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ON HADAMARD ALGEBRAS

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Topological algebras of sequences of complex numbers are introduced, endowed with a Hadamard product type. The complex homomorphisms on these algebras are characterized, and units, prime cyclic ideals, prime closed ideals, and prime minimal ideals, discussed. Existence of closed and maximal ideals are investigated, and it is shown that the Jacobson and nilradicals are both trivial.

1. Introduction and topological preliminaries.

Nowadays there is an increasing attention in research in Fractional Calculus applied to certain classes of analytic functions. Fractional operators such as Riemann - Liouville, Weyl and Kober, and their various generalizations among others have successfully been applied in obtaining characterization properties, coefficient estimates, distortion inequalities, convolution structures for various subclasses of analytic functions, etc., ([2], [4], [5], [6]). Recently various distortion theorems were established on certain subclasses of the class A of functions $f(z)$ defined by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0,\ n \in \mathbb{N} \tag{1}$$

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which are analytic in the complex unit disk. Moreover, if $f(z)$ is given by (1) and belongs to one of these subclasses, and

$$g(z) = z - \sum_{n=2}^{\infty} b_n\, z^n$$

belongs to another subclass, then their Hadamard product

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n\, b_n\, z^n$$

belongs to a third such subclass (see [1]). More generally, let us consider the set $\mathcal{A}_0$ of complex sequences $a = (a_n)_{n\geq 0}$ such that the number $\rho(a) = \overline{\lim}\, |a_n|^{1/n}$ is finite [1]. Given $a = (a_n)_{n\geq 0}$, $b = (b_n)_{n\geq 0} \in \mathcal{A}_0$, $\zeta \in \mathbb{C}$ and a sequence $\varepsilon = (\varepsilon_n)_{n\geq 0}$ such that $|\varepsilon_n| = 1$ for each $n \geq 0$ we define $\zeta \cdot a + b \in \mathcal{A}_0$ and $a \odot_\varepsilon b \in \mathcal{A}_0$ as

$$\zeta \cdot a + b = (\zeta\, a_n + b_n)_{n\geq 0} \qquad \text{and} \qquad a \odot_\varepsilon b = (\varepsilon_n\, a_n\, b_n)_{n\geq 0}.$$

Proposition 1. *The set $(\mathcal{A}_0, +, \cdot, \odot_\varepsilon)$ is an abelian - unit - complex algebra.*

Proof. With the above notation, will show that if $a,\ b \in \mathcal{A}_0$ and ζ is complex then

$$\rho(a + b) \leq \rho(a) + \rho(b), \tag{2}$$

$$\rho(\zeta \cdot a) = \rho(a) \quad \text{if} \quad \zeta \neq 0, \tag{3}$$

$$\rho(a \odot_\varepsilon b) \leq \rho(a)\, \rho(b). \tag{4}$$

For (2) we observe that $|a_n + b_n|^{1/n} \leq |a_n|^{1/n} + |b_n|^{1/n}$ for all $n \in \mathbb{N}$ and hence

$$\overline{\lim}\, |a_n + b_n|^{1/n} \leq \overline{\lim}\left[|a_n|^{1/n} + |b_n|^{1/n}\right] \leq \overline{\lim}\, |a_n|^{1/n} + \overline{\lim}\, |b_n|^{1/n}.$$

Since $\overline{\lim}\, |a_n\, b_n|^{1/n} \leq \overline{\lim}\, |a_n|^{1/n}\, \overline{\lim}\, |b_n|^{1/n}$ the inequality (4) holds. Moreover, if $\{s_n\}_{n\geq 0}$, $\{t_n\}_{n\geq 0}$ are sequences of non negative numbers, the first convergent to a finite limit and $\overline{\lim}\, t_n < +\infty$ then $\overline{\lim}\,(s_n\ t_n) = \lim s_n\ \overline{\lim}\, t_n$. Thus (3) follows. Now it is clear that we get a complex abelian algebra which has the element $\varepsilon^* = (\overline{\varepsilon_n})_{n\geq 0}$ as unit. $\square$

Remark 1. If r is nonnegative let $An(r)$ be the set of analytic functions in the open circle $D(0,\ 1/r)$ with center zero and radius $1/r$. In particular, $An(0)$ denotes the set of entire functions. Since the family $\{An(r)\}_{r\geq 0}$ is increasing with respect to inclusion we can write $\mathcal{A}_0 = \lim An(r)$.

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[1] Since we are concerned with superior limits the evaluation of nth radicals causes no trouble in the case $n = 0$.

