LE MATEMATICHE
Vol. LXXIX (2024) – Issue I, pp. 79–92
doi: 10.4418/2024.79.1.4

LINE IDEMPOTENT GRAPH OF SOME COMMUTATIVE RINGS

O. GANI MONDAL - SK. MD. ABU NAYEEM

Let X be a finite commutative ring with unity and $Id(X)$ be the set of idempotent elements of X. The idempotent graph $G_{Id}(X)$ of X is a simple undirected graph with all elements of X as vertices and two distinct vertices x, y are adjacent if and only if $x + y \in Id(X)$. In this paper, we have considered the idempotent graph of some commutative rings and investigated those graph and their complement for being line graphs.

1. Introduction

After the introduction of the notion of zero divisor graphs of a commutative ring [4] in 1999, research on graphs from algebraic structures has proliferated manyfold in the last decade [1, 2, 8, 10, 12, 15, 18]. Akbari et al. [2] proposed the idea of an idempotent graph of a ring X in 2013 and found some graph theoretical parameters for it. According to their definition, a graph whose vertices are the points of X and if two separate points x and y of X are such that $xy = yx = 0$, then they are adjacent, is called an "idempotent graph of X." Recently, Razzaghi and Sahebi [20] have associated another graph with a commutative ring X with a non-zero identity. They have assumed the ring X as the set of vertices and that two distinct vertices x and y are adjacent if and only if $(x+y)^2 = (x+y)$, that is,

Received on February 1, 2024

AMS 2010 Subject Classification: Primary: 05C25, Secondary: 05C07, 05C09
Keywords: Idempotent, congruence relation, Artinian ring, local ring, line graph, idempotent graph, path graph.

if and only if $x+y \in Id(X)$. Here, $Id(X)$ is the set of all idempotent elements of X. They have also named this graph the idempotent graph of X and denoted the graph by $G_{Id}(X)$. They have determined the necessary and sufficient conditions for the graph to be a connected graph and found several graph theoretic parameters like chromatic index, diameter, and girth of that graph. Very recently, Sharma and Dutta [21] have further studied that graph and obtained some other graph theoretic parameters like radius, independence number, chromatic number, and the condition for planarity and Hamiltonicity. In the present paper, we aim to study that graph furthermore.

While characterizing different graphs associated with algebraic structures, researchers have concentrated on the algebraic properties of the structure for which the obtained graph belongs to some special classes of graphs, such as line graphs. In 2021, Barati [7] studied the line graph of the zero divisor graph of a commutative ring, and derived the conditions under which the zero divisor graph is a line graph or complement of a line graph. In 2022, Khojasteh [19] carried out a similar study on the cozero-divisor graph of commutative ring. Pranjali et al. [17] have studied the line graph of unit graphs associated with finite commutative rings. Pirzada and Altaf [18] have also studied the line graph of unit graphs of a commutative ring. Motivated by these studies, in this paper, we investigate the idempotent graphs of some commutative rings and find the conditions under which those graphs or their complements are line graphs.

Let G be a finite simple graph. Each vertex of the line graph $L(G)$ of the graph G corresponds to an edge of G. Two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident to a common vertex in G. The graph with the same set of vertices as of G is called its *complement graph* $\bar{G}$, when two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. A *bipartite graph* is such a graph whose vertex set is the union of two disjoint sets of vertices U and V in a manner that each edge in the graph links a vertex in U to a vertex in V. We denote a complete bipartite graph with $|U| = m$ and $|V| = n$ by $K_{m,n}$. K_n, C_n, P_n denote the complete graph, cycle graph, and path graph with n vertices respectively. A subset of the vertex set of a graph is called an *independent set* if no two elements of that set are adjacent.

When ring X contains precisely one maximal ideal, it is referred to as a local ring. The descending chain condition on ideals is satisfied by a ring known as an Artinian ring.

2. Preliminaries

Let X be a finite commutative ring with unity. An element $x \in R$ is said to be an idempotent element of X if it satisfies the condition $x^2 = x$. The total number

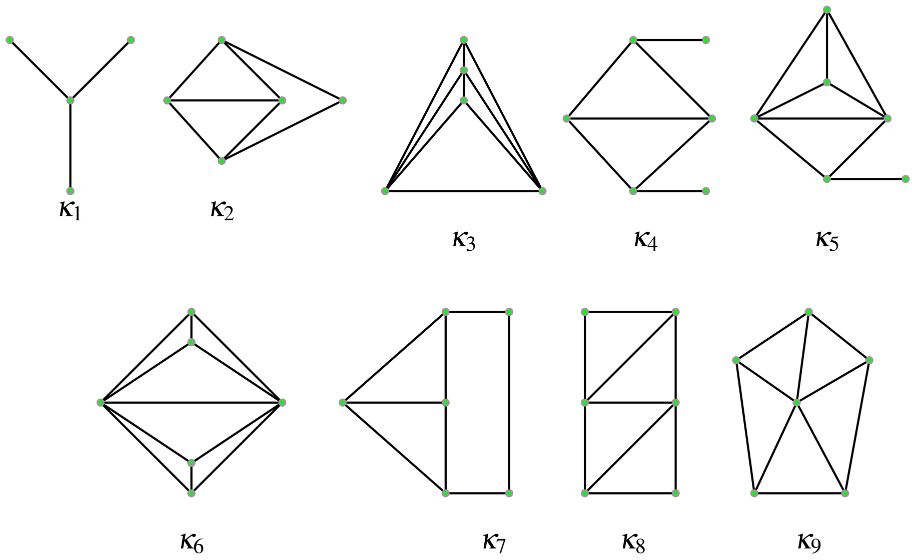

Figure 1: Nine induced forbidden subgraphs.

