory-Lipschitz type, assuming the corresponding composition Nemytskii operator of the nonlinear function F in equation (1) of the linear part of the corresponding linear evolution equation of the type:

$$\dot{y}(t)+Sy(t)=f(t), \quad t\in\mathbb{R}^{+}, \tag{3}$$

with initial data $y(0)=y_0\in$ B, where $f:\mathbb{R}^{+}\to$ B is a given function.

For semilinear evolution equations in Banach spaces we refer to Prüss [24] and for periodic solutions of semilinear evolution equations in [25], while for stability of (C_0)-semigroups we refer to Phóng [26] and to Ruess and Phóng [27] for asymptotically almost periodic solutions of evolution equations in Banach spaces. We note that Arendt and Batty [1] studied almost periodic solutions of first- and second-order Cauchy problems. Also, we refer to the articles by Chepyzhov and Vishik [9], and Pankov [22] for non-autonomous evolution equations with almost periodic symbols, and almost periodic functions, Bohr compactification, and differential equations, respectively. Further, we refer to Bokalo and Lorenzi in [8] for linear first-order evolution problems without initial conditions, and to Horani, Favini and Tanabe in [17] for parabolic first and second order differential equations.

Moreover, in our previous works [4] and [5] we studied functional evolution equations (without an initial condition) with delay, in the case where the operator is infinitesimal generator of an analytic semigroup, and we achieved almost periodic effects. We also refer to Schechter's [28] book on fundamental concepts and especially on solutions to the Carathéodory type for the initial value problem. On the other hand, the problem of almost periodicity and its generalization in the class of almost automorphic functions is of particular interest, as is shown in the work of Benkhalti, Es-Sebbar and Ezzinbi [7] on the Bohr-Neugebauer property. Also, Batty, Hutter and Räbiger [6] presented results on almost periodicity of mild solutions of inhomogeneous periodic Cauchy problems. Furthermore, Fan and Li [12] presented results for analyticity and exponential stability of semigroups for the elastic systems with structural damping in Banach spaces.

Also, we refer to classical literature Hille and Phillips [15], and Yosida [30], for the semigroup theory and expesially for the Bochner integration. For the notions of mild, weak or generalized, strong and classical solution which may be local or global in time we refer to Engel and Nagel [11] and Pazy in [23]. Particularly in [11] included many applications of differential evolution equations to physical problems. We also refer to Yoshizawa's classic book [29] on stability theory and periodic solutions, to Daleckii and Krein [10] as well as to Pazy [23] and Arendt, Batty, Hieber, and Neubrander [2], for classical existence and regularity results in Banach spaces.

Section 2 includes the basic concepts and symbolisms we need, as well as the preliminary results for the solution spaces we introduce in the case of almost periodicity that follow.

In section 3 we present our results about the initial value problem (1) when f in the linear evolution equation (3) is a uniform bounded continuous function as well as when it is almost periodic (Bohl-Bohr or Stepanov) function. Specifically we find solution spaces that are uniformly closed subsets of the space of continuous and bounded functions (for X a metric space a subset of the space $\mathrm{C}(E,X)$ of continuous maps from E to X is uniformly closed if it contains the limit of every uniformly convergent sequence in it), are Baire spaces in the strong topology (i.e. have the property that the intersection of a countable family of open and dense sets is dense) and are uniformly complete and the solutions obtained are strong and global. Moreover by continuous pertubation of the linear part of (3) we obtain strong global solution of (1) in the completion of the above solution spaces. An asymptotic stability property is also established for the solutions of the nonlinear initial value problem.

As an application of the abstract results in section 4 we consider a Navier-Stokes-Maxwell system. We refer to the works of Giga and Yoshida [13], [14] for classical presentation and results. Further, refer to the article by Arsénio and Gallagher in [3] for results on solutions of Navier-Stokes-Maxwell systems in large energy spaces. For the Navier-Stokes-Ohm type equations of one fluid we refer in our previous works [4] and [5] and references therein.

2. Notations and Preliminary Results

Let $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda^1)$ be the classical Borel-Lebesgue measure on $\mathbb{R}$ and the classical space $\mathrm{L}^1(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda^1)$ consists of all functions $f:\mathbb{R}\to\mathbb{R}$ for which there exists the Lebesgue integral $\int_{\mathbb{R}}|f(x)|d\lambda^1(x)$, equipped with the norm $\|f\|_1 := \int_{\mathbb{R}}|f(x)|d\lambda^1(x)$.

As usual we denote by $\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$ the Banach space of bounded continuous functions $\varphi:\mathbb{R}^+\to\mathrm{B}$ endowed with supremum norm

$$|\varphi| := \sup\left\{\|\varphi(t)\|_{\mathrm{B}} : t\in\mathbb{R}^+\right\},$$

and let $\mathrm{C}(\mathbb{R}^+,\mathrm{B})$ be the Fréchet space of continuous functions with domain $\mathbb{R}^+$ and range B.

Also, $\mathrm{L}(\mathbb{R}^+,\mathrm{B})$ denotes the space of Lipschitz functions $\phi:\mathbb{R}^+\to\mathrm{B}$, i.e. there exists a constant $l\geq 0$ such that $\|\phi(t)-\phi(s)\|_{\mathrm{B}}\leq l\cdot|t-s|$, for every $t,s\in\mathbb{R}^+$, while the infimum of such constants l is called the Lipschitz constant of ϕ and it is denoted by $\ell(\phi)$. Then, the set of all bounded Lipschitz maps from $\mathbb{R}^+$ into B is provided with the norm $\|\phi\|_{\ell(\phi)} := \max\{\ell(\phi),|\phi|\}$.

By $\mathrm{M}^1(\mathbb{R}^+,\mathrm{B})$ it is denoted the Banach space of Bochner measurable functions $g:\mathbb{R}^+\to\mathrm{B}$ such that $\int_t^{t+1}\|g(s)\|_{\mathrm{B}}\,d\lambda^1(s)<+\infty$ for all $t\in\mathbb{R}^+$ under the norm

$$|g|_{\mathrm{M}^1}:=\sup\left\{\left(\int_t^{t+1}\|g(s)\|_{\mathrm{B}}\,d\lambda^1(s)\right):t\in\mathbb{R}^+\right\}.$$

Following Levitan and Zhikov [20] for classic concepts and results of almost periodic functions we give the next definitions and important properties.

Definition 2.1. An element $u\in\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$ is said to be Bohl-Bohr almost periodic if, for every $\varepsilon>0$, there is a positive real number $\ell:=\ell(\varepsilon)$ such that any interval of $\mathbb{R}^+$ of length ℓ contains at least one point τ (called an ε-almost period of u) for which

$$|u_\tau-u|=\sup\left\{\|u(t+\tau)-u(t)\|_{\mathrm{B}}:t\in\mathbb{R}^+\right\}<\varepsilon.$$

The sum of two almost periodic functions is an almost periodic function, as well as their product. Also, if a sequence of almost periodic functions is uniformly convergent then the limit function is also almost periodic function. Further, it is noted that in classical theory almost periodic functions have the important properties of being bounded, uniformly continuous and the mean value $M\{f\}$ exists. For the proofs of the above, we refer, for example, to the standard book [20].