Remark 2. We shall consider $An(r)$ with the topology of uniform convergence on compact subsets, i.e. the compact open topology. On the other hand, let $\mathcal{A}_0(r)$ be the set of those elements $a \in \mathcal{A}_0$ such that $\rho(a) \leq r$. There is a natural map

$$\Gamma_r : \mathcal{A}_0(r) \to An(r),$$

$$\Gamma_r(a)(z) = \sum_{n=0}^{\infty} a_n \; z^n,$$

which is clearly injective. If $\rho\big[(f^{(n)}(0)/n!)_{n\geq 0}\big] > r > 0$ for $f \in An(r)$, we would choose a number t such that $1/\big(\rho\big[(f^{(n)}(0)/n!)_{n\geq 0}\big]\big) < t < 1/r$ and $|z|\,\rho\big[(f^{(n)}(0)/n!)_{n\geq 0}\big] > 1$ if $|z| = t$, which contradicts the absolute convergence of f on compact subsets of $D(0, 1/r)$. Hence $\rho\big[(f^{(n)}(0)/n!)_{n\geq 0}\big] \leq r$ and $\Gamma_r\big[(f^{(n)}(0)/n!)_{n\geq 0}\big] = f$. Analogously, $\rho\big[(f^{(n)}(0)/n!)_{n\geq 0}\big] = 0$ if f is an entire function [2]. Now, given $(a_n)_{n\geq 0} \subseteq \mathcal{A}_0(r)$, $a \in \mathcal{A}_0(r)$ we shall write $a_n \to a$ if and only if $\Gamma_r(a_n) \to \Gamma_r(a)$ in $An(r)$, i.e. $\mathcal{A}_0(r)$ has the topology with respect to which Γ_r becomes an homeomorphism. In particular, $\mathcal{A}_0(r)$ is a complete space as can be seen by standard arguments.

Remark 3. If $r < s$ the topology of $An(r)$ is stronger than that induced on it by $An(s)$. In particular, the inclusion $\iota_{r,s} : An(r) \hookrightarrow An(s)$ is continuous. Of course, the same holds if we replace $An(r)$ and $An(s)$ by $\mathcal{A}_0(r)$ and $\mathcal{A}_0(s)$ respectively.

Remark 4. Given an unbounded strictly increasing sequence $\{r_n\}_{n\geq 1}$ of non negative numbers we have $\mathcal{A}_0 = \lim \mathcal{A}_0(r_n)$. We shall consider $\mathcal{A}_0$ as a countable union space in the sense of Gelfand and Shilov [3]. So, a sequence $\{a_k\}_{k\geq 1}$ is said to converge to a in $\mathcal{A}_0$ if all the a_k and a belong to some particular $\mathcal{A}_0(r_n)$ and $\{a_k\}_{k\geq 1}$ converges to a in $\mathcal{A}_0(r_n)$. As usual, this concept of convergence is independent of the sequence $\{r_n\}_{n\geq 1}$. Indeed, the space $\mathcal{A}_0$ has a Fréchet structure [3].

Theorem 1. *Let $a_n = (a_{nm})_{m\geq 0}$, $n = 0, 1, \ldots$ be a sequence of elements of $\mathcal{A}_0$. The following assertions are equivalent:*

(i) $a_n \to 0$ *in* $\mathcal{A}_0$.

[2] In particular, we can improve (2) by observing that $\rho(a+b) \leq \max\{\rho(a),\ \rho(b)\}$ if $a,\ b \in \mathcal{A}_0$.

[3] The same conclusion holds for the space $\mathcal{U}_0 = \lim An(r)$ of analytic functions at zero.

(ii) $\Gamma_r(a_n) \to 0$ *in* $An(r)$, $r \geq 0$.

(iii) *There is* $r > 0$ *such that* $\rho(a_n) \leq r$ *for* $n \geq 0$ *and for all* $\delta > 0$ *there exist* n_0 *such that* $|a_{nm}| \leq \delta\, r^m$ *if* $n \geq n_0$ *and* $m \in \mathbb{N}_0$.

Proof. Clearly (i) $\Leftrightarrow$ (ii). Moreover, if (i) holds there is $r > 0$ such that $\rho(a_n) \leq r$ for $n \geq 0$ and $\Gamma_r(a_n) \to 0$ in $\mathcal{A}_0(r)$. So, given $\delta > 0$ and $0 < s < 1/r$ there exist n_0 such that $|\Gamma_r(a_n)(z)| \leq \delta$ if $|z| = s$ and $n \geq n_0$. If $m \geq 0$ and $n \geq n_0$ we get

$$|a_{nm}| = \left| \frac{1}{2\pi s^m} \int_{-\pi}^{\pi} \Gamma_r(a_n)(s\ \exp(ix))\ \exp(-imx)\, dx \right|$$

$$\leq \frac{1}{2\pi s^m} \int_{-\pi}^{\pi} |\Gamma_r(a_n)(s\ \exp(ix))|\, dx \leq\ \delta\, s^{-m}.$$

Letting $s \to 1/r$ on the right, we see that (iii) holds. Finally, let us assume (iii), we show that (ii) holds. For, let $\xi > 0$, $(m_n)_{n \geq 1}$ be an increasing sequence of positive integers such that $\sup\limits_{m \geq m_n} |a_{nm}|^{1/m} \leq r$ for $n = 1, 2, \ldots$ Therefore, if $|z| \leq s < 1/r$, we can write

$$\left|\Gamma_r(a_n)(z)\right| \leq \sum_{m=0}^{m_n - 1} |a_{nm}|\, |z|^m + \sum_{m=m_n}^{\infty} (rs)^m. \tag{5}$$