of idempotent elements in a ring $\mathbb{Z}_n$ is 2^k where k represents the number of different prime factors of n [13].

Since we are concerned with line graphs, the following lemma will be very useful.

Lemma 2.1. *[9] The necessary and sufficient condition for a graph G to be a line graph of some graph is that it does not contain the graphs depicted in Figure 1 as induced subgraphs.*

3. Line graph of an idempotent graph over the ring $\mathbb{Z}_n$

Let $G_{Id}(X)$ be the idempotent graph of the commutative ring X. We start with the following useful lemmas.

Lemma 3.1. *[20] Let X be a ring with $Id(X)$ being trivial and $x \in X$. Then the following conditions hold in $G_{Id}(X)$.*

(a) If $Id(X)$ does not contain the element $2.x$, then $deg(x) = 2$.

(b) If $Id(X)$ contains the element $2.x$, then $deg(x) = 1$.

Lemma 3.2. *[20] If X is a ring such that $Id(X)$ is trivial and it generates $(X, +)$, then $G_{Id}(X)$ is a path graph and vice versa.*

Figure 2: $G_{Id}(\mathbb{Z}_n)$ is a line graph when n is even prime power.

Figure 3: $G_{Id}(\mathbb{Z}_n)$ is a line graph when n is odd prime power.

Lemma 3.3. *[13] If* $\gcd(a,m) = d$ *and* $d|b$ *then the congruence relation* $ax \equiv b$ *has exactly* d *number of solutions. Again, if* $d \nmid b$ *then there does not exist any solution of the congruence relation.*

Now we put forward the following theorem characterizing $G_{Id}(\mathbb{Z}_n)$.

Theorem 3.4. *If* n *is a prime power, then* $G_{Id}(\mathbb{Z}_n)$ *is a line graph.*

Proof. Let p be a prime and $n = p^m$, where m is any non-zero positive integer. Then the set of vertices of $G_{Id}(\mathbb{Z}_n)$ is $\{0,1,...,p^m - 1\}$. Since n has only one prime factor, namely p, there are two idempotent elements of $\mathbb{Z}_n$, viz. the trivial idempotents 0 and 1. By above Lemma 3.1, if an element $x \in \mathbb{Z}_n$ satisfies one of the relations $2.x \equiv 0 \pmod{n}$ or $2.x \equiv 1 \pmod{n}$ then the degree of x is 1. Now, we have the following two cases.
Case 1. (n is even) Let us consider $n = 2k$, $k \in \mathbb{N}$.
Subcase 1.1: ($2.x \equiv 1 \pmod{n}$) Since $\gcd(2,2k) = 2$ and $2 \nmid 1$, there does not exist a solution.
Subcase 1.2: ($2.x \equiv 0 \pmod{n}$) Since $\gcd(2,2k) = 2$ and $2|2$, there are two distinct solutions, namely 0 and $\frac{p^m}{2}$. Now from Lemma 3.1, the degree of each of the vertices $\frac{p^m}{2}$ and 0 is 1 and the degree of every other vertex is 2. This suggests that the idempotent graph $G_{Id}(\mathbb{Z}_n)$ (where $n = p^m$) is a path graph, as depicted in Figure 2. Since a path graph is a line graph, $G_{Id}(\mathbb{Z}_n)$ is a line graph.
Case 2. (n is odd) Let us consider $n = 2k+1$, $k \in \mathbb{N}$.
Subcase 2.1: ($2.x \equiv 1 \pmod{n}$) Since $\gcd(2,2k+1) = 1$, there is a unique solution that is $x = \frac{p^m+1}{2}$.
Subcase 2.2: ($2.x \equiv 0 \pmod{n}$) Since $\gcd(2,2k+1) = 1$, there is a unique solution that is 0. Again from Lemma 3.1, we have that the degree of each of the vertices $\frac{p^m+1}{2}$ and 0 is 1, and that of every other vertex is 2. Therefore, they form a path graph as depicted in Figure 3.