Bochner's criterion asserts that $u\in\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$ is almost periodic if the set of all translates of u, i.e. $\mathcal{T}(u):=\{u_\tau:\tau\in\mathbb{R}^+\}$, where $u_\tau(t):=u(t+\tau)$, is relatively compact in $\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$.

Let $\mathrm{A_B}(\mathbb{R}^+,\mathrm{B})$ denotes the closed subspace of $\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$ of all almost periodic functions in $\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$.

Definition 2.2. An element $v\in\mathrm{M}^1(\mathbb{R}^+,\mathrm{B})$ is said to be Stepanov almost periodic, if for every $\varepsilon>0$ there exists $\ell:=\ell(\varepsilon)>0$ such that any interval of $\mathbb{R}$ of length ℓ contains at least one point $\tau:=\tau_\varepsilon$ satisfying

$$|v_\tau-v|_{\mathrm{M}^1}=\sup\left\{\int_t^{t+1}\|v(s+\tau)-v(s)\|_{\mathrm{B}}\,d\lambda^1(s):t\in\mathbb{R}^+\right\}<\varepsilon.$$

Then we define the sets in which we will achieve solutions to the initial value problem we study when the linear part is an almost periodic function.

Definition 2.3. (i) Let $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ denotes the set of all elements $\psi\in\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$ such that for all $\varepsilon>0$ there exists $\ell:=\ell(\varepsilon)>0$, for every $(\xi,\xi+\ell)\subseteq\mathbb{R}^+$ there exists $\tau:=\tau_\varepsilon\in(\xi,\xi+\ell)$ satisfying $\lim_{t\to+\infty}\|\psi(t+\tau)-\psi(t)\|_{\mathrm{B}}<\varepsilon$.

(ii) Let $\mathcal{B}(\mathbb{R}^+,\mathrm{B})$ denotes the set of all $g\in\mathrm{M}^1(\mathbb{R}^+,\mathrm{B})$ such that for every $\varepsilon>0$ there exists $\ell:=\ell(\varepsilon)>0$, for every $(\xi,\xi+\ell)\subseteq\mathbb{R}^+$ there exists $\tau:=\tau_\varepsilon\in(\xi,\xi+\ell)$ satisfying $\lim_{t\to+\infty}\int_t^{t+1}\|g(s+\tau)-g(s)\|_{\mathrm{B}}\,d\lambda^1(s)<\varepsilon$.

The next Lemma is a well-known result in differential topology (cf. Theorem 4.2; p. 59 in Hirsch [16]), which we will need to apply.

Lemma 2.4. *If E is a paracompact space and X is a complete metric space, then every uniformly closed subset K of the space $\mathrm{C}(E,X)$ of continuous maps from E to X is a Baire space in the strong topology.*

Recall that a paracompact space is a topological space in which every open cover has an open refinement that is locally finite. Moreover, every compact space is paracompact, while every paracompact Hausdorff space is normal, and a Hausdorff space is paracompact if and only if it admits partitions of unity subordinate to any open cover.

Then, in the next proposition we show the properties of $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ that we need.

Proposition 2.5. $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ *is uniformly closed subset of* $\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$*, Baire space in the strong topology and uniformly complete.*

Proof. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ and $\psi_n \stackrel{u}{\longrightarrow} \psi$. Then, $\psi_n \in \mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$ such that for every $\varepsilon > 0$ there exists $\ell := \ell(\varepsilon) > 0$, for all $(\xi,\xi+\ell) \subseteq \mathbb{R}^+$ there exists $\tau^* := \tau^*_\varepsilon \in (\xi,\xi+\ell)$ satisfying $\lim\limits_{t\to+\infty} \|\psi_n(t+\tau^*) - \psi_n(t)\|_{\mathrm{B}} < \frac{\varepsilon}{3}$.

Also, for every $\varepsilon > 0$ there exists $n_0 := n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq n_0$ holds $\|\psi_n(t) - \psi(t)\|_{\mathrm{B}} < \frac{\varepsilon}{3}$, for all $t \in \mathbb{R}^+$.

Then, considering the norm of the $\psi(t+\tau^*) - \psi(t)$ difference, adding and subtracting the $\psi_n(t+\tau^*)$ and $\psi_n(t)$ terms, and using the triangle inequality we have:

$$\begin{aligned}
&\|\psi(t+\tau^*) - \psi(t)\|_{\mathrm{B}} \\
&\leq \|\psi(t+\tau^*) - \psi_n(t+\tau^*)\|_{\mathrm{B}} + \|-\psi(t) + \psi_n(t)\|_{\mathrm{B}} + \|\psi_n(t+\tau^*) - \psi_n(t)\|_{\mathrm{B}} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|\psi_n(t+\tau^*) - \psi_n(t)\|_{\mathrm{B}}\,.
\end{aligned}$$

Thus,

$$\lim_{t\to+\infty} \|\psi(t+\tau^*) - \psi(t)\|_{\mathrm{B}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence $\psi \in \mathcal{A}(\mathbb{R}^+,\mathrm{B})$, i.e. $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ is uniformly closed subset of $\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$.

Since $\mathbb{R}^+$ is paracompact, B is complete, $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ is uniformly closed and $\mathcal{A}(\mathbb{R}^+,\mathrm{B}) \subseteq \mathrm{C_b}(\mathbb{R}^+,\mathrm{B}) \subseteq \mathrm{C}(\mathbb{R}^+,\mathrm{B})$ combining the last result and the Lemma 2.4 it follows that $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ is a Baire space in the strong topology.

Now let $(\psi_n)_{n\in\mathbb{N}}$ be a uniform Cauchy sequence in $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ (cf. [21]). Thus for all $\varepsilon > 0$ exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\|\psi_n(t) - \psi_m(t)\|_{\mathrm{B}} < \varepsilon$, for all $n,m \geq n_0$ and $t \in \mathbb{R}^+$. Since $\mathrm{C_b}(\mathbb{R}^+,\mathrm{B}) \supseteq \mathcal{A}(\mathbb{R}^+,\mathrm{B})$, the sequence $(\psi_n)_{n\in\mathbb{N}}$ is also a uniform Cauchy sequence in $\mathrm{C_b}(\mathbb{R}^+,\mathrm{B})$ and hence it must converge in an element $\psi \in C_b(\mathbb{R}^+,B)$, i.e. $\psi_n \xrightarrow{u} \psi$. Then, as above we conclude that $\psi \in \mathcal{A}(\mathbb{R}^+,B)$, i.e. $\mathcal{A}(\mathbb{R}^+,B)$ is uniformly complete. □

With similar arguments it is shown that set $\mathcal{B}(\mathbb{R}^+,B)$ also has the same properties.

In the following we also denote by $\overline{\mathcal{A}}(\mathbb{R}^+,B)$ the completion of $\mathcal{A}(\mathbb{R}^+,B)$ (i.e. $\overline{\mathcal{A}}(\mathbb{R}^+,B)$ is a complete metric space which contains $\mathcal{A}(\mathbb{R}^+,B)$ as a dense subspace) and by $\overline{\mathcal{B}}(\mathbb{R}^+,B)$ the completion of $\mathcal{B}(\mathbb{R}^+,B)$.