There is $n_0 \in \mathbb{N}$ such that

$$\sum_{m=m_n}^{\infty} (rs)^m \leq \xi/2 \quad \text{and} \quad |a_{nm}| \leq \frac{(1 - rs)\xi}{2}\, r^m \tag{6}$$

for $m \in \mathbb{N}_0$ and $n \geq n_0$. Using (5) and (6), we have

$$\left|\Gamma_r(a_n)(z)\right| \leq \frac{(1 - rs)\xi}{2} \sum_{m=0}^{m_n - 1} r^m\, s^m + \frac{\xi}{2} \leq \xi$$

for $n \geq n_0$ and $|z| \leq s < 1/r$. Since s is arbitrary, therefore $\Gamma_r(a_n) \to 0$ uniformly on compact subsets of the circle with center zero and radius $1/r$. $\square$

Proposition 2. $(\mathcal{A}_0, +, \cdot, \odot_\varepsilon)$ *is a topological algebra.*

Proof. Let $r, s \in \mathbb{R}_{\geq 0}$, $a \in \mathcal{A}_0(r)$, $b \in \mathcal{A}_0(s)$ and W be a neighborhood of $a+b$ in $\mathcal{A}_0$. There are positive numbers δ_j and compact subsets K_j of $D(0, 1/(r+s))$, $1 \leq j \leq n$, such that

$$\bigcap_{j=1}^{n} \left\{ h \in An(r+s) : \sup_{K_j} |\Gamma_r(a) + \Gamma_s(b) - h| < \delta_j \right\} \subseteq \Gamma_{r+s}(W).$$

But $An(r) + An(s) \subseteq An(r+s)$ and if we write

$$U = \bigcap_{j=1}^{n} \left\{ h \in An(r) \ : \ \sup_{K_j} |\Gamma_r(a) - h| < \frac{\delta_j}{2} \right\},$$

$$V = \bigcap_{j=1}^{n} \left\{ h \in An(s) : \ \sup_{K_j} |\Gamma_s(b) - h| < \frac{\delta_j}{2} \right\},$$

then $(\Gamma_r)^{-1}(U)$ and $(\Gamma_s)^{-1}(V)$ are neighborhoods of a in $\mathcal{A}_0(r)$ and of b in $\mathcal{A}_0(s)$ such that $(\Gamma_r)^{-1}(U) + (\Gamma_s)^{-1}(V) \subseteq W$. Now, let $\gamma \in \mathbb{C}$, $c \in \mathcal{A}_0(t)$ and S be a neighborhood of $\gamma \cdot c$ in $\mathcal{A}_0(t)$. Let $\{C_k\}_{1 \leq k \leq m}$ a be family of compact subsets of $D(0, 1/t)$ such that

$$\bigcap_{k=1}^{m} \left\{ h \in An(t) : \ \sup_{C_k} |\Gamma_t(\gamma \cdot c) - h| < \eta_k \right\} \subseteq \ \Gamma_t(S)$$

for certain positive numbers $\eta_1, \ldots, \eta_m$. If $\beta \in \mathbb{C}$ and $b \in \mathcal{A}_0(t)$, then we have

$$|\Gamma_t(\gamma \cdot c) - \Gamma_t(\beta \cdot b)| \leq |\beta - \gamma| \, [|\Gamma_t(c) - \Gamma_t(b)| + |\Gamma_t(c)|] + |\gamma(\Gamma_t(c) - \Gamma_t(b))|.$$

Given ε any positive number, let

$$T = \bigcap_{k=1}^{m} \left\{ h \in An(t) : \ \sup_{C_k} |\Gamma_t(c) - h| < \frac{\eta_k}{\varepsilon + |\gamma|} \right\} \quad \text{if } c = 0,$$

or let

$$\varepsilon = \frac{\min_k \ \eta_k}{2 \ \max_{\cup C_k} |\Gamma_t(c)|} \ ,$$

$$T = \bigcap_{k=1}^{m} \left\{ h \in An(t) : \ \sup_{C_k} |\Gamma_t(c) - h| < \frac{\eta_k}{2(\varepsilon + |\gamma|)} \right\} \quad \text{if } c \neq \ 0.$$