□

Lemma 3.5. *[15] In the ring* $\mathbb{Z}_{p_1^m p_2^n}$*, there are four idempotent elements* 0, 1, $p_1^{r(p_2-1)p_2^{n-1}}$, *and* $p_2^{t(p_1-1)p_1^{m-1}}$ *where* r, t *are the smallest positive integers such* *that $rp_2^{n-1}(p_2-1)-m$ and $tp_1^{m-1}(p_1-1)-n$ are both positive, and p_1 and p_2 are distinct primes.*

Lemma 3.6. *If p_1 and p_2 are distinct primes, then*

$$p_1^{r(p_2-1)p_2^{n-1}}+p_2^{t(p_1-1)p_1^{m-1}} \equiv 1 \pmod{p_1^m p_2^n}\cdot$$

Proof. By Euler's theorem, we have, $p_1^{\phi(p_2^n)} \equiv 1 \pmod{p_2^n}$ where $\gcd(p_1,p_2)=1$. Thus $p_2^n|p_1^{r(p_2-1)p_2^{n-1}}-1$ and obviously, $p_2^n|p_2^{t(p_1-1)p_1^{m-1}}$.
Therefore, $p_2^n|p_1^{r(p_2-1)p_2^{n-1}}+p_2^{t(p_1-1)p_1^{m-1}}-1$. Similarly, we obtain that $p_1^m|p_1^{r(p_2-1)p_2^{n-1}}+p_2^{t(p_1-1)p_1^{m-1}}-1$. These together imply that $p_1^m p_2^n|p_1^{r(p_2-1)p_2^{n-1}}+p_2^{t(p_1-1)p_1^{m-1}}-1$ or,

$$p_1^{r(p_2-1)p_2^{n-1}}+p_2^{t(p_1-1)p_1^{m-1}} \equiv 1 \pmod{p_1^m p_2^n}.$$

□

Corollary 3.7. *In the graph $G_{Id}(\mathbb{Z}_{p_1^m p_2^n})$, $p_1^{r(p_2-1)p_2^{n-1}}$ is adjacent to $p_2^{t(p_1-1)p_1^{m-1}}$.*

Theorem 3.8. *If $X=\mathbb{Z}_{p_1^m p_2^n}$, where p_1, p_2 are distinct primes and m, n are both positive integers, then the graph $G_{Id}(X)$ can not be a line graph.*

Proof. Let us denote all vertices of $G_{Id}(X)$ as the set V. In this case $V=\mathbb{Z}_{p_1^m p_2^n}$. From Lemma 3.5, the idempotent elements of $\mathbb{Z}_{p_1^m p_2^n}$ are $\left\{0,1,p_1^{r(p_2-1)p_2^{n-1}},p_2^{t(p_1-1)p_1^{m-1}}\right\}$, where r, t are the smallest positive integers such that $rp_2^{n-1}(p_2-1)-m$ and $tp_1^{m-1}(p_1-1)-n$ are positive integers. Now, we consider the subset $S=\left\{1,p_1^m p_2^n-1,p_1^{r(p_2-1)p_2^{n-1}}-1,p_2^{t(p_1-1)p_1^{m-1}}-1\right\}$ of V. We show below that except for $p_2=3$, $p_1=2$ with $n=1,m=1$, the subset $D=\left\{p_1^m p_2^n-1,p_1^{r(p_2-1)p_2^{n-1}}-1,p_2^{t(p_1-1)p_1^{m-1}}-1\right\}$ of S is an independent set and the element $1\in S$ is adjacent to every element in D, i.e., the subgraph induced by S is $K_{1,3}$. The following three cases may arise.

Case 1. $\Big(p_1^m p_2^n-1+p_1^{r(p_2-1)p_2^{n-1}}-1 \equiv 0$ or 1 or $p_1^{r(p_2-1)p_2^{n-1}}$ and $p_2^{t(p_1-1)p_1^{m-1}}$ $\pmod{p_1^m p_2^n}\Big)$

Subcase 1.1: If $p_1^m p_2^n-1+p_1^{r(p_2-1)p_2^{n-1}}-1\equiv 0 \pmod{p_1^m p_2^n}$, then $p_1^m p_2^n|(p_1^{r(p_2-1)p_2^{n-1}}-2)$. It is possible only when $p_1=2$ and then $r=\frac{1}{(p_2-1)p_2^{(n-1)}}$. As such, r will be a positive integer only when $n=1$ and $p_2=2$, which contradicts that $\gcd(p_1,p_2)=1$.

