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AN ADDITIVE PERTURBATION LEMMA FOR LINEAR M-ACCRETIVE OPERATORS IN HILBERT SPACES

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In this paper, we give a new sufficient condition for the sum of a linear maximal accretive operator and an accretive one to be maximal accretive in Hilbert spaces setting. As an application, an extended result to the operator-norm error bound estimate for the exponential Trotter-Kato product formula is given.

1. introduction

A linear operator T with domain $\mathcal{D}(T)$ in a complex Hilbert space $\mathcal{H}$ is said to be accretive if

$$\mathrm{Re} < Tx, x > \geq 0 \qquad \text{for all } x \in \mathcal{D}(T)$$

or, equivalently if

$$\|(\lambda + T)x\| \geq \lambda \|x\| \qquad \text{for all } x \in \mathcal{D}(T) \text{ and } \lambda > 0.$$

Further, if $\mathcal{R}(\lambda + T) = \mathcal{H}$ for some (and hence for every) $\lambda > 0$, we say that T is maximal accretive, or m-accretive for short, where $\mathcal{R}(T)$ denote the range of

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an operator T. In particular, every m-accretive operator is accretive and closed densely defined, its adjoint is also m-accretive (cf. [9], p. 279). Furthermore,

$$(\lambda+T)^{-1}\in\mathcal{B}(\mathcal{H}) \quad \text{and} \quad \|(\lambda+T)^{-1}\|\leq\frac{1}{\lambda} \text{ for all } \lambda>0,$$

where, $\mathcal{B}(\mathcal{H})$ denote the Banach space of all bounded linear operators on $\mathcal{H}$. In particular, a bounded accretive operator is m-accretive. An operator T is called dissipative (resp. m-dissipative) if $-T$ is accretive (resp. m-accretive). A normal operator T (bounded or not) is m-accretive if and only if its spectrum is contained in the half complex plane $\overline{\mathbb{C}_+}$. Hence a normal accretive operator is m-accretive. For more details about linear accretive operators, see for instance [9–11].

Consider two linear operators T and A in Hilbert space $\mathcal{H}$, such that $\mathcal{D}(T)\subset\mathcal{D}(A)$. Assume furthermore that T is m-accretive and A is an accretive operator. Then the question is:

Under which conditions the sum $T+A$ is m-accretive?

Many papers have been devoted to this problem and most results treat pairs T, A of relatively bounded or resolvent commuting operators. We refer the reader to [4, 5, 7, 8, 17, 19, 20, 22–24]. Since T is closed it follows that there are two nonnegative constants a, b such that

$$\|Ax\|^2\leq a\|x\|^2+b\|Tx\|^2, \quad \text{for all } x\in\mathcal{D}(T)\subset\mathcal{D}(A). \tag{1}$$

In this case, A is called relatively bounded with respect to T or simply T-bounded, and refer to b as a relative bound.

Gustafson, [6], showed that that $T+A$ is also m-accretive if A is T-bounded, with $b<1$ (see [6, Theorem 2.]). This is an extension of the basic work of Rellich, Kato, and others (cf. [9]), form selfadjoint operators to m-accretive one. Okazawa showed in [16] that the closure of the sum $T+A$ is m-accretive, if the bounded operator $A(t+T)^{-1}$ on $\mathcal{H}$ is a contraction for some $t>0$, [16, Theorem 1.]. In particular, he also showed that the validity of (1) with $b=1$ implies that the closure of $T+A$ is m-accretive, [16, Corollary 1.]. Later, the same author in [15] gave a variant of perturbation by assumed the existence of two nonnegative constants a and $\beta\leq 1$ such that

$$\operatorname{Re}<Tx,Ax>+a\|x\|^2+\beta\|Tx\|^2\geq 0, \quad \text{for all } x\in\mathcal{D}(T). \tag{2}$$

If $\beta<1$, then $T+A$ is m-accretive and also the closure of $T+A$ is m-accretive for $\beta=1$, [15, Theorem 4.1]. Note that this result cover the case of relatively bounded perturbation, see [15, Remark 4.4]. There are many papers on the question of such perturbation, see [17–19, 21, 23] for more results. In recent years, a prominent development out of this is the applications to various fields, see [1–3, 8, 12–14, 19].