In both cases we get $D(\gamma, \varepsilon) \cdot (\Gamma_t)^{-1}(T) \subseteq S$, and the binary operation $\cdot$ is continuous. In order to show that the Hadamard product is bicontinuous, let $a_n \to a$ in $\mathcal{A}_0(r)$, $b_n \to b$ in $\mathcal{A}_0(s)$, where $a_n = (a_n^m)_{m\geq 0}$, $b_n = (b_n^m)_{m\geq 0}$, $a = (a^m)_{m\geq 0}$ and $b = (b^m)_{m\geq 0}$. Since $\rho(a) \leq r$ and $\rho(b) \leq s$ there are positive constants A, B such that $|a^m| \leq A\, r^m$ and $|b^m| \leq B\, s^m$ for $m \in \mathbb{N}_0$. Given $\delta > 0$, we can choose $\xi > 0$ such that $\xi(\xi + A + B) < \delta$. Furthermore, let N be a positive integer such that $|a_n^m - a^m| \leq \xi\, r^m$ and $|b_n^m - b^m| \leq \xi\, s^m$ if $n \geq N$ and $m \geq 0$. Then

$$
\begin{aligned}
(7) \qquad |(a_n \odot_\varepsilon b_n - a \odot_\varepsilon b)^m| &\leq |a_n^m - a^m|\, |b_n^m| + |a^m|\, |b_n^m - b^m| \\
&\leq \xi[|b_n^m|\, r^m + |a^m|\, s^m] \\
&\leq \xi[(|b^m| + \xi\, s^m)\, r^m + |a^m|\, s^m] \\
&\leq \xi[(\xi + A)\, s^m + B\, r^m]\, r^m.
\end{aligned}
$$

Since the Hadamard product is commutative we can assume that $r \leq s$, so that (7) becomes

$$|(a_n \odot_\varepsilon b_n - a \odot_\varepsilon b)^m| \leq \xi(\xi + A + B)\, (rs)^m \leq \delta\, (rs)^m$$

and the conclusion follows by Theorem 1. $\square$

2. Linear forms and Hadamard homomorphisms.

Proposition 3. *Let $\varphi : \mathcal{U}_0 \to \mathbb{C}$ be a linear form. Then φ is continuous if and only if for all $r \geq 0$ and all $s > r$ there is a positive constant $C_{r,s}$ such that $|\langle f, \varphi\rangle| \leq C_{r,s} \sup_{|z|\leq 1/s} |f(z)|$.*

Proof. The condition is obviously sufficient. Let us suppose the existence of $r \geq 0$, $s > r$ and of a sequence $\{f_n\} \subseteq An(r)$ such that

$$|\langle f_n, \varphi\rangle| > n^2 \sup_{|z|\leq 1/s} |f_n(z)|.$$

Since φ is linear each $f_n \neq 0$. Let

$$g_n(z) = n^{-1}\Big(\sup_{|z|\leq 1/s} |f_n(z)|\Big)^{-1} f_n(z), \quad n \geq 1, \ |z| < 1/s.$$

Then $\{g_n\} \subseteq An(s)$, and if $s < t$ we have $\sup_{|z|\leq 1/t} |g_n(z)| \leq 1/n$, i.e. $g_n \to 0$ in $An(s)$. But $|\langle g_n, \varphi\rangle| > n$ for all n and so φ is not continuous. $\square$

Lemma 1. *Let $0 < r < 1$, $a = (a_n)_{n\geq 0}$, $a \in \mathcal{A}_0(r)$ such that $|a_n| < 1$ for all n. Then the element $\varepsilon^* - a$ is invertible and $(\varepsilon^* - a)^{-1} = \sum_{m=0}^{\infty} a^m$.*

Proof. It will suffice to show that $\left(\sum_{m=0}^{M} a^m \right)_{M \geq 0}$ is a Cauchy sequence because $\mathcal{A}_0$ is a Fréchet space. For, let $a_{M,P} = \left(\sum_{m=M}^{M+P} (\varepsilon_n\, a_n)^m \right)_{n \geq 0}$, with $M, P \in \mathbb{N}_0$. We have $a_{M,P} \in \mathcal{A}_0(r)$ because $\mathcal{A}_0(r^m) \subseteq \mathcal{A}_0(r)$ for all m. Given $\delta > 0$ let N be a positive integer sufficiently large so that $|a_n| \leq r^n$ and $r^{n-1} \leq \delta(1-r)$ if $n \geq N$. On writing $\eta = \sup_{0 \leq n < N} |a_n|$ we can choose $M_0 \in \mathbb{N}$ such that $2\eta^m \leq \delta(1-\eta) r^N$ if $m \geq M_0$. Thus, if $n \geq N$, $P \geq 0$ we have

$$|a_{M,P}(n)| \leq \sum_{m=M}^{M+P} |a_n|^m \leq \sum_{m=M}^{M+P} r^{nm} \leq r^{nM} \sum_{k=0}^{P} r^{Nk} \leq \frac{r^{n+M-1}}{1-r} \leq \delta\, r^n.$$

If $0 \leq n < N$, $P \geq 0$, $M \geq M_0$ then

$$|a_{M,P}(n)| = \left| \frac{a_n^M [1 - (\varepsilon_n\, a_n)^{P+1}]}{1 - \varepsilon_n\, a_n} \right| \leq \frac{2|a_n|^M}{1 - |a_n|} \leq \frac{2\eta^M}{1-\eta} \leq \delta\, r^n.$$

By Th. 1 we deduce that $\left(\sum_{m=0}^{M} a^m \right)_{M \geq 0}$ is a Cauchy sequence and the result follows. $\square$

Proposition 4. *If r, s are positive then $\mathcal{A}_0(r)$ and $\mathcal{A}_0(s)$ are linearly homeomorphic.*