Subcase 1.2: If $p_1^m p_2^n - 1 + p_1^{r(p_2-1)p_2^{n-1}} - 1 \equiv 1 \pmod{p_1^m p_2^n}$, then $p_1^m p_2^n | p_1^{r(p_2-1)p_2^{n-1}} - 3$. It is possible only when $p_1 = 3$, $p_2 = 2$ with $n = 1, m = 1$.
Subcase 1.3: If $p_1^m p_2^n - 1 + p_1^{r(p_2-1)p_2^{n-1}} - 1 \equiv p_1^{r(p_2-1)p_2^{n-1}} \pmod{p_1^m p_2^n}$, then $p_1^m p_2^n | p_1^m p_2^n - 2$ which is not possible.
Subcase 1.4: If $p_1^m p_2^n - 1 + p_1^{r(p_2-1)p_2^{n-1}} - 1 \equiv p_2^{t(p_1-1)p_1^{m-1}} \pmod{p_1^m p_2^n}$, then $p_1^m p_2^n | p_1^{r(p_2-1)p_2^{n-1}} - p_2^{t(p_1-1)p_1^{m-1}} - 2$ which is possible only when $p_1^{r(p_2-1)p_2^{n-1}} - p_2^{t(p_1-1)p_1^{m-1}} - 2 = 0$,
i.e., $p_1^{r(p_2-1)p_2^{n-1}} - 1 = 1 + p_2^{t(p_1-1)p_1^{m-1}}$. By Euler's theorem, $p_2^n | p_1^{r(p_2-1)p_2^{n-1}} - 1$ that is $p_2^n | 1 + p_2^{t(p_1-1)p_1^{m-1}}$. This is possible only when $p_2 = 1$ which contradicts the fact that p_2 is a prime.

Case 2. $\left(p_1^m p_2^n - 1 + p_2^{t(p_1-1)p_1^{m-1}} - 1 \equiv 0 \text{ or } 1 \text{ and } p_1^{r(p_2-1)p_2^{n-1}} \text{ or } p_2^{t(p_1-1)p_1^{m-1}} \pmod{p_1^m p_2^n}\right)$ Similar to Case 1 above, we can show that D is an independent set except for $p_2 = 3$, $p_1 = 2$ and $n = 1, m = 1$.

Case 3. $\left(p_1^{r(p_2-1)p_2^{n-1}} - 1 + p_2^{t(p_1-1)p_1^{m-1}} - 1 \equiv 0 \text{ or } 1 \text{ and } p_1^{r(p_2-1)p_2^{n-1}} \text{ or } p_2^{t(p_1-1)p_1^{m-1}} \pmod{p_1^m p_2^n}\right)$

Subcase 3.1: $\left(p_1^{r(p_2-1)p_2^{n-1}} - 1 + p_2^{t(p_1-1)p_1^{m-1}} - 1 \equiv 0 \pmod{p_1^m p_2^n}\right)$.

From the above Lemma 3.6 we have,

$$p_1^{r(p_2-1)p_2^{n-1}} + p_2^{t(p_1-1)p_1^{m-1}} \equiv 1 \pmod{p_1^m p_2^n}.$$

So, $p_1^m p_2^n | p_1^{r(p_2-1)p_2^{n-1}} + p_2^{t(p_1-1)p_1^{m-1}} - 1$,

$$\text{i.e.,} \qquad p_1^m p_2^n \nmid (p_1^{r(p_2-1)p_2^{n-1}} + p_2^{t(p_1-1)p_1^{m-1}} - 1) - 1.$$

Thus, $p_1^{r(p_2-1)p_2^{n-1}} - 1 + p_2^{t(p_1-1)p_1^{m-1}} - 1 \equiv 0 \pmod{p_1^m p_2^n}$ which is not possible.
Subcase 3.2: If $p_1^{r(p_2-1)p_2^{n-1}} - 1 + p_2^{t(p_1-1)p_1^{m-1}} - 1 \equiv 1$, that is, $p_1^{r(p_2-1)p_2^{n-1}} + p_2^{t(p_1-1)p_1^{m-1}} - 1 \equiv 2 \pmod{p_1^m p_2^n}$, then by above Lemma 3.6 $p_1^m p_2^n | p_1^m p_2^n - 2$, which is not possible.
Subcase 3.3: If $p_1^{r(p_2-1)p_2^{n-1}} - 1 + p_2^{t(p_1-1)p_1^{m-1}} - 1 \equiv p_1^{r(p_2-1)p_2^{n-1}} \pmod{p_1^m p_2^n}$, then
$p_1^m p_2^n | p_2^{t(p_1-1)p_1^{m-1}} - 2$. It is possible only when $p_2 = 2$ and then $t = \frac{1}{(p_1-1)p_1^{(m-1)}}$. As such, t will be positive integer only when $m = 1$ and $p_1 = 2$, which contradicts the fact that $\gcd(p_1, p_2) = 1$.
Subcase 3.4: If $p_1^{r(p_2-1)p_2^{n-1}} - 1 + p_2^{t(p_1-1)p_1^{m-1}} - 1 \equiv p_2^{t(p_1-1)p_1^{m-1}} \pmod{p_1^m p_2^n}$, then

$p_1^m p_2^n | p_1^{r(p_2-1)p_2^{n-1}} - 2$. It is possible only when $p_1 = 2$ and then $r = \frac{1}{(p_2-1)p_2^{(n-1)}}$. As such, r is a positive integer only when $n = 1$ and $p_2 = 2$ which contradicts the fact that $\gcd(p_1, p_2) = 1$.