The aim of this paper is to establish a new perturbation results on the m-accretivity of the operator sum $T+A$. This may be viewed as a slight improvement and generalization of the perturbation results, particularly, those of Okazawa, [15, 17]. Namely, we prove the following lemma.

Lemma 1.1. *Let T and A two operators such that $\mathcal{D}(T) \subset \mathcal{D}(A)$. Assume that T is m-accretive, A is accretive and there exists $c \geq 0$, such that*

$$\text{Re} < Tx, Ax > \geq c\,\|Ax\|^2, \quad \textit{for all } x \in \mathcal{D}(T). \tag{3}$$

If we take $b = \max\{c \geq 0 : (3) \text{ holds }\}$, we have,

1. *if $0 \leq b \leq 1$, then $T+A$ is also m-accretive,*
2. *if $b > 1$ then $T+A$ is m-ω-accretive, with $\omega = \pi/2 - \arcsin(\frac{b-1}{b})$.*

Here, T is m-ω-accretive if $e^{\pm i\theta}T$ is m-accretive for $\theta = \frac{\pi}{2} - \omega$, $0 < \omega \leq \pi/2$ (or m-sectorial as it was introduced, e.g., in the Kato book [9, Ch.IX, §1]). In this case, $-T$ generates an holomorphic contraction semigroup on the sector $|arg(\lambda)| < \omega$. In this connection, we note that for any $\varepsilon > 0$

$$\left\|(\lambda+T)^{-1}\right\| \leq \frac{M_\varepsilon}{|\lambda|}, \qquad \text{for } |arg(\lambda)| \leq \frac{\pi}{2} + \omega - \varepsilon$$

with M_ε is independent of λ (see [9, pp. 490]).

The novelty of the lemma is the optimality of b such that (3) holds. Clearly, (3) implies Re $< Tx, Ax > \geq 0$ for all $x \in \mathcal{D}(T)$, this exactly the assumption of [16, Theorem 2.]. Hence, we conclude that $T+A$ is also m-accretive. Our result is a refinement of this result by given a more precise sector containing the numerical range $W(T+A)$ of operator $T+A$ as of function of the constant b. Also, from (3), we have for $b > 0$,

$$\|Ax\| \leq \frac{1}{b}\|Tx\|, \quad \text{for all } x \in \mathcal{D}(T). \tag{4}$$

Thus the assumption (3) is stronger than the relative boundedness with respect to T. In particular, if $b > 1$ the lower bound $\frac{1}{b} < 1$, so according to [6, Theorem 2.], $T+A$ is m-accretive. Here, we say more, $T+A$ is m-ω-accretive with ω depends of the lower bound $\frac{1}{b} < 1$.

2. Proof

Proof of Lemma 1.1. Let $b = \max\{c \geq 0 : (3) \text{ holds }\}$. If $b = 0$, this exactly the [16, Theorem 2.]. Assume that $0 < b \leq 1$. We obtain from (3)

$$\begin{aligned} 0 &\leq \mathrm{Re} < Tx, Ax > -b\,\|Ax\|^2 \\ &\leq \mathrm{Re} < Tx, Ax > +(\alpha - b)\,\|Ax\|^2 \end{aligned}$$

for some $\alpha > 1$. Using (4), we get

$$0 \leq \mathrm{Re} < Tx, Ax > + \frac{\alpha - b}{b^2}\,\|Tx\|^2.$$

Choosing α such that $\beta = \dfrac{\alpha - b}{b^2} < 1$, by (2) we conclude that $T + A$ is m-accretive (cf.[15, Theorem 4.1]).