Proof. We set $\Phi_{r,s} : \mathcal{A}_0(r) \to \mathcal{A}_0(s)$, $\Phi_{r,s}[(a_n)_{n \geq 0}] = ((s/r)^n a_n)_{n \geq 0}$. Since $\rho[\Phi_{r,s}(a)] = s/r\, \rho(a) \leq s$ the function $\Phi_{r,s}$ is well defined and it is clearly linear. Indeed, let $a_n \to 0$ in $\mathcal{A}_0(r)$. If $\delta > 0$, there is N such that if $n \geq N$ and $m \in \mathbb{N}_0$ then $|a_{nm}| \leq \delta\, r^m$, i.e. $|(\Phi_{r,s}(a_n))_m| \leq \delta\, s^m$ and $\Phi_{r,s}(a_n) \to 0$ in $\mathcal{A}_0(s)$. Finally, we have $[\Phi_{r,s}]^{-1} = \Phi_{s,r}$. $\square$

Corollary 1. *Every complex valued Hadamard homomorphism is continuous.*

Proof. First assume that $0 < r < 1$. We shall denote U_r for the set of elements $a = (a_n)_{n \geq 0}$, $a \in \mathcal{A}_0(r)$, such that $|a_n| < 1$ for all n. Let us suppose that $|\chi(a)| > 1$ for a Hadamard homomorphism $\chi : \mathcal{A}_0 \to \mathbb{C}$ and $a \in U_r$. Since $\rho(a/\chi(a)) = \rho(a)$, then $a/\chi(a) \in \mathcal{A}_0(r)$ and $|a_n/\chi(a)| < 1$ for $n \geq 0$, i.e. $\varepsilon^* - a/\chi(a) \in \ker(\chi)$ and it is invertible, which is absurd. Thus $|\chi(a)| \leq 1$ if $a \in U_r$. We observe that $\mu(a) = \sup_{n \geq 0} |a_n|$ is finite if $a \in \mathcal{A}_0(r)$. Moreover, if $a \neq 0$ then $|a_n|/\mu(a) \leq 1$ for all n, i.e. $[r/\mu(a)] \cdot a \in U_r$, i.e. $|\chi(a)| \leq \mu(a)/r$. In particular, this inequality holds even if $a = 0$. By applying Th. 1 we see that

$\mu(a_n) \to 0$ if $a_n \to 0$ in $\mathcal{A}_0(r)$ and so $\chi(a_n) \to 0$. For the general case, if $a_n \to 0$ in $\mathcal{A}_0(r)$ we choose $0 < s < 1/r$. We can write

$$\chi(a_n) = \chi[(s^{-m})_{m\geq 0} \odot_\varepsilon \Phi_{r,rs}(\overline{\varepsilon}_m\, a_{nm})_{m\geq 0}]$$

$$= \chi[(s^{-m})_{m\geq 0}]\, \chi[\Phi_{r,rs}(\overline{\varepsilon}_m\, a_{nm})_{m\geq 0}].$$

Indeed, by Prop. 4 we know that $\Phi_{r,rs}(\overline{\varepsilon}_m\, a_{nm})_{m\geq 0} \to 0$ in $\mathcal{A}_0(rs)$, i.e. $\chi(a_n) \to 0$. $\square$

Remark 5. With the above notation, if $f \in An(r)$, $g \in An(s)$ it is natural to define $f \odot_\varepsilon g = \Gamma_{rs}[(\Gamma_r)^{-1} f \odot_\varepsilon (\Gamma_s)^{-1} g]$ (see (4) and Remark 2). If $r \leq s$ we already know that $\Gamma_s|_{\mathcal{A}_0(r)} = \Gamma_r$, so that $f \odot_\varepsilon g$ is a well defined element of $\mathcal{U}_0$.

Corollary 2. *For every non zero Hadamard homomorphism $\varkappa : \mathcal{U}_0 \to \mathbb{C}$ there is a unique $p \in \mathbb{N}_0$ such that $\langle f, \varkappa\rangle = \varepsilon_p\, f^{(p)}(0)/p!$, $f \in \mathcal{U}_0$.*

Proof. Given $f \in A_0$, let $r > 0$ such that $a \in An(r)$. Since its Taylor expansion $f(z) = \sum_{n=0}^{\infty} f^{(n)}(0)/n!\, z^n$ is convergent on compact subsets of $D(0, 1/r)$, we have

$$\langle f, \varkappa\rangle = \sum_{n=0}^{\infty} f^{(n)}(0)/n!\, \langle \varkappa, z^n\rangle. \tag{8}$$