From all the above cases, we conclude that except for $p_2 = 3$, $p_1 = 2$ with $n = 1, m = 1$, D is an independent set, and $K_{1,3}$ being an induced subgraph, induced by S, $G_{Id}(\mathbb{Z}_{p_1^m p_2^n})$ can not be a line graph. For $p_2 = 3$, $p_1 = 2$ with $n = 1, m = 1$, i.e., when $p_1^m p_2^n = 6$, we consider the subset $S = \{0, 1, 2, 3, 4\}$ of the vertex set $V = \{0, 1, 2, 3, 4, 5\}$ of $G_{Id}(\mathbb{Z}_{p_1^m p_2^n})$. Clearly, S forms a subgraph which is isomorphic to κ_2 shown in Figure 1. Hence, $G_{Id}(\mathbb{Z}_{p_1^m p_2^n})$ can not be a line graph. □

4. Line graph of the idempotent graph of Artinian ring

Consider X to be a finite Artinian ring. Then $X \cong X_1 \times X_2 \times \cdots \times X_n$ where each X_i, $1 \le i \le n$ is a local ring [6].

Lemma 4.1. *If r is a positive integer and $r \ge 3$ then $r-1$ is not an idempotent element of an Artinian Ring $X \cong \mathbb{Z}_r$.*

Proof. If possible, let $r - 1 \in Id(\mathbb{Z}_r)$. Then $(r-1)^2 \equiv (r-1) \pmod r \Rightarrow r - 1 \equiv 1 \pmod r \Rightarrow r | r-2$, which is not possible for $r \ge 3$. Thus, $r - 1 \notin Id(\mathbb{Z}_r)$. □

Lemma 4.2. *If r is a positive integer and $r \ge 4$, then $2(r-1)$ is not an idempotent element of an Artinian ring $X \cong \mathbb{Z}_r$.*

Proof. If possible, let $2(r-1) \in Id(\mathbb{Z}_r)$. Then $[2(r-1)]^2 \equiv 2(r-1) \pmod r$ $\Rightarrow 2(r-1) \equiv 4 \pmod r \Rightarrow r-1 \equiv 2 \pmod r \Rightarrow r | r-3$, which is not possible for $r \ge 4$. Thus, $2(r-1) \notin Id(\mathbb{Z}_r)$ for $r \ge 4$. □

Theorem 4.3. *Let $X = X_1 \times X_2$ be an Artinian ring. Then $G_{Id}(X)$ is a line graph if only if $X \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. We examine the following instances.

Case 1. $\Big(|X_i| = r_i \ge 4$ for both $i = 1, 2\Big)$

Since $|Id(X_1)| \ge 2$ and $|Id(X_2)| \ge 2$, $|Id(X_1 \times X_2)| \ge 4$, $r_i - 1$, $2(r_i - 1)$, $i = 1, 2$ are not idempotents. We consider the vertex subset $S = \{(1,1), (0, r_2 - 1), (r_1 - 1, r_2 - 1), (r_1 - 1, 0)\}$ of $G_{Id}(X)$ and then S forms the subgraph $K_{1,3}$ which is a forbidden subgraph for a line graph. Hence, $G_{Id}(X)$ can not be a line graph.

Case 2. $\Big(|X_1| = r_1 < 4$ and $|X_2| = r_2 \ge 4\Big)$

Then we choose a vertex subset $S = \{(1,1), (0, r_2 - 1), (r_1 - 1, r_2 - 1), (r_1 -$

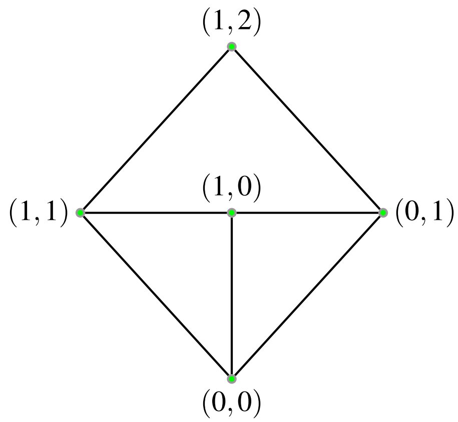

Figure 4: Induced subgraph of $G_{Id}(\mathbb{Z}_2 \times \mathbb{Z}_3)$.

$1,0)\}$ which forms an induced subgraph $K_{1,3}$ in $G_{Id}(X)$. So, $G_{Id}(X)$ can not be a line graph.
Case 3. $\left(|X_1| = r_1 \geq 4 \text{ and } |X_2| = r_2 < 4\right)$
Same as Case 2.
Case 4. $\left(X \cong \mathbb{Z}_3 \times \mathbb{Z}_3\right)$
In this case, we consider the subset $S = \{(0,0),(1,1),(0,2),(2,0)\}$ which again forms an induced subgraph $K_{1,3}$ in $G_{Id}(X)$. So $G_{Id}(X)$ can not be a line graph.
Case 5. $\left(X \cong \mathbb{Z}_2 \times \mathbb{Z}_3\right)$
Then we take the subset $S = \{(0,0),(0,1),(1,1),(1,0),(1,2)\}$ of vertices which forms an induced subgraph κ_2, as depicted in Figure 4. Therefore, $G_{Id}(X)$ can not be a line graph.
Case 6. $\left(X \cong \mathbb{Z}_3 \times \mathbb{Z}_2\right)$
Same as Case 5.
Case 7. $\left(X \cong \mathbb{Z}_2 \times \mathbb{Z}_2\right)$
In this case, $\{(1,0),(0,1),(1,1),(0,0)\}$ are the vertices, each of which is adjacent to the other. Thus, the graph is the complete graph K_4 and $L(K_{1,4}) \cong K_4$. So it implies that $G_{Id}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a line graph of some graph.