Now, suppose that that $b > 1$. Let $x \in \mathcal{D}(T)$, then for every $t > 0$, we have

$$\begin{aligned} \mathrm{Re} < tx + Tx, Ax > &= t\mathrm{Re} < x, Ax > + \mathrm{Re} < Tx, Ax > \\ &\geq b\,\|Ax\|^2. \end{aligned}$$

Thus we have

$$\|Ax\| \leq \frac{1}{b}\,\|tx + Tx\|. \tag{5}$$

Since T is m-accretive, then

$$\left\|A(t+T)^{-1}x\right\| \leq \frac{1}{b}\,\|x\|, \quad \text{for all } x \in \mathcal{H}.$$

Hence it follows that

$$\left\|A(t+T)^{-1}\right\| \leq \frac{1}{b} < 1. \tag{6}$$

Then the operator $I + A(t+T)^{-1}$ is invertible and

$$\left\|(I + A(t+T)^{-1})^{-1}\right\| \leq \frac{b}{b-1}.$$

The fact that

$$t + T + A = [I + A(t+T)^{-1}](t+T),$$

it follows that $-t \in \rho(T+A)$ and

$$\left\|t(t+T+A)^{-1}\right\| \leq \frac{b}{b-1} = M, \quad \text{for all } t > 0,$$

with $M > 1$. Since $T+A$ is accretive, $\rho(T+A)$ contains also the half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. Put $S = \{z \in \mathbb{C} : z \neq 0; |arg(z)| < \pi/2 - \arcsin(\frac{1}{M}) = \theta\}$ and $M' := 1/\sin(\pi/2 - \theta')$ with $0 < \theta < \theta' < \pi/2$, clearly $M' > M$. Let $\mu \in \mathbb{C}$ such that $|arg(\mu)| \leq \theta'$ and fix λ with $\mathrm{Re}\lambda = -t < 0$. Let $|\mu - \lambda| \leq \frac{|\lambda|}{M'}$, we have that $\left\|(\mu-\lambda)(t+T+A)^{-1}\right\| \leq \frac{M}{M'} < 1$. Hence it follows that $\mu \in \rho(T+A)$ and

$$(\mu+T+A)^{-1} = (\lambda+T+A)^{-1}[I+(\mu-\lambda)(\lambda+T+A)^{-1}]^{-1}.$$

Thus

$$\begin{aligned}\left\|\mu(\mu+T+A)^{-1}\right\| &\leq \frac{|\mu|}{|\lambda|}\frac{1}{1-\frac{M}{M'}}M \\ &\leq (1+\frac{1}{M'})\frac{1}{1-\frac{M}{M'}}M.\end{aligned}$$

On the other hand,

$$\begin{aligned}(1+\frac{1}{M'})\frac{1}{1-\frac{M}{M'}}M &= \frac{1+\sin(\pi/2-\theta')}{\sin(\pi/2-\omega)-\sin(\pi/2-\theta')} \\ &\leq \frac{1}{\sin((\theta'-\theta)/2)\sin((\theta'+\theta)/2)} \\ &\leq \frac{1}{\sin(\theta'-\theta)\sin(\theta)} \\ &\leq \frac{1}{\sin(\theta'-\theta)\sin(\pi/2-\arcsin(\frac{1}{M}))} \\ &\leq \frac{1}{\sin(\theta'-\theta)\cos(\arcsin(\frac{1}{M}))} \\ &\leq \frac{1}{\sin(\theta'-\theta)\sqrt{1-\frac{1}{M^2}}} \\ &\leq \frac{M}{\sin(\theta'-\theta)\sqrt{M^2-1}}.\end{aligned}$$

This implies that

$$\left\|(\mu+T+A)^{-1}\right\| \leq \frac{M}{|\mu|\sin(\theta'-\theta)\sqrt{M^2-1}}.$$

This shows that the sector S belongs to $\rho(T+A)$ and for any $\varepsilon > 0$,

$$\left\|(\mu+T+A)^{-1}\right\| \leq \frac{M_\varepsilon}{|\mu|} \quad \text{for} \quad |arg(\mu)| \leq \pi/2 - \arcsin(\frac{1}{M}) + \varepsilon,$$

with $M_\varepsilon = \dfrac{M}{\sin(\varepsilon)\sqrt{M^2-1}}$ and $\theta' - \theta = \varepsilon$. Clearly, M_ε is independent of μ. Hence, $T+A$ is m-ω-accretive, with $\omega = \pi/2 - \arcsin(\dfrac{b-1}{b})$. □

Remark 2.1. 1. As seen in the last paragraph of the proof, the condition (2) implies (3) at least for $0 \leq b \leq 1$. Thus [15, Theorem 4.1] is covered by Lemma 1.1.