But $z^k \odot_\varepsilon z^h = \varepsilon_k\, \delta_{kh}\, z^h$, where δ denotes Kronecker's delta function. Thus

$$\langle \varkappa, z^k\rangle\, \langle \varkappa, z^h\rangle = \langle \varkappa, z^k \odot_\varepsilon z^h\rangle = \varepsilon_k\, \delta_{kh}\, \langle \varkappa, z^h\rangle,$$

so for $k, h \in \mathbb{N}_0$ we have $\langle \varkappa, z^k\rangle\, \langle \varkappa, z^h\rangle = 0$ if $k \neq h$ and $\langle \varkappa, z^k\rangle^2 = \varepsilon_k\, \langle \varkappa, z^k\rangle$. Since $\varkappa \neq 0$ there must be a unique $p \in \mathbb{N}_0$ such that $\langle \varkappa, z^k\rangle = \varepsilon_p\, \delta_{kp}$, $k \in \mathbb{N}_0$ and the conclusion follows from (8). $\square$

Remark 6. Let $\chi_p : \mathcal{A}_0 \to \mathbb{C}$ be the complex homomorphism $\chi_p(a) = \varepsilon_p\, a_p$, $p \geq 0$. Then $\ker(\chi_p)$ is a maximal closed ideal of $\mathcal{A}_0$. Indeed, it is cyclic and $\ker(\chi_p) = \langle((1 - \delta_{pn})\, \overline{\varepsilon}_n)_{n\geq 0}\rangle$.

Corollary 3. *The Hadamard ring $(\mathcal{U}_0, +, \odot_\varepsilon)$ has a trivial Jacobson radical.*

3. On Hadamard units.

Theorem 2. *An element $a \in \mathcal{A}_0$ is a unit, i.e. $a \in \mathbb{U}(\mathcal{A}_0)$, if and only if $a_n \neq 0$ for all n and $\underline{\lim}\, |a_n|^{1/n} > 0$.*

Proof. If the condition holds we write $b = (\overline{\varepsilon}_n/(\varepsilon_n\, a_n)_{n\geq 0}$. Since

$$\overline{\lim}\, |b_n|^{1/n} = \overline{\lim}\, |a_n|^{-1/n} = (\underline{\lim}\, |a_n|^{1/n})^{-1} < +\infty,$$

it is clear that b is the multiplicative inverse of a. Reciprocally, if the equation $a \odot_\varepsilon b = \varepsilon^*$ is solvable in $\mathcal{A}_0$ it is immediate that $a_n \neq 0$ for all n. If $\underline{\lim}\, |a_n|^{1/n} = 0$ there is an infinite sequence (n_k) such that $|a_{n_k}|^{1/n_k} \to 0$. But it yields $\overline{\lim}\, |b_n|^{1/n} = \overline{\lim}\, |a_n|^{-1/n} = +\infty$, a contradiction. $\square$

Example 1. Let $f \in \mathcal{U}_0$ be given as $f(z) = z^{-1} \log(1-z)^{-1}$. If $|\omega| = 1$ and $\varepsilon = (\omega^n)_{n\geq 1}$ then $f \in \mathbb{U}(\mathcal{U}_0)$, its inverse being $f^{-1}(z) = 1/(1 - z/\omega^2)^2$. Here we have $f \in An(1)$ and $\Gamma_1(\varepsilon^*)(z) = \omega/(w-z)$ is the Hadamard neutral product element, i.e. $g \odot_\varepsilon \Gamma_1(\varepsilon^*) = g$ for all $g \in \mathcal{U}_0$.

Proposition 5. *Let $a \in \mathcal{A}_0$, $M_a(b) = a \odot_\varepsilon b$, $b \in \mathcal{A}_0$. If $\rho(a) > 0$ and $s \geq 0$ then*

(i) $M_a : \mathcal{A}_0(s) \to \mathcal{A}_0(\rho(a)\, s)$ is injective if and only if $a_n \neq 0$ for all $n \geq 0$.
(ii) It is surjective if and only if $a \in \mathbb{U}(\mathcal{A}_0$.

Proof. If $a_k = 0$ for some $k \geq 0$ then $M_a\big[(\delta_{nk})_{n\geq\, 0}\big] = 0$ and M_a is not injective. Since the condition is obviously sufficient (i) holds. On the other hand, let $a \in \mathbb{U}(\mathcal{A}_0)$. Given $c \in \mathcal{A}_0(\rho(a)\, s)$ let b be the Hadamard inverse of a. Thus $\rho(b \odot_\varepsilon c) \leq (\overline{\lim}\, |c_n|^{1/n})/\rho(a) \leq s$, $M_a(b \odot_\varepsilon c) = c$ and the condition in (ii) is sufficient. If $M_a : \mathcal{A}_0(s) \to \mathcal{A}_0(\rho(a)\, s)$ is surjective it is immediate that $a_n \neq 0$ for all n. Let us suppose $\underline{\lim}\, |a_n|^{1/n} = 0$ and let (n_j) be an strictly increasing subsequence of $\mathbb{N}$ such that $\lim |a_{n_j}|^{1/n_j} = 0$. If $s > 0$ there is $b \in \mathcal{A}_0(s)$ such that $M_a(b) = ((rs)^n)_{n\geq 0}$. So $|b_{n_j}|^{1/n_j} = (rs)/|a_{n_j}|^{1/n_j}$, i.e. $|b_{n_j}|^{1/n_j} \to +\infty$. Nevertheless $\rho(b) \leq s$, i.e. there is $N \in \mathbb{N}$ such that $|b_n|^{1/n} \leq s$ if $n \geq N$ and so $\underline{\lim}\, |a_n|^{1/n} > 0$. Finally, if $s = 0$ let $c = (c_p)_{p\geq 0}$, where $c_p = a_{n_j}$ if $p = n_j$, $c_p = 0$ if $p \neq n_j$ for all j, $p = 0, 1, \dots$ Then $c \in \mathcal{A}_0(0) - \mathrm{Im}(M_a)$. $\square$