□

Theorem 4.4. *If $X \cong X_1 \times X_2 \times \cdots \times X_n$, $n \geq 3$ is an Artinian ring, then $G_{Id}(X)$ is a line graph if and only if $X_1 \cong X_2 \cong \cdots \cong X_n \cong \mathbb{Z}_2$.*

Proof. Let, $|X_i| \geq 3$ for at least one i. Without loss of generality, we consider that $|X_n| \geq 3$. Here, all elements of X are the vertices of $G_{Id}(X)$. Since 2 can not be an idempotent element of X_n, any n-tuple of the form $(-,-,\ldots,2)$ can

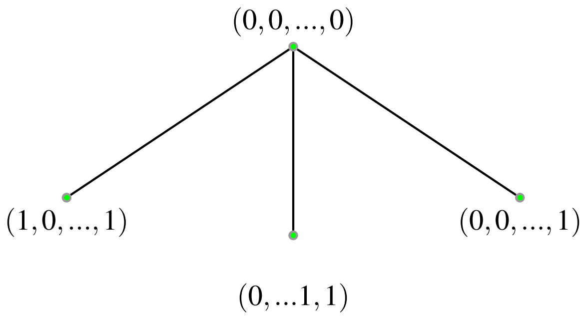

Figure 5: $K_{1,3}$.

not be an idempotent element of X. Thus, the set of vertices $S = \{(0,0,...0), (1,0,...,1), (0,0,...,1), (0,...1,1)\}$ forms an induced subgraph $K_{1,3}$ as depicted in Figure 5. So $G_{Id}(X)$ can not be a line graph.

Again, if $X_1 \cong X_2 \cong \cdots \cong X_n \cong \mathbb{Z}_2, G_{Id}(X)$ being the complete graph K_{2^n}, is a line graph of $K_{1,2^n}$. Hence, the theorem follows. □

In [22], Mathil et al. have studied the planarity of the idempotent graph of a ring. They have established the below lemma.

Lemma 4.5. *The idempotent graph of a non-local commutative ring* $X \cong X_1 \times X_2 \times \cdots \times X_n$ *is planar if and only if* $X \cong X_1 \times X_2$ *and either of the following holds:*

(i) $(X_1,+) = \langle Id(X_1)\rangle$ *and* $(X_2,+) = \langle Id(X_2)\rangle$

(ii) $(X_1,+) = \langle Id(X_1)\rangle$ *and* $char(X_2) = 2$

(iii) $char(X_1) = 2$ *and* $char(X_2) = 2$.

Although the idempotent graph $G_{Id}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is both a line graph and a planar graph, it may be noted that neither every line idempotent graph is planar, nor every planar idempotent graph is a line graph. For example, $G_{Id}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ being the complete graph K_8 is a line graph, but not a planar graph. On the other hand, if $X \cong X_1 \times X_2$ satisfies the conditions $(X_1,+) = \langle Id(X_1)\rangle$ and $(X_2,+) = \langle Id(X_2)\rangle$, then $G_{Id}(X)$ is a planar graph but not a line graph as it contains $K_{1,3}$ as an induced subgraph (see Figure 3, [22]). If $X \cong X_1 \times X_2$ satisfies the conditions $(X_1,+) = \langle Id(X_1)\rangle$ and $char(X_2) = 2$ then also $G_{Id}(X)$ is a planar graph, but not a line graph. This is evidenced by considering the idempotent graph $G_{Id}(\mathbb{Z}_3 \times \mathbb{Z}_2)$ which contains κ_2 as an induced subgraph.

We summarize the above observation in the form of a theorem as given below.