2. If the assumptions of Lemma 1.1 are satisfied, we can see that Re $< tx + Tx, Ax > \geq 0$ for all $x \in \mathcal{D}(T)$. Therefore $A(t+T)^{-1}$ is bounded accretive operator for any $t > 0$.

Corollary 2.2. *Let T and A as in Lemma 1.1 obeying* (3)*. Then*

1. *$-(T+A)$ generates contractive one-parameter semigroup for* $0 \leq b \leq 1$.
2. *$-(T+A)$ generates contractive holomorphic one-parameter semigroup with angle* $\omega = \arcsin(\dfrac{b-1}{b})$ *for* $b > 1$.

3. An application

One of interest is the operator-norm error bound estimate for the exponential Trotter-Kato product formula in the case of accretive perturbations, see [3] and a recent book [25] Ch.9, Sect.9.4, or [14] for a short survey. Let A be a semi-bounded from below densely defined self-adjoint operator and B an m-accretive operator in a Hilbert space $\mathcal{H}$.

In [3, Theorem 3.4] it has been shown that if B is A-bounded with lower bound < 1 and

$$\mathcal{D}((A+B)^\alpha) \subset \mathcal{D}(A^\alpha) \cap \mathcal{D}((B^*)^\alpha) \neq \{0\} \quad \text{for some } \alpha \in (0.1], \tag{7}$$

then there is a constant $L_\alpha > 0$ such that the estimates

$$\left\| \left(e^{-tB/n} e^{-tA/n}\right)^n - e^{-t(A+B)} \right\| \leq L_\alpha \frac{\ln n}{n^\alpha} \tag{8}$$

and

$$\left\| \left(e^{-tA^*/n} e^{-tB^*/n}\right)^n - e^{-t(A+B)^*} \right\| \leq L_\alpha \frac{\ln n}{n^\alpha} \tag{9}$$

hold for some $\alpha \in (0.1]$ and $n = 1, 2, \ldots$ uniformly in $t \geq 0$. Here T^α denotes the fractional powers of an m-accretive operator, see [10, 11].

The aim of the present result is to extend [3, Theorem 3.4]. This extension is accomplished by replacing the relative boundedness by the assumption (3). More precisely, we have

Theorem 3.1. *Let A be a semibounded from below densely defined self-adjoint operator and B an m-accretive operator with* (3) *for some* $b > 1$. *Assume that* (7) *holds. Then there is a constant* $L_\alpha > 0$ *such that the estimates* (8) *and* (9) *hold for some* $\alpha \in (0.1]$ *and* $n = 1, 2, \ldots$ *uniformly in* $t \geq 0$.

Proof. From (3), we have for $b > 1$,

$$\|Bx\| \leq a \|Ax\|, \quad \text{for all } x \in \mathcal{D}(A), \tag{10}$$

with $a = \frac{1}{b} < 1$. Hence B is A-bounded with lower bound $a < 1$. Also, by lemma 1.1, $A + B$ is m-ω-accretive, with $\omega = \pi/2 - \arcsin(\dfrac{b-1}{b})$. Now, all assumptions of [3, Theorem 3.4] are fulfilled. Hence we obtain the desired result. ☐

Remark 3.2. It well known that, for an m-accretive operator T, the fractional powers T^α are m-$(\alpha\pi)/2$-accretive and, if $\alpha \in (0, 1/2)$, then $\mathcal{D}(T^\alpha) = \mathcal{D}(T^{*\alpha})$, see [11, Theorem 1.1]. Since A, B and $A + B$ are m-accretive operators, we deduce that

$$\mathcal{D}((A+B)^{*\alpha}) = \mathcal{D}((A+B)^\alpha) \subset \mathcal{D}(A^\alpha) \cap \mathcal{D}(B^\alpha) = \mathcal{D}(A^\alpha) \cap \mathcal{D}((B^*)^\alpha),$$

for some $\alpha \in (0, 1/2[$. Thus, the condition (7) may be omitted in Theorem 3.1 if we take $\alpha \in (0, 1/2[$ (cf. [3, Theorem 4.1]).

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