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LIE PRODUCT AND FIXED POINTS PRESERVERS

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Let $B(X)$ be the algebra of all bounded linear operators on a complex Banach space X. In this paper, it is determined the form of surjective maps $\phi : B(X) \to B(X)$ that satisfy $F(\phi(A)\phi(B) - \phi(B)\phi(A)) = F(AB - BA)$ for every $A, B \in B(X)$, where $F(A)$ denotes the set of all fixed points of an operator $A \in B(X)$.

1. Introduction

Preserving problems on operator algebras have attracted attention of many authors in the last decades. These problems concern the question of characterizing the form of all maps on operator algebras that leave invariant a certain property, and many results exposing the algebraic structure of such maps are obtained. Recently, some preserver problems concern certain properties of different types of products of operators (cf. [1, 2, 4–13]).

Let $B(X)$ denote the algebra of all bounded linear operators on a complex Banach space X. Let $A \in B(X)$. Recall that a vector $x \in X$ is a fixed point of A, whenever we have $Ax = x$. It is clear that the set of all fixed points of A is a subspace of X. Denote by $F(A)$ and dim $F(A)$ the set of all fixed points of A and the dimension of $F(A)$, respectively.

Given a vector $x \in X$ and a linear functional $f \in X^*$. The rank at most one operator, $x \otimes f$ is defined by $(x \otimes f)z = f(z)x$ for all $z \in X$. Note that

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$$x \otimes f \text{ is nilpotent if and only if } f(x) = 0,$$

and

$$x \otimes f \text{ is idempotent if and only if } f(x) = 1.$$

Denote by $P_1(X)$ the set of all rank one idempotent operators.

Recall the lattice, $Lat(A)$, is the set of all invariant subspaces of A and so $F(A) \in Lat(A)$ for every $A \in B(X)$ (see [3]). Recall also that the set of fixed points of the operator $x \otimes f$ is given by:

$$F(x \otimes f) = \begin{cases} span\{x\} & \text{if } x \otimes f \text{ is idempotent,} \\ \{0\} & \text{if } x \otimes f \text{ is not idempotent.} \end{cases}$$

Authors in [11] characterized the forms of surjective maps on $B(X)$ which preserve the dimension of fixed points of products of operators. More precisely, it was shown that if $\phi : B(X) \to B(X)$ is a surjective map which satisfies dim $F(AB) =$ dim $F(\phi(A)\phi(B))$, for every $A, B \in B(X)$, then there exists an invertible operator $S \in B(X)$ such that $\phi(A) = SAS^{-1}$ for all $A \in B(X)$ or $\phi(A) = -SAS^{-1}$ for all $A \in B(X)$. Authors in [13], considered the maps $\phi : B(X) \to B(X)$ and $\phi : M_n B(X) \to B(X)$ satisfying $F(A+B) = F(\phi(A)+\phi(B))$ and $\dim F(A+B) = \dim F(\phi(A)+\phi(B))$, respectively. Moreover, authors in [12], considered the forms of surjective maps on $B(X)$ which preserve the fixed points of triple Jordan products of operators, i.e., $F(ABA) = F(\phi(A)\phi(B)\phi(A))$, for all $A, B \in B(X)$.

Authors in [4] characterized the forms of surjective maps on $B(X)$ which preserve the Jordan product. More precisely, it was shown that if $\phi : B(X) \to B(X)$ is a surjective map which satisfies

$$F(\phi(T)\phi(A) + \phi(A)\phi(T)) = F(TA + AT)$$

for all $A, T \in B(X)$, then there exists a nonzero scalar $\alpha \in \mathbb{C}$ with $\alpha^2 = 1$ such that $\phi(T) = \alpha T$ for all $T \in B(X)$.

In [7] author showed that if $\phi : B(X) \to B(X)$ is a surjective additive map which satisfies

$$F(AB + BA) \subseteq F(\phi(A)\phi(B) + \phi(B)\phi(A)),$$

for every $A, B \in B(X)$, then $\phi(A) = A$, for every $A \in B(X)$ or $\phi(A) = -A$, for every $A \in B(X)$.

The Lie product of $A, B \in B(X)$ is defined as $[A, B] = AB - BA$. The aim of this paper is to continue these works by studying maps on $B(X)$ which preserve the fixed points of Lie products of operators. The complete form of our main

result is as following:
Main theorem. Let X be a complex Banach space with $\dim X \geq 4$. Let $\phi : B(X) \to B(X)$ be a surjective map. Then ϕ satisfies

$$F\left(\phi(A)\phi(B)-\phi(B)\phi(A)\right)=F\left(AB-BA\right),$$

for all $A,B \in B(X)$ if and only if there exist a nonzero scalar $\gamma \in \mathbb{C}$ with $\gamma^2 = 1$ and a scalar function $\tau : B(X) \to \mathbb{C}$ such that $\phi(A) = \gamma A + \tau(A)I$ for all $A \in B(X)$.

2. Preliminaries and Notations

In this text, we denote by $F_1(X)$ and $N_1(X)$ the set of all rank at most one operators and the set of all rank one nilpotent operators on X, respectively on X. For every operator $T \in B(X)$, let $N(T)$ be the kernel of T, and $R(T)$ be its range.

Lemma 2.1. *Let $A \in B(X)$. The following statements are equivalent:*
(i) $A \in \mathbb{C}I$,
(ii) $F\left(AT-TA\right)=\{0\}$, for all $T \in P_1(X)$.