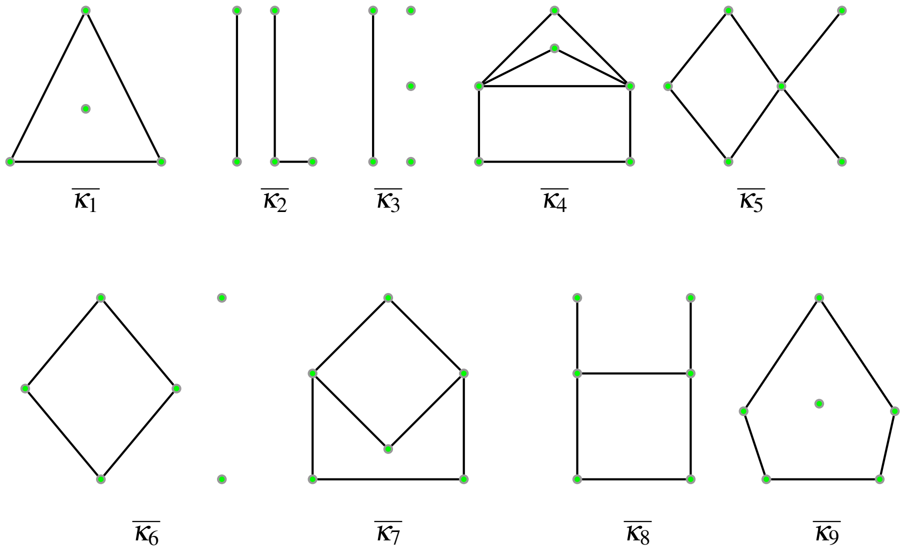

Figure 6: Forbidden induced subgraphs of complements of line graphs.

Theorem 4.6. *A line idempotent graph $G_{Id}(X)$ of a non-local commutative ring $X \cong X_1 \times X_2 \times \cdots \times X_n$ is planar if and only if $X \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

5. When the idempotent graphs are the line graph's complement

We study the case when a line graph's complement is isomorphic to the graph $G_{Id}(\mathbb{Z}_n)$ for some positive integer n.

Lemma 5.1. *[9] The graph $G_{Id}(\mathbb{Z}_n)$ is a line graph's complement if and only if none of the following graphs $\bar{\kappa}_i$ is contained in $G_{Id}(\mathbb{Z}_n)$ as an induced subgraph.*

Lemma 5.2. *[11] The path graph $P_n, n \geq 6$, is not a line graph's complement.*

Proof. Let us take a path graph with at least 6 vertices, as depicted in Figure 7. If we delete the vertex v_4 from the graph, then the remaining parts of the graph form a subgraph that is isomorphic to $\overline{\kappa_2}$. Hence, P_n can not be a line graph's complement.

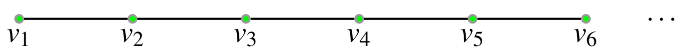

Figure 7: $P_n, n \geq 6$.

□

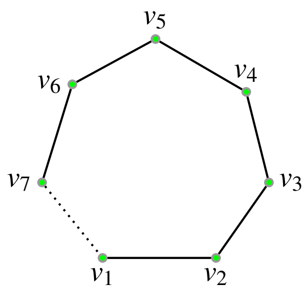

Figure 8: $C_n, n \geq 7$.

Lemma 5.3. *[11] The cycle graph $C_n, n \geq 7$, is not a line graph's complement.*

Proof. Let us take a cycle graph with at least 7 vertices, as depicted in Figure 8. If we delete the vertices v_4 and v_7 from the graph, then the remaining part of the graph forms a subgraph that is isomorphic to $\overline{\kappa_2}$. So, $C_n, n \geq 7$ is not a line graph's complement.

□

Theorem 5.4. *Let X be a finite commutative ring and $X \cong X_1 \times X_2 \times \cdots \times X_k$, $k \geq 3$, where each $X_i, 1 \leq i \leq k$ is a local ring and $|X_i| \geq 3$ for at least one $i \in \{1, 2, \ldots, k\}$. Then $G_{Id}(X)$ is not a line graph's complement.*

Proof. Let us consider $|X_i| = n$. Choose the subset $S = \{(0,0,...,0), (1,0,...,0), (0,1,...,0), (0,0,...,n-1)\}$ of the vertex set V of $G_{Id}(X)$. Clearly, S induces the subgraph $\overline{\kappa_1}$. Therefore, $G_{Id}(X)$ is not a line graph's complement. □

Theorem 5.5. *Let $X \cong X_1 \times X_2$ be a finite commutative ring, where $X_i, i = 1, 2$ are local rings. Then $G_{Id}(X)$ is a line graph's complement if and only if $X \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$.*

Proof. We consider the following instances.
Case 1. $\Big(|X_1| = n_1 \geq 3$ and $|X_2| = n_2 \geq 3\Big)$
Consider the vertex subset $S = \{(0,0), (1,0), (0,1), (n_1 - 1, n_2 - 1)\}$. Clearly, S induces the graph $\overline{\kappa_1}$. So, for this case, $G_{Id}(X)$ is not a line graph's complement.
Case 2. $\Big(|X_1| = 2$ and $|X_2| = n > 3\Big)$
There are two possibilities.
Subcase 2.1: $\Big(3 \notin Id(X_2)\Big)$
Consider the subset $S = \{(0,0), (1,0), (0,1), (1,2)\}$ which induces the subgraph $\overline{\kappa_1}$.

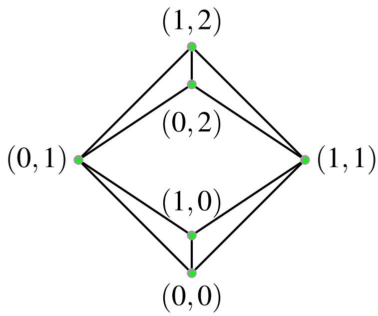

Figure 9: $G_{Id}(X = \mathbb{Z}_2 \times \mathbb{Z}_3)$.