3. Main Results

3.1. Strong global continuous bounded solutions of the evolution equations

In the following Proposition we show the existence of a strong solution for the linear initial value problem (3).

Recall that a function $\varphi : \mathbb{R}^+ \to B$ is a strong solution on $\mathbb{R}^+$ of the initial value problem (3), if it is strongly differentiable for λ^1-almost every $t \in \mathbb{R}^+$, satisfies the differential equation for λ^1-almost every $t \in \mathbb{R}^+$, and verifies the initial value as well.

Proposition 3.1. *Let $f \in C_b(\mathbb{R}^+,B)$ and satisfies a Lipschitz condition. Then for any $\varphi_0 \in B$ there is at least one strong solution φ of the linear initial value problem* (3) *in $C_b(\mathbb{R}^+,B)$ with $\varphi(0) = \varphi_0$. Specifically $y : \mathbb{R}^+ \to B$, with $y(t) := \int_0^t e^{-sS} f(t-s)d\lambda^1(s)$ belongs in $C_b(\mathbb{R}^+,B)$ and $\varphi(t) := e^{-tS}\varphi_0 + y(t)$ is the solution of the initial value problem* (3).

Proof. Consider the function $y : \mathbb{R}^+ \to B$, and observe that

$$y(t) := \int_0^t e^{-sS} f(t-s)d\lambda^1(s) = \int_0^t e^{-(t-s)S} f(s)d\lambda^1(s).$$

Estimating the norm of $y(t)$ and using inequality (2) we have:

$$\|y(t)\|_B \leq c\int_0^t e^{-as}\|f(t-s)\|_B\, d\lambda^1(s) \leq c|f|\int_0^{+\infty} e^{-as}d\lambda^1(s) \leq \frac{c}{a}\|f\|_{\ell(f)},$$

i.e. $y \in C_b(\mathbb{R}^+,B)$.

Then, for $h > 0$, applying the above formula for y, adding and subtracting the terms $\int_t^{t+h} e^{-(t+h-s)S} f(s)d\lambda^1(s)$ and $\int_t^{t+h} f(t)d\lambda^1(s)$, and using standard properties of the Bochner integral we have:

$$
\begin{aligned}
&\frac{1}{h}(y(t+h)-y(t))\\
&=\frac{1}{h}\int_0^{t+h} e^{-(t+h-s)S} f(s)d\lambda^1(s)-\frac{1}{h}\int_0^{t} e^{-(t-s)S} f(s)d\lambda^1(s)\\
&=\frac{1}{h}\Bigg(\int_0^{t+h} e^{-(t+h-s)S} f(s)d\lambda^1(s)-\int_0^{t} e^{-(t-s)S} f(s)d\lambda^1(s)\\
&\quad+\int_t^{t+h} e^{-(t+h-s)S} f(t)d\lambda^1(s)-\int_t^{t+h} e^{-(t+h-s)S} f(t)d\lambda^1(s)\\
&\quad+hf(t)-\int_t^{t+h} f(t)d\lambda^1(s)\Bigg)\\
&=\frac{1}{h}\Bigg(\int_0^{t} e^{-(t+h-s)S} f(s)d\lambda^1(s)+\int_t^{t+h} e^{-(t+h-s)S} f(s)d\lambda^1(s)\\
&\quad-\int_0^{t} e^{-(t-s)S} f(s)d\lambda^1(s)+\int_t^{t+h} e^{-(t+h-s)S} f(t)d\lambda^1(s)\\
&\quad-\int_t^{t+h} e^{-(t+h-s)S} f(t)d\lambda^1(s)+hf(t)-\int_t^{t+h} f(t)d\lambda^1(s)\Bigg)\\
&=\frac{1}{h}\left(\int_0^{t} e^{-(t+h-s)S} f(s)d\lambda^1(s)-\int_0^{t} e^{-(t-s)S} f(s)d\lambda^1(s)\right)\\
&\quad+\frac{1}{h}\left(\int_t^{t+h} e^{-(t+h-s)S} f(s)d\lambda^1(s)-\int_t^{t+h} e^{-(t+h-s)S} f(t)d\lambda^1(s)\right)\\
&\quad+\frac{1}{h}\left(\int_t^{t+h} e^{-(t+h-s)S} f(t)d\lambda^1(s)-\int_t^{t+h} f(t)d\lambda^1(s)+hf(t)\right)\\
&=\frac{1}{h}\int_0^{t}\left(e^{-(t+h-s)S}-e^{-(t-s)S}\right) f(s)d\lambda^1(s)\\
&\quad+\frac{1}{h}\int_t^{t+h} e^{-(t+h-s)S}\left(f(s)-f(t)\right)d\lambda^1(s)\\
&\quad+\frac{1}{h}\int_t^{t+h}\left(e^{-(t+h-s)S}-I\right) f(t)d\lambda^1(s)+f(t)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{h}(y(t+h)-y(t)) =&\frac{1}{h}\left(e^{-hS}-I\right)\int_0^{t} e^{-(t-s)S} f(s)d\lambda^1(s)\\
&+\frac{1}{h}\int_t^{t+h} e^{-(t+h-s)S}\left(f(s)-f(t)\right)d\lambda^1(s)\\
&+\frac{1}{h}\int_t^{t+h}\left(e^{-(t+h-s)S}-I\right) f(t)d\lambda^1(s)+f(t). \qquad (4)
\end{aligned}
$$

At this point we claim that if a Lipschitz function $f \in \mathrm{C_b}(\mathbb{R}^+, \mathrm{B})$, then for λ^1-almost every $t \in \mathbb{R}^+$ holds:

(i) $\lim\limits_{h \to 0^+} \frac{1}{h} \int_t^{t+h} e^{-(t+h-s)S} (f(s) - f(t))\, d\lambda^1(s) = 0$, and

(ii) $\lim\limits_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \left(e^{-(t+h-s)S} - I\right) f(t) d\lambda^1(s) = 0.$

Indeed, applying classical results about Bochner integral in Hille and Phillips [15] for (i) we have:

$$\begin{aligned}
&\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} e^{-(t+h-s)S} (f(s) - f(t))\, d\lambda^1(s) \\
&= \lim_{h \to 0^+} e^{-hS} \frac{1}{h} \int_t^{t+h} e^{-(t-s)S} (f(s) - f(t))\, d\lambda^1(s) \\
&= e^0 G_1(t) = 0,
\end{aligned}$$

with $G_1(s) := e^{-(t-s)S} (f(s) - f(t))$,

and for (ii):

$$\begin{aligned}
&\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \left(e^{-(t+h-s)S} - I\right) f(t) d\lambda^1(s) \\
&= \lim_{h \to 0^+} \left(\frac{1}{h} e^{-hS} \int_t^{t+h} e^{-(t-s)S} f(t) d\lambda^1(s) - \frac{1}{h} \int_t^{t+h} I f(t) d\lambda^1(s) \right) \\
&= I G_2(t) - \lim_{h \to 0^+} \frac{1}{h} I f(t)(t + h - t) = 0,
\end{aligned}$$

with $G_2(s) := e^{-(t-s)S} f(t)$.

Passing to the limit for $h \to 0^+$ in (4) and applying the above claims (i) and (ii) yields

$$\dot{y}(t) + S y(t) = f(t).$$

Now we define the function φ by the formula $\varphi(t) := e^{-tS} \varphi_0 + y(t)$.