Proof. Since $(i) \Rightarrow (ii)$ is obvious, we need only to prove the implication $(ii) \Rightarrow (i)$. To prove this claim, we show that for every $x \in X$, x and Ax are linearly dependent. Suppose it is not, so there exists $x \in X$ such that x and Ax are linearly independent. We distinguish two cases:

Case 1. Let x, Ax and A^2x are linearly independent, it follows that there exists $f \in X^*$ such that

$$f(x)=1, \quad f(Ax)=0, \quad f\left(A^2x\right)=-2.$$

Set $x \otimes f$. So if $T = x \otimes f$ then $(AT-TA)(x+Ax) = x+Ax$, hence we get $x+Ax \in F(AT-TA)$. This is a contradiction.

Case 2. If not, then there exist $a \neq 0$ and $b \in \mathbb{C}$ such that $Ax = aA^2x + bx$. Let $f \in X^*$ such that $f(x) = 1$ and $f(Ax) = 1-\eta$, where η is a complex scalar satisfying $\eta^2 a + \eta(1-2a) + b - 1 = 0$. Consider an operator $T \in B(X)$ such that $T = x \otimes f$. Hence we have $(AT-TA)(\eta x + Ax) = \eta x + Ax$ which is a contradiction. So for every $x \in X$, x and Ax are linearly dependent and hence according to the statement of [8, Lemma 2.4] should be included there exists a scalar $\lambda \in \mathbb{C}$ such that $A = \lambda I$. □

Lemma 2.2. *Let A and B be two operators. The following statements are equivalent:*
(i) There exists a scalar $\lambda \in \mathbb{C}$ such that $A = B + \lambda I$,
(ii) $F(AT - TA) = F(BT - TB)$ for all $T \in P_1(X)$.

Proof. $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$. If $F(AT - TA) = F(BT - TB) = \{0\}$ for all $T \in P_1(X)$, then from Lemma 2.1 we have $A, B \in \mathbb{C}I$. It is easily shown that $A = B + \lambda I$, for some scalar $\lambda \in \mathbb{C}$.

Assume that $F(AT - TA) \neq \{0\}$ for some $T \in P_1(X)$. It follows from Lemma 2.1, that there exists $x \in X$ such that x and Ax are linearly independent. Suppose x, Ax and Bx are linearly independent. Similar to proof of Lemma 2.1, we obtain $\eta x + Ax \in F(AT - TA)$, where $\eta \in \mathbb{C}$. On the other hand $F(BT - TB) \subseteq span\{x, Bx\}$ which follows $\eta x + Ax \in span\{x, Bx\}$. This is a contradiction.
Now, x, Ax and Bx are linearly dependent for all $x \in X$. Lemma 2.4 in [9] tell us that there exist $\beta, \lambda \in \mathbb{C}$ such that $A = \beta B + \lambda I$. By hypothesis, $F(BT - TB) = F((\beta B + \lambda I)T - T(\beta B + \lambda I))$. Hence $F(BT - TB) = F(\beta(BT - TB)$. Since $F(BT - TB) \neq \{0\}$, we conclude $\beta = 1$. □

Lemma 2.3. *For a nonzero operator $A \in B(X)$, the following statements are equivalent:*
(i) $A \in F_1(X) + \mathbb{C}I$ (ii) $\dim\ F(AT - TA) \leq 1, for\ \ all\ \ T \in B(X)$

Proof. For $(i) \Rightarrow (ii)$, let $T \in B(X)$ be an arbitrary operator, and consider a operator $A = x \otimes f + \lambda I$ where $x \in X$, $f \in X^*$ and $\lambda \in \mathbb{C}$. Note that for every $y \in X$ we have $(AT - TA)y = f(Ty)x - f(y)Tx$.

If $A = x \otimes f$ is non nilpotent we have $x \notin F(AT - TA)$. If it's nilpotent then one of two vectors x or Tx is not in $F(AT - TA)$. Hence, we obtain that $\dim\ F(AT - TA) \leq 1$.

Conversely, if $A = \lambda I$ where λ is a nonzero scalar, then the sentence is complete. Suppose that there exists a vector $x \in X$ such that x, Ax and A^2x are linearly independent. Let $T \in B(X)$ such that

$$Tx = 0, \quad TAx = -x \quad \text{and } T\left(A^2x\right) = -2Ax.$$

Then

$$\begin{cases} (AT - TA)x = x \\ (AT - TA)Ax = Ax \end{cases}$$

which implies that $span\{x, Ax\} \subseteq F(AT - TA)$ is a contradiction.
Therefore x, Ax and A^2x are linearly dependent for all $x \in X$, then from Lemma

2.4 in [9] we have

$$A^2 = \lambda A + aI, \quad for \ some \ scalars \ \lambda, a \in \mathbb{C}. \tag{1}$$

Case 1. Let A be a non-scalar operator such that A is injective. We have $a \neq 0$. Indeed, if $a = 0$ then $A(A - \lambda I)x = 0$ for every $x \in X$. From $\dim N(A) = 0$ follows $A = \lambda I$ which is a contradiction. Thus A is invertible and so it is surjective. Since A is a non-scalar operator there exist linearly independent vectors $x_i \in X, i = 1,2$ such that $Ax_1 = x_2$. From $\dim X \geq 4$ and invertibility of A there exist $x_i \in X, i = 3,4$ such that $x_i \in X, i = 1,2,3,4$ are linearly independent vectors and $Ax_3 = x_4$. We choose $T \in B(X)$ to be an operator satisfying

$$Tx_i = 0, i = 1,3 \quad and \quad Tx_j = -x_{j-1}, j = 2,4