Corollary 4. *If $\rho(a) > 0$ and $s \geq 0$ then $M_a : \mathcal{A}_0(s) \to \mathcal{A}_0(\rho(a)\, s)$ is a linear homeomorphism if and only if it is surjective.*

Remark 6. If $a \in \mathcal{A}_0$ will write $Supp(a)$ for the set of those $n \in \mathbb{N}_0$ such that $a_n \neq 0$. Then

$$\langle a\rangle = \left\{ b \in \mathcal{A}_0 : Supp(b) \subseteq Supp(a) \text{ and } \overline{\lim}_{n/a_n\neq 0} \left|\frac{b_n}{a_n}\right|^{1/n} < +\infty \right\}.$$

4. On certain ideals of $(\mathcal{A}_0, +, \odot_\varepsilon)$.

We already know that kernels of non zero complex Hadamard homomorphisms are closed maximal cyclic ideals. In particular, all maximal ideals that contain those elements $a \in \mathcal{A}_0$ such that $a_n = 0$ for all but a finite number of indexes $n's$ are dense in $\mathcal{A}_0$.

Proposition 6. *Every prime cyclic ideal of $\mathcal{A}_0$ is the kernel of a complex Hadamard homomorphism.*

Proof. Let $a \in \mathcal{A}_0$ such that $\langle a \rangle$ is a prime ideal. If there are indexes $p,\ q$ such that $a_p = a_q = 0,\ p \neq q$, then $(\delta_{np})_{n\geq 0} \odot_\varepsilon (\delta_{nq})_{n\geq 0} \in \langle a \rangle$, $(\delta_{np})_{n\geq 0} \notin \mathcal{A}_0$ and $(\delta_{nq})_{n\geq 0} \notin \mathcal{A}_0$. If $a_n \neq 0$ for all n then $\underline{\lim}\, |a_n|^{1/n} = 0$. As before, let (n_j) be an strictly increasing subsequence of $\mathbb{N}$ such that $\lim |a_{n_j}|^{1/n_j} = 0$. If S is an infinite subsequence of (n_j) such that its complement is also infinite, we consider $b = (\varkappa_S(n)\, \varepsilon_n)_{n\geq 0}$, where $\varkappa_S(n) = 1$ or 0 according as n belongs or does not belong to S respectively. Thus $b \odot_\varepsilon (\varepsilon - b) \in \langle a \rangle$, but b nor $\varepsilon - b$ do not belong to $\langle a \rangle$. E.g. if $b \in \langle a \rangle$ let $c \in \mathcal{A}_0$ such that $b = a \odot_\varepsilon c$. Then $a_n\, c_n = 1$ if $n \in S$, i.e. $\lim_{n\in S, n\to\infty} |c_n|^{1/n} = \lim_{n\in S, n\to\infty} |a_n|^{-1/n} = +\infty$. But $(|c_n|^{1/n})_n$ should be bounded because $c \in \mathcal{A}_0$. So there must be an only index r such that $a_r = 0$ and $\langle a \rangle = \ker(\chi_r)$. $\square$

Proposition 7. *All proper closed prime ideals π of $\mathcal{A}_0$ are maximal.*

Proof. Let π be a proper closed prime ideal π of $\mathcal{A}_0$, $m \in \mathbb{N}_0$ such that $\delta_m \notin \pi$, where $\delta_m = (\delta_{nm})_{n\geq 0}$. Since π is prime we have $\{\delta_n : n \neq m\} \subseteq \pi$. Since π is closed $\ker(\chi_m) \subseteq \pi$ and π becomes maximal. $\square$

Remark 7. In what follows, we shall analyze relationships between prime ideals of $\mathcal{A}_0$ and ultrafilters of the class $\mathcal{P}(\mathbb{N}_0)$ of subsets of $\mathbb{N}_0$. Recall that if X is a non empty set and $\mathcal{F} \subseteq \mathcal{P}(X)$, $\mathcal{F} \neq \emptyset$, then $\mathcal{F}$ is a filter if the following two conditions hold:

(i) If $A \in \mathcal{F}$, $B \subseteq A$ then $B \in \mathcal{F}$.

(ii) If $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$.

It is known that a necessary and sufficient condition in order that $\mathcal{F}$ be a maximal filter (or ultrafilter) is

(iii) For all $A \in \mathcal{P}(X)$, $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Will denote $\Pr(\mathcal{A}_0)$ and $U\, f(\mathbb{N}_0)$ for the classes of prime ideals of $\mathcal{A}_0$ and ultrafilters of $\mathbb{N}_0$ respectively.