Then, $\varphi(0) = e^{-0S} \varphi_0 + y(0) = \varphi_0$, while

$$\dot{\varphi}(t) = -S e^{-tS} \varphi_0 + \dot{y}(t) = -S\left(e^{-tS} \varphi_0 + y(t)\right) + f(t),$$

i.e. $\dot{\varphi}(t) + S\varphi(t) = f(t)$. Further, since $e^{-at} \leq 1$ for $t \geq 0$ and $a > 0$, we have:

$$\|\varphi(t)\|_{\mathrm{B}} \leq c e^{-at} \|\varphi_0\|_{\mathrm{B}} + \frac{c}{a} \|f\|_{\ell(f)} \leq c \|\varphi_0\|_{\mathrm{B}} + \frac{c}{a} \|f\|_{\ell(f)}.$$

Hence, the function φ is a strong bounded continuous solution of the initial value problem. □

Let Φ be the corresponding composition Nemytskii operator of the nonlinear function F in equation (1) defined by $\Phi y(t) := F(t, y(t))$, $t \in \mathbb{R}^+$.

Proposition 3.2. *If* $\Phi y \in \mathrm{C_b}(\mathbb{R}^+, \mathrm{B})$ *for every Lipschitz function* $y \in \mathrm{C_b}(\mathbb{R}^+, \mathrm{B})$ *and there exists a constant* $\eta > 0$ *such that* $\|\Phi y_1 - \Phi y_2\|_{\mathrm{B}} \leq \eta \|y_1 - y_2\|_{\mathrm{B}}$*, for every Lipschitz functions* $y_1, y_2 \in \mathrm{C_b}(\mathbb{R}^+, \mathrm{B})$*, and* $\frac{c\eta}{a} < 1$*, then for each* $x_0 \in \mathrm{B}$ *the nonlinear initial value problem* (1) *has at least one strong solution* $\phi \in C_b(\mathbb{R}^+, \mathrm{B})$*, with* $\phi(0) = x_0$*.*

Proof. We consider the operator $Z : \mathrm{C_b}(\mathbb{R}^+, \mathrm{B}) \to \mathrm{C_b}(\mathbb{R}^+, \mathrm{B})$, which to any Lipschitz function $y \in \mathrm{C_b}(\mathbb{R}^+, \mathrm{B})$ maps according to Proposition 3.1 a strong solution $Zy(t)$ in $\mathrm{C_b}(\mathbb{R}^+, \mathrm{B})$ of the linear initial value problem $\frac{d}{dt}x(t) + Sx(t) = \Phi y(t)$, $x(0) = x_0$, such that $Zy(t) := e^{-tS}x_0 + \int_0^t e^{-sS}\Phi y(t-s)d\lambda^1(s)$, $t \in \mathbb{R}^+$.

If $y_1, y_2 \in \mathrm{C_b}(\mathbb{R}^+, \mathrm{B})$ are also Lipschitz functions and $t \in \mathbb{R}^+$, then

$$\begin{aligned} \|Zy_1(t) - Zy_2(t)\|_{\mathrm{B}} &\leq \int_0^t \left\|e^{-sS}\right\| \|\Phi y_1(t-s) - \Phi y_2(t-s)\|_{\mathrm{B}}\, d\lambda^1(s) \\ &\leq c\eta \int_0^t e^{-as} \|y_1(t-s) - y_2(t-s)\|_{\mathrm{B}}\, d\lambda^1(s) \\ &\leq c\eta \left(\int_0^{+\infty} e^{-as} d\lambda^1(s)\right) |y_1 - y_2| \\ &= \frac{c\eta}{a} |y_1 - y_2|. \end{aligned}$$

Hence, operator Z is contraction and its fixed point ϕ is a solution of the nonlinear problem in $\mathrm{C_b}(\mathbb{R}^+, \mathrm{B})$ with $\phi(0) = x_0$. □

Concluding this section we note that the Banach space reflectance hypothesis and the further normality hypothesis for f to be Lipschitz can be omitted in the case of bounded generators. To this end, we refer to our previous work [18] and [19] where we studied evolution equations in four-dimensional space-time continuum (Minkowski space) with results of existence and uniqueness of strong and classical solutions respectively. Also, in these works we combine the order structure in a way compatible with the topology of the underlying space.

3.2. Solutions in a Baire space of the evolution equations

We show that if the linear part of the evolution equation (3) is almost periodic, then the solution of the initial value problem is in a Baire space in the strong topology as set out in section 2.

Proposition 3.3. *Let* $f \in \mathrm{A_B}(\mathbb{R}^+, \mathrm{B})$ *and* $y_0 \in \mathrm{B}$*. Then equation* (3) *has at least one strong solution* ψ *in* $\mathcal{A}(\mathbb{R}^+, \mathrm{B})$ *with* $\psi(0) = y_0$*.*

Proof. By Proposition 3.1 the function $\psi : \mathbb{R}^+ \to \mathrm{B} : t \to \psi(t) := e^{-tS}y_0 + \upsilon(t)$, with $\upsilon : \mathbb{R}^+ \to \mathrm{B} : t \to \upsilon(t) := \int_0^t e^{-(t-s)S} f(s) d\lambda^1(s)$ is a strong solution of the problem.

Estimating the norm of the difference $\psi(t+\tau)-\psi(t)$ we have:

$$\begin{aligned}\|\psi(t+\tau)-\psi(t)\|_{\mathrm{B}} &= \left\|e^{-(t+\tau)S}y_0+\upsilon(t+\tau)-e^{-tS}y_0-\upsilon(t)\right\|_{\mathrm{B}}\\ &\leq \left\|e^{-(t+\tau)S}y_0-e^{-tS}y_0\right\|_{\mathrm{B}}+\|\upsilon(t+\tau)-\upsilon(t)\|_{\mathrm{B}}. \qquad (5)\end{aligned}$$

For the first term on the right-hand side of inequality (5), making use of the triangle inequality and of condition (2), we have:

$$\begin{aligned}\left\|e^{-(t+\tau)S}y_0-e^{-tS}y_0\right\|_{\mathrm{B}} &= \left\|\left(e^{-(t+\tau)S}-e^{-tS}\right)y_0\right\|_{\mathrm{B}}\\ &\leq \left(\left\|e^{-(t+\tau)S}\right\|+\left\|e^{-tS}\right\|\right)\|y_0\|_{\mathrm{B}}\\ &\leq \left(ce^{-a(t+\tau)}+ce^{-at}\right)\|y_0\|_{\mathrm{B}}\\ &= ce^{-at}\left(e^{-a\tau}+1\right)\|y_0\|_{\mathrm{B}}. \qquad (6)\end{aligned}$$