Subcase 2.2: $\left(3 \in Id(X_2)\right)$
Choose the subset $S = \{(0,1),(1,0),(1,1),(0,2),(0,3),(1,3)\}$ of the vertex set of $G_{Id}(X)$. Clearly, it induces the subgraph $\overline{\kappa_6}$.
Case 3. $\left(|X_1| = 2 \text{ and } |X_2| = 3, \text{ i.e., } X \cong \mathbb{Z}_2 \times \mathbb{Z}_3\right)$
In Figure 9, the graph is shown.
Case 4. $\left(|X_1| = 2 \text{ and } |X_2| = 2, \text{ i.e., } X \cong \mathbb{Z}_2 \times \mathbb{Z}_2\right)$
The graph is a complete graph. Obviously, $G_{Id}(X)$ is the line graph's complement.

Evidently, $G_{Id}(X)$ is a line graph's complement in all possible cases. □

Acknowledgement

Osman Gani Mondal is supported by UGC, India for this research work. Both the authors thankfully acknowledge the suggestions received from Prof. Ayman Badawi regarding some improvements in the article.

Competing interest

The authors declare that there are no interests, financial or non-financial, directly or indirectly related to this work.

REFERENCES

[1] M. Afkhami, F. Khosh-Ahang, Unit graphs of rings of polynomials and power series, Arab. J. Math. 2 (3) (2013) 233—246.

[2] S. Akbari, M. Habibi, A. Majidinya, R. Manaviyat, On the idempotent graph of a ring, J. Algebra Appl. 12(6) (2013) 1350003 (14 pages).

[3] D. F. Anderson, A. Badawi, The total graph of a commutative ring, J. Algebra 320(7) (2008) 2706–2719.

[4] D. F. Anderson, P. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217 (1999) 434–447.

[5] N. Ashrafi, H. R. Maimani, M. R. Pournaki, S. Yassemi, Unit graphs associated to rings, Commun. Algebra 38 (8) (2010) 2851–2871.

[6] M. F. Atiyah, I. G. McDonald, Introduction to Commutative Algebra, Addison Wesley, 1969.

[7] Z. Barati, Line zero divisor graphs, J. Algebra Appl. 20(9) (2021) 2150154 (13 pages).

[8] Z. Barati, K. Khashyarmanesh, F. Mohammadi, K. Nafar, On the associated graphs to a commutative ring, J. Algebra Appl. 11 (2012) 1250037 (17 pages).

[9] L. W. Beineke, Characterizations of the derived graph, J. Comb. Theory 9 (1970) 129–135.

[10] T. T. Chelvam, T. Asir, On the total graph and its complement of a commutative ring, Commun. Algebra 41 (2013) 3820-–3835.

[11] D. Cvetkovic, S. Simic, Some remarks on the complement of a line graph, Publications de l'Institut Mathématique (Beograd) 17(31) (1974) 37–44.

[12] D. Dolžan, The metric dimension of the total graph of a finite commutative ring, Canad. Math. Bull., 59(4) (2016) 748–759.

[13] I. N. Herstein, Topics in Algebra, 2nd edition, John Willey and Sons, 1975.

[14] F. Heydari, M. J. Nikmehr, The unit graph of a left Artinian ring, Acta Math. Hungar. 139 (1–2) (2013), 134–146.

[15] P. Kanwar, M. Khatkar, R. K. Sharma, Idempotents and units of matrix rings over polynomial rings, Int. Electron. J. Algebra 22 (2017) 147–169.

[16] P. M. Magi, Sr. M. Jose, A. Kishore, Adjacency matrix and eigenvalues of the zero divisor graph $\Gamma(\mathbb{Z}_n)$, J. Math. Comput. Sci. 10 (2020) 1285–1297.

[17] Pranjali, A. Kumar, P. Sharma, Line graph of unit graphs associated with finite commuta-tive rings, Proyecciones (Antofagasta, On line), 40 (4) 919–926, 2021.

[18] S. Pirzada, A. Altaf, Line graphs of unit graphs associated with the direct product of rings, Korean J. Math. 30 (2022) 53–60 .

[19] S. Khojasteh, Line cozero-divisor graphs, Le Matematiche, 77 (2022), 293–306.

[20] S. Razzaghi, S. Sahebi, A graph with respect to idempotents of a ring, J. Algebra Appl. 20(6) (2021) 2150105 (11 pages).

[21] P. Sharma, S.Dutta, On idempotent graph of rings, Palestine J. Math. 12(1) (2023)

883–891.

[22] P. Mathil, B. Baloda, J. Kumar, On the idempotent graph of a ring, https://arxiv.org/abs/2306.08327.

O. GANI MONDAL
Department of Mathematics and Statistics
Aliah University, Kolkata – 700 160
India.
e-mail: osmanganimondal@gmail.com

SK. MD. ABU NAYEEM
Department of Mathematics and Statistics
Aliah University, Kolkata – 700 160
India.
e-mail: nayeem.math@aliah.ac.in