Theorem 3. *The following functions are well defined*

$$\begin{cases} \Phi : \Pr(\mathcal{A}_0) \to U\, f(\mathbb{N}_0) \\ \Phi(\pi) = \{A \in \mathcal{P}(\mathbb{N}_0) : a \in \pi \ \ if\ a \in \mathcal{A}_0 \ \ and\ \ Supp(a) \subseteq A\}, \end{cases}$$

$$\begin{cases} \Psi : Uf(\mathbb{N}_0) \to \Pr(\mathcal{A}_0) \\ \Psi(\mathcal{F}) = \{a \in \mathcal{A}_0 : Supp(a) \in \mathcal{F}\}. \end{cases}$$

Proof. Clearly $\Phi(\pi)$ satisfies condition (i) of filters. If $A, B \in \Phi(\pi)$ then $A \cup B \in \Phi(\pi)$. In fact, if $a \in \mathcal{A}_0$ and $Supp(a) \subseteq A \cup B$ we have

$$a = (\varkappa_{A-B}(n)\, a_n)_{n\geq 0} + (\varkappa_{B-A}(n)\, a_n)_{n\geq 0} + (\varkappa_{A\cap B}(n)\, a_n)_{n\geq 0},$$

i.e. a can be written as the sum of three elements of π and therefore belongs to π. Now, if $C \in \mathcal{P}(\mathbb{N}_0)$, $C \notin \Phi(\pi)$ and $C^c \notin \Phi(\pi)$ there exist elements b, c of $\mathcal{A}_0$ not belonging to π such that $Supp(b) \subseteq C$, $Supp(c) \subseteq C^c$. But $b \odot_\varepsilon c = 0$ and it is contradicted the primality of π. On the other hand, let $\mathcal{F}$ be an ultrafilter of $\mathbb{N}_0$, $a, b \in \Psi(\mathcal{F})$. Since $Supp(a+b) \subseteq Supp(a) \cup Supp(b)$ and $Supp(a) \cup Supp(b) \in \mathcal{F}$ then $Supp(a+b) \in \mathcal{F}$, i.e. $a+b \in \Psi(\mathcal{F})$. If $c \in \Psi(\mathcal{F})$, $d \in \mathcal{A}_0$ then

$$Supp(c \odot_\varepsilon d) = Supp(c) \cap Supp(d) \subseteq Supp(c)$$

and $Supp(c) \in \mathcal{F}$. So $Supp(c \odot_\varepsilon d) \in \mathcal{F}$ and $c \odot_\varepsilon d \in \Psi(\mathcal{F})$. Finally, let $a, b \in \mathcal{A}_0$ such that $a \odot_\varepsilon b \in \Psi(\mathcal{F})$. If they not belong to $\Psi(\mathcal{F})$ their supports do not belong to $\mathcal{F}$. Consequently $Supp(a)^c$ and $Supp(b)^c$ belong to $\mathcal{F}$ and

$$Supp(a)^c \cup Supp(b)^c = Supp(a \odot_\varepsilon b)^c$$

is an element of $\mathcal{F}$. But $Supp(a \odot_\varepsilon b) \in \mathcal{F}$ and so $\mathcal{F}$ is not a proper subclass of $\mathcal{P}(\mathbb{N}_0)$. Thus $\Psi(\mathcal{F})$ is a prime ideal of $\mathcal{A}_0$. $\square$

Theorem 4. *Given $\pi \in \Pr(\mathcal{A}_0)$ the following assertions are equivalent:*

(i) $\Psi[\Phi(\pi)] = \pi$.
(ii) $\pi \in \mathrm{Im}(\Psi)$.
(iii) Every element of π is supported in a proper subset of $\mathbb{N}_0$.
(iv) π is minimal.

Proof. Clearly $(i) \Rightarrow (ii)$. If $\pi = \Psi(\mathcal{F})$ then $\mathcal{F}$ contains the supports of the elements of π. Since $\mathcal{F}$ is a filter all these supports must be proper subsets of $\mathbb{N}_0$ and so $(ii) \Rightarrow (iii)$. To see that $(iii) \Rightarrow (iv)$ let $\mathfrak{p}$ be a prime ideal such that there is $a \in \pi - \mathfrak{p}$. Since $Supp(a) \notin \Phi(\mathfrak{p})$ we have $Supp(a)^c \in \Phi(\mathfrak{p})$, i.e. $\mathfrak{p}$ contains those elements of $\mathcal{A}_0$ whose supports are contained in $Supp(a)^c$. Thus $a + (\varkappa_{Supp(a)^c}(n)\, \varepsilon_n)_{n\geq 0} \in \pi$ and its support is $\mathbb{N}_0$. Finally if $\Psi[\Phi(\pi)] \neq \pi$ then $\Psi[\Phi(\pi)]$ is a proper prime ideal of π and $(iv) \Rightarrow (i)$. $\square$

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