For the second term on the right-hand side of inequality (5), writing the integral with ends from 0 to $t+\tau$ as the sum of two successive integrals from 0 to t and from t to $t+\tau$, using the triangle inequality and condition (2), we have:

$$\begin{aligned}&\|\upsilon(t+\tau)-\upsilon(t)\|_{\mathrm{B}}\\ &= \left\|\int_0^{t+\tau} e^{-(t+\tau-s)S}f(s)d\lambda^1(s)-\int_0^t e^{-(t-s)S}f(s)d\lambda^1(s)\right\|_{\mathrm{B}}\\ &= \left\|\int_0^{t+\tau} e^{-sS}f(t+\tau-s)d\lambda^1(s)-\int_0^t e^{-sS}f(t-s)d\lambda^1(s)\right\|_{\mathrm{B}}\\ &= \left\|\int_0^{t} e^{-sS}f(t+\tau-s)d\lambda^1(s)+\int_t^{t+\tau} e^{-sS}f(t+\tau-s)d\lambda^1(s)\right.\\ &\quad \left.-\int_0^t e^{-sS}f(t-s)d\lambda^1(s)\right\|_{\mathrm{B}}\\ &\leq \left\|\int_0^{t} e^{-sS}f(t+\tau-s)d\lambda^1(s)-\int_0^t e^{-sS}f(t-s)d\lambda^1(s)\right\|_{\mathrm{B}}\\ &\quad +\left\|\int_t^{t+\tau} e^{-sS}f(t+\tau-s)d\lambda^1(s)\right\|_{\mathrm{B}}\\ &\leq \int_0^t \left\|e^{-sS}\right\|\|f(t+\tau-s)-f(t-s)\|_{\mathrm{B}}\,d\lambda^1(s)\\ &\quad +\int_t^{t+\tau}\left\|e^{-sS}\right\|\|f(t+\tau-s)\|_{\mathrm{B}}\,d\lambda^1(s)\end{aligned}$$

$$\begin{aligned}
&\leq c\int_0^t e^{-as}\,\|f(t+\tau-s)-f(t-s)\|_{\mathrm{B}}\,d\lambda^1(s)\\
&\quad + c\int_t^{t+\tau} e^{-as}\,\|f(t+\tau-s)\|_{\mathrm{B}}\,d\lambda^1(s)\\
&\leq c\,|f_\tau - f|\int_0^{+\infty} e^{-as}d\lambda^1(s) + c\,|f_\tau|\int_t^{+\infty} e^{-as}d\lambda^1(s)\\
&= \frac{c}{a}\,|f_\tau - f| + \frac{c}{a}\,|f_\tau|\,e^{-at}.
\end{aligned} \tag{7}$$

Therefore, due to inequalities (6) and (7), inequality (5) gives:

$$\|\psi(t+\tau)-\psi(t)\|_{\mathrm{B}} \leq ce^{-at}\left(\left(e^{-a\tau}+1\right)\|y_0\|_{\mathrm{B}} + \frac{1}{a}\,|f_\tau|\right) + \frac{c}{a}\,|f_\tau - f|. \tag{8}$$

For $t \to +\infty$ the right side converges to $\frac{c}{a}\,|f_\tau - f| < \frac{c}{a}\frac{a}{c}\varepsilon = \varepsilon$, applying with $\frac{a}{c}\varepsilon$ in the definition for f to be almost periodic. Hence $\psi \in \mathcal{A}(\mathbb{R}^+,\mathrm{B})$. □

Proposition 3.4. *Let* $f \in \mathrm{A}_{\mathrm{B}}(\mathbb{R}^+,\mathrm{B})$ *and* $y_0 \in \mathrm{B}$. *Then the linear initial value problem* (3) *has at least one strong solution* ψ *in* $\mathcal{B}(\mathbb{R}^+,\mathrm{B})$ *with* $\psi(0) = y_0$.

Proof. Applying similar arguments as in the previous Proposition 3.3 the function $\psi : \mathbb{R}^+ \to \mathrm{B} : t \to \psi(t) := e^{-tS}y_0 + \upsilon(t)$, with $\upsilon : \mathbb{R}^+ \to \mathrm{B} : t \to \upsilon(t) := \int_0^t e^{-(t-s)S} f(s)d\lambda^1(s)$ is a strong solution of the problem. Also, by inequality (8) yields: $\|\psi(s+\tau)-\psi(s)\|_{\mathrm{B}} \leq ce^{-as}\left(\left(e^{-a\tau}+1\right)\|y_0\|_{\mathrm{B}} + \frac{1}{a}\,|f_\tau|\right) + \frac{c}{a}\,|f_\tau - f|$. Thus,

$$\begin{aligned}
&\int_t^{t+1} \|\psi(s+\tau)-\psi(s)\|_{\mathrm{B}}\,d\lambda^1(s)\\
&\leq \int_t^{t+1}\left(ce^{-as}\left(\left(e^{-a\tau}+1\right)\|y_0\|_{\mathrm{B}} + \frac{1}{a}\,|f_\tau|\right) + \frac{c}{a}\,|f_\tau - f|\right)d\lambda^1(s)\\
&= c\left(\left(e^{-a\tau}+1\right)\|y_0\|_{\mathrm{B}} + \frac{1}{a}\,|f_\tau|\right)\int_t^{t+1} e^{-as}d\lambda^1(s) + \frac{c}{a}\,|f_\tau - f|\int_t^{t+1} d\lambda^1(s)\\
&= -\frac{c}{a}\left(\left(e^{-a\tau}+1\right)\|y_0\|_{\mathrm{B}} + \frac{1}{a}\,|f_\tau|\right)\left(e^{-a}-1\right)e^{-at} + \frac{c}{a}\,|f_\tau - f|.
\end{aligned}$$

For $t \to +\infty$ the right side converges to $\frac{c}{a}\,|f_\tau - f| < \frac{c}{a}\frac{a}{c}\varepsilon = \varepsilon$, applying with $\frac{a}{c}\varepsilon$ in the definition for f to be almost periodic. Hence $\psi \in \mathcal{B}(\mathbb{R}^+,\mathrm{B})$. □

Corollary 3.5. *If the solution of the problem* (3) *is classical* (*and thus unique*) *in* $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ *or in* $\mathcal{B}(\mathbb{R}^+,\mathrm{B})$, *and there exists* $\omega \in \mathbb{R}^+$ *such that* $f(t+\omega) = f(t)$ *for all* $t \in \mathbb{R}^+$, *then* $\lim_{t\to+\infty} \|\psi(t+\omega)-\psi(t)\| = 0$.

Proof. With the further assumptions we make about the solution ψ (resulting from Proposition 3.3 or 3.4 respectively), and the function f, then we have:

$$\begin{aligned}
&\|\psi(t+\omega)-\psi(t)\|_{\mathrm{B}}\\
&=\left\|e^{-(t+\omega)S}y_0+\int_0^{t+\omega}e^{-sS}f(t+\omega-s)d\lambda^1(s)-e^{-tS}y_0\right.\\
&\qquad\left.-\int_0^t e^{-sS}f(t-s)d\lambda^1(s)\right\|_{\mathrm{B}}\\
&=\left\|e^{-(t+\omega)S}y_0+\int_0^t e^{-sS}f(t+\omega-s)d\lambda^1(s)+\int_t^{t+\omega}e^{-sS}f(t+\omega-s)d\lambda^1(s)\right.\\
&\qquad\left.-e^{-tS}y_0-\int_0^t e^{-sS}f(t-s)d\lambda^1(s)\right\|_{\mathrm{B}}\\
&=\left\|e^{-(t+\omega)S}y_0-e^{-tS}y_0+\int_t^{t+\omega}e^{-sS}f(t-s)d\lambda^1(s)\right\|_{\mathrm{B}}\\
&\le\left\|e^{-(t+\omega)S}\right\|\|y_0\|_{\mathrm{B}}+\left\|e^{-tS}\right\|\|y_0\|_{\mathrm{B}}+\int_t^{t+\omega}\left\|e^{-sS}\right\|\|f(t-s)\|_{\mathrm{B}}\,d\lambda^1(s)\\
&\le ce^{-a(t+\omega)}\|y_0\|_{\mathrm{B}}+ce^{-at}\|y_0\|_{\mathrm{B}}+\int_t^{+\infty}ce^{-as}\|f(t-s)\|_{\mathrm{B}}\,d\lambda^1(s)\\
&\le ce^{-a(t+\omega)}\|y_0\|_{\mathrm{B}}+ce^{-at}\|y_0\|_{\mathrm{B}}+c|f|\int_t^{+\infty}e^{-as}d\lambda^1(s)\\
&=c\|y_0\|_{\mathrm{B}}\left(e^{-a\omega}+1\right)e^{-at}+\frac{c}{a}|f|e^{-at}\longrightarrow 0,
\end{aligned}$$

as $t\to+\infty$. □

Proposition 3.6. *Let* $\Phi y\in\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ *for every* $y\in\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ *and there exists a constant* $\eta>0$ *such that* $\|\Phi y_1-\Phi y_2\|_{\mathrm{B}}\le\eta\|y_1-y_2\|_{\mathrm{B}}$, *for all* $y_1,y_2\in\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ *and* $\frac{c\eta}{a}<1$. *Then for each* $x_0\in\mathrm{B}$ *the nonlinear initial value problem* (1) *has at least one strong solution* $\phi\in\overline{\mathcal{A}}(\mathbb{R}^+,\mathrm{B})$, *with* $\phi(0)=x_0$.

Proof. We consider the operator $\Xi:\overline{\mathcal{A}}(\mathbb{R}^+,\mathrm{B})\to\overline{\mathcal{A}}(\mathbb{R}^+,\mathrm{B})$, which to any element $y\in\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ maps a strong solution $\Xi y(t)$ in $\overline{\mathcal{A}}(\mathbb{R}^+,\mathrm{B})$ of the linear initial value problem $\frac{d}{dt}x(t)+Sx(t)=\Phi y(t)$, $x(0)=x_0$, such that

$$\Xi y(t):=e^{-tS}x_0+\int_0^t e^{-sS}\Phi y(t-s)d\lambda^1(s),\ t\in\mathbb{R}^+.$$

If $y_1,y_2\in\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ and $t\in\mathbb{R}^+$, then as in the proof of Proposition 3.2 we conclude $\|\Xi y_1(t)-\Xi y_2(t)\|_{\mathrm{B}}\le\frac{c\eta}{a}|y_1-y_2|$. Thus operator Ξ is contraction and its fixed point ϕ is a solution of the nonlinear problem in $\overline{\mathcal{A}}(\mathbb{R}^+,\mathrm{B})$ with $\phi(0)=x_0$. □

Also, the corresponding result is applied by replacing spaces $\mathcal{A}(\mathbb{R}^+,\mathrm{B})$ and $\overline{\mathcal{A}}(\mathbb{R}^+,\mathrm{B})$ with $\mathcal{B}(\mathbb{R}^+,\mathrm{B})$ and $\overline{\mathcal{B}}(\mathbb{R}^+,\mathrm{B})$ respectively.

3.3. Asymptotic stability property for the solutions of the nonlinear initial value problem

At this point we recall the concepts of stability and asymptotic stability. Specifically, a solution $\tilde{\psi}(t)$ of the initial value problem (1) is Lyapunov stable if for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$ there exists $\delta := \delta(\varepsilon, t_0) > 0$ such that if $\psi(t)$ is a solution of the initial value problem (1) and $\|\psi(t_0) - \tilde{\psi}(t_0)\|_{\mathrm{B}} < \delta$ then $\|\psi(t) - \tilde{\psi}(t)\|_{\mathrm{B}} < \varepsilon$, for all $t \geq t_0$. Moreover, a solution $\tilde{\psi}(t)$ of the initial value problem (1) is asymptotically stable if it is Lyapunov stable and if for every $t_0 \in \mathbb{R}^+$ there exists $\delta := \delta(t_0) > 0$ such that if $\psi(t)$ is a solution of the initial value problem (1) and $\|\psi(t_0) - \tilde{\psi}(t_0)\|_{\mathrm{B}} < \delta$ then $\lim_{t\to+\infty} \|\psi(t) - \tilde{\psi}(t)\|_{\mathrm{B}} = 0$.

An asymptotic stability property for the solutions of the nonlinear initial value problem (1) is established in the following proposition.

Proposition 3.7. *Under the same assumptions as in the Propositions* 3.2 *and* 3.6 *respectively, the corresponding solution* $\psi(t)$ *of the nonlinear initial value problem* (1) *satisfy the asymptotic stability property* $\lim_{t\to+\infty} \|\psi(t) - \tilde{\psi}(t)\|_{\mathrm{B}} = 0$.

Proof. Let Z and Ξ be the operators defined in the proofs of Propositions 3.2 and 3.6 respectively. We then set $\Lambda = Z$ or $\Lambda = \Xi$ and observe that the corresponding solutions are fixed points of these operators. Also, if Φ is the corresponding Nemytskii operator of the nonlinear function F appearing in equation (1), then $\Phi\psi(t-s) = F(t-s, \psi(t-s))$ holds.

Then, making use of the triangle inequality and (2), we have:

$$\begin{aligned}
&\|\psi(t) - \tilde{\psi}(t)\|_{\mathrm{B}} = \|\Lambda\psi(t) - \Lambda\tilde{\psi}(t)\|_{\mathrm{B}} \\
&= \left\| e^{-tS}\psi(t_0) + \int_0^t e^{-sS} F(t-s, \psi(t-s))\, d\lambda^1(s) \right. \\
&\qquad \left. - e^{-tS}\tilde{\psi}(t_0) - \int_0^t e^{-sS} F(t-s, \tilde{\psi}(t-s))\, d\lambda^1(s) \right\|_{\mathrm{B}} \\
&\leq \left\| e^{-tS}\left(\psi(t_0) - \tilde{\psi}(t_0)\right) \right\|_{\mathrm{B}} \\
&\qquad + \left\| \int_0^t e^{-sS} \left(F(t-s, \psi(t-s)) - F(t-s, \tilde{\psi}(t-s))\right) d\lambda^1(s) \right\|_{\mathrm{B}} \\
&\leq c e^{-at} \|\psi(t_0) - \tilde{\psi}(t_0)\|_{\mathrm{B}} \\
&\qquad + c \int_0^t e^{-as} \|F(t-s, \psi(t-s)) - F(t-s, \tilde{\psi}(t-s))\|_{\mathrm{B}}\, d\lambda^1(s) \\
&\leq c e^{-at} \|\psi(t_0) - \tilde{\psi}(t_0)\|_{\mathrm{B}} + c \int_0^t e^{-as} \eta \, \|\psi(t-s) - \tilde{\psi}(t-s)\|_{\mathrm{B}}\, d\lambda^1(s) \\
&= c \|\psi(t_0) - \tilde{\psi}(t_0)\|_{\mathrm{B}} e^{-at} + \int_0^t c\eta e^{-a(t-s)} \|\psi(s) - \tilde{\psi}(s)\|_{\mathrm{B}}\, d\lambda^1(s).
\end{aligned}$$

Applying Grönwall inequality in the last expression we have:

$$\begin{aligned}
&\|\psi(t)-\tilde{\psi}(t)\|_{\mathrm{B}}\\
&\leq c\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\\
&\quad+\int_0^t c||\psi(t_0)-\tilde{\psi}(t_0)||_{\mathrm{B}}e^{-as}c\eta e^{-a(t-s)}e^{\int_s^t c\eta e^{-a(t-r)}dr}d\lambda^1(s)\\
&= c\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\\
&\quad+c^2\eta\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\int_0^t e^{c\eta e^{-at}\int_s^t e^{ar}dr}d\lambda^1(s)\\
&= c\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\\
&\quad+c^2\eta\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}e^{\frac{c\eta}{a}}\int_0^t e^{-\frac{c\eta}{a}e^{-a(t-s)}}d\lambda^1(s)
\end{aligned}$$

Then, with a change of variable $\omega=-\frac{c\eta}{a}e^{-a(t-s)}$ in the integral, and using the inequality $e^x\leq 1+xe^x$, for every $x\in\mathbb{R}$ from calculus, we have:

$$\begin{aligned}
&\|\psi(t)-\tilde{\psi}(t)\|_{\mathrm{B}}\\
&\leq c\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}+c^2\eta\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}e^{\frac{c\eta}{a}}\frac{1}{a}\int_{-\frac{c\eta}{a}e^{-at}}^{-\frac{c\eta}{a}}\frac{e^{\omega}}{\omega}d\omega\\
&\leq c\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}+c^2\eta\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}e^{\frac{c\eta}{a}}\frac{1}{a}\int_{-\frac{c\eta}{a}e^{-at}}^{-\frac{c\eta}{a}}\frac{e^{\omega}}{|\omega|}d\omega\\
&\leq c||\psi(t_0)-\tilde{\psi}(t_0)||_{\mathrm{B}}e^{-at}\\
&\quad+c^2\eta\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}e^{\frac{c\eta}{a}}\frac{1}{a}\int_{-\frac{c\eta}{a}e^{-at}}^{-\frac{c\eta}{a}}\frac{1+\omega e^{\omega}}{|\omega|}d\omega\\
&= c\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\\
&\quad-\frac{c^2\eta}{a}e^{\frac{c\eta}{a}}\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\int_{-\frac{c\eta}{a}e^{-at}}^{-\frac{c\eta}{a}}\left(\frac{1}{\omega}+e^{\omega}\right)d\omega\\
&= c\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\\
&\quad-\frac{c^2\eta}{a}e^{\frac{c\eta}{a}}\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\left(at+e^{-\frac{c\eta}{a}}-e^{-\frac{c\eta}{a}e^{-at}}\right)\\
&= c\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\\
&\quad-c^2\eta e^{\frac{c\eta}{a}}\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,te^{-at}-\frac{c^2\eta}{a}\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\\
&\quad+\frac{c^2\eta}{a}e^{\frac{c\eta}{a}}e^{-\frac{c\eta}{a}e^{-at}}\,\|\psi(t_0)-\tilde{\psi}(t_0)\|_{\mathrm{B}}\,e^{-at}\longrightarrow 0,
\end{aligned}$$

as $t\to+\infty$, and the result is complete. □

4. Application

The evolution of a highly ionized gas, which is a mixture of two fluids (electrons and ions), is considered, i.e. the interaction of two fluids with the electromagnetic field. In this case the evolution of the plasma can be described by the system of equations:

$$\begin{cases} nm_1 \frac{\partial}{\partial t}\upsilon_1 = \nu_1\Delta\upsilon_1 - m_1 n(\upsilon_1\cdot\nabla)\upsilon_1 - en(E+\upsilon_1\times B) - R - \nabla p_1 \\ nm_2 \frac{\partial}{\partial t}\upsilon_2 = \nu_2\Delta\upsilon_2 - m_2 n(\upsilon_2\cdot\nabla)\upsilon_2 + ezn(E+\upsilon_2\times B) + R - \nabla p_2, \end{cases} \tag{9}$$

with $\mathrm{div}(\upsilon_1) = \mathrm{div}(\upsilon_2) = 0$, $\frac{\partial}{\partial t}B = -\mathrm{curl}(E)$, $\mathrm{div}(B) = \mathrm{div}(E) = 0$, and $\frac{\partial}{\partial t}\varepsilon_0 E = \mathrm{curl}(\mu_0^{-1}B) - ne(z\upsilon_2 - \upsilon_1)$. Here, for $j = 1,2$ respectively: υ_j is the macroscopic fluid velocity, p_j is the thermal pressure, B is the magnetic flux density, E is the electric field, m_j is the mass of a particle, ν_j is the kinematic viscosity, n is the number density, e is the elementary charge, z is the charge number, ε_0 is the vacuum dielectric constant, μ_0 is the vacuum permeability and $R = -\zeta(\upsilon_2 - \upsilon_1)$, with constant $\zeta > 0$, represents the momentum transfer between the two components of the plasma under consideration.

Following Giga and Yoshida [13] and [14], the above system (9) (Navier-Stokes-Maxwell system) can be written in the evolution equation form.

Let Ω be a smoothly bounded domain in $\mathbb{R}^3$ with boundary conditions $\upsilon_1 = \upsilon_2 = 0$, $E\times n = 0$ on $\partial\Omega$, where n is the normal vector to $\partial\Omega$.

Applying projection operator $P : L^2(\Omega) \to H_0$ to both sides the above equations gives, in dimensionless analysis form, the next evolution equations

$$\frac{d}{dt}\upsilon_j = -A\upsilon_j - P(\upsilon_j\cdot\nabla)\upsilon_j + (-1)^j\left(P(E+\upsilon_j\times B) + R\right),$$

for almost every $t \geq 0$, $j = 1,2$.

Consider the space $H_0 := \left\{\upsilon \in L^2(\Omega) : div(\upsilon) = 0 \text{ on } \Omega, \upsilon\cdot n = 0 \text{ on } \partial\Omega\right\}$, where $L^2(\Omega)$ is the Hilbert space of square integrable functions on Ω equipped with the usual inner product $<,>$ and norm $||\cdot||$.

Also, consider the space $H := H_0 \times H_0 \times L^2(\Omega) \times L^2(\Omega)$ equipped with the usual inner product $< v, \tilde{v} > := \int_\Omega v\cdot\tilde{v}dx$, where $v = (\upsilon_1, \upsilon_2, B, E)$, $\tilde{v} = (\tilde{\upsilon}_1, \tilde{\upsilon}_2, \tilde{B}, \tilde{E})$ are elements of H, and $\upsilon_1, \upsilon_2 \in H_0$, $B, E \in L^2(\Omega)$ and $||u||$ is the norm of $u \in H$.

As usual $A := -P\Delta$ is Stokes operator, which is positive and selfadjoint on H_0, with domain $\mathrm{D}(A) := \left\{v \in W^2(\Omega)\cap H_0 : v = 0 \text{ on } \partial\Omega\right\}$, where $W^2(\Omega)$ is the corresponding Sobolev space and Δ is the Laplacian.

The problem (9) can also be written in the following form of the differential evolution equation

$$\dot{u} = -Lu + Nu + Ru,$$

where $-Nu := (P(\upsilon_1 \cdot \nabla)\upsilon_1, P(\upsilon_2 \cdot \nabla)\upsilon_2, 0, 0) + (P(\upsilon_1 \times B), -P(\upsilon_2 \times B), 0, 0)$, $Ru := (-R, -R, 0, 0)$ and $Lu := (L_1 + L_2)\upsilon$.

Here $\mathrm{D}(L) = \mathrm{D}(L_1) \cap \mathrm{D}(L_2)$, where $\mathrm{D}(L_1) = \mathrm{D}(A) \times \mathrm{D}(A) \times L^2(\Omega) \times L^2(\Omega)$ and $\mathrm{D}(L_2) = \big\{(\upsilon_1, \upsilon_2, B, E) : \upsilon_1, \upsilon_2 \in H_0, \mathrm{curl}(E), \mathrm{curl}(B) \in L^2(\Omega), E \times n = 0 \text{ on } \partial\Omega\big\}$. Also, $L_1 u := (A\upsilon_1, A\upsilon_2, 0, 0)$ with $\upsilon = (\upsilon_1, \upsilon_2, B, E)$, and $L_2 u := (PE, -PE, \mathrm{curl}(E), -\mathrm{curl}(E) + \upsilon_2 - \upsilon_1)$.

Considering the initial value $u(0) = u_0 \in \mathrm{D}(L)$, with $u_0 = (\upsilon_{1,0}, \upsilon_{2,0}, B_0, E_0)$, then we have the corresponding Cauchy problem of the type (1).

Assuming the conditions of exponential stability and small constant in the Lipschitz condition, we can then apply the previous results and propose new generalized solutions for the magnetohydrodynamic system.

REFERENCES

[1] W. Arendt and C. J. K. Batty, *Almost Periodic Solutions of First- and Second-Order Cauchy Problems*, J. Differ. Equations 137 (1997), 363–383.

[2] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics Vol.96 (second edition), Birkhäuser 2011.

[3] D. Arsénio and I. Gallagher, *Solutions of Navier-Stokes-Maxwell systems in large energy spaces*, Trans. Amer. Math. Soc. 373 (2020), 3853–3884.

[4] E. S. Athanasiadou, V. F. Dionysatos, P. N. Koumantos and P. K. Pavlakos, *A semigroup approach to functional differential evolution equations* Math. Methods Appl. Sci. 37(2) (2014), 217–222.

[5] E. S. Athanasiadou, V. F. Dionysatos, P. N. Koumantos and P. K. Pavlakos, *Almost periodic solutions of Navier-Stokes-Ohm type equations in Magnetohydrodynamics*, In: Daras, Nicholas J. (Ed.) Applications of Mathematics and Informatics to Science and Engineering. Springer Optimization and Its Applications Vol. 91, pp.43–57 Springer 2014.

[6] C. J. K. Batty, W. Hutter and F. Räbiger, *Almost Periodicity of Mild Solutions of Inhomogeneous Periodic Cauchy Problems*, J. Differ. Equations 156 (1999), 309–327.

[7] R. Benkhalti, B. Es-Sebbar, K. Ezzinbi, *On a Bohr-Neugebauer Property for some Almost Automorphic Abstract Delay Equations*, J. Integral Equations Appl. 30 (3) (2018), 313-345.

[8] M. Bokalo and A. Lorenzi, *Linear First-Order Evolution Problems Without Initial Conditions*, Milan J. Math. 77 (2009), 437–494.

[9] V. V. Chepyzhov and M. I. Vishik, *Non-autonomous evolution equations with almost periodic symbols*, Rend. Semin. Mat. Fis. Milano 62 (1992), 185–213.

[10] Ju. L. Daleckii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Spaces*. Translations of Mathematical Monographs Vol. 43, American Mathematical Society, Providence, Rhode Island 1974.

[11] K. -J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*. Springer-Verlag, New York, Inc. 2000.

[12] H. Fan and Y. Li, *Analyticity and exponential stability of semigroups for the elastic systems with structural damping in Banach spaces*, J. Math. Anal. Appl. 410 (2014), 316–322.

[13] Y. Giga and Z. Yoshida, *On the Ohm-Navier-Stokes system in MHD*, J. Math. Phys. **24**(12) (1983), 2860–2864.

[14] Y. Giga and Z. Yoshida, *On the equations of the two-component theory in magnetohydrodynamics* Commun. Partial Differ. Equations 9 (1984), 503–522.

[15] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*. American Mathematical Society, Colloquium Publications, Volume 31, AMS 1957.

[16] M. W. Hirsch, *Differential Topology*, Corrected fifth printing. Springer-Verlag, New York 1994.

[17] M. Al Horani, A. Favini and H. Tanabe, *Parabolic First and Second Order Differential Equations*, Milan J. Math. 84 (2016), 299–315.

[18] P. N. Koumantos, *Nonlinear evolution equations in Minkowski space-time*, Rep. Math. Phys. 84 (2) (2019), 187–199.

[19] P. N. Koumantos, *Order topology in Minkowski space and applications*, Rep. Math. Phys. 87 (2) (2021), 209–223.

[20] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*. Cambridge Univ. Press, Cambridge 1982.

[21] W. A. J Luxemburg and A. C. Zaanen, *Riesz Spaces Vol. I*. North-Holland Publishing Company, Amsterdam, London 1971.

[22] A. Pankov, *Almost periodic functions, Bohr compactification, and differential equations*, Rend. Semin. Mat. Fis. Milano 66 (1998), 149–158.

[23] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag New York, Berlin, Heidelberg, Tokyo 1983.

[24] J. Prüss, *On semilinear evolution equations in Banach spaces*, J. Reine Angew. Math. 0303_0304 (1978), 144–158.

[25] J. Prüss, *Periodic solutions of semilinear evolution equations*, Nonlinear Anal. Theory Methods Appl. 3 (5) (1979), 601–612.

[26] V. Q. Phóng, *On Stability of* C_0*-semigroups*, Proc. Am. Math. Soc. 129 (10) (2001), 2871–2879.

[27] W. M. Ruess and V. Q. Phóng, *Asymptotically Almost Periodic Solutions of Evolution Equations in Banach Spaces*, J. Differ. Equations 122 (1995), 282–301.

[28] E. Schechter, *Handbook of Analysis and Its Foundations*. Academic Press, San Diego, London, Boston, New York, Sydney, Tokyo, Toronto 1997.

[29] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost*

Periodic Solutions. Springer-Verlag, New York Heidelberg Berlin 1975.

[30] K. Yosida, *Functional Analysis*, Sixth Edition, Springer, Berlin, Heidelberg, New York 1980.

P. KOUMANTOS
Institute of Applied and Computational Mathematics
Foundation of Technology and Research (IACM-FORTH)
70013 Heraklion, Crete, Greece
e-mail: pnkoumantos@gmail.com
e-mail: pkoumantos@iacm.forth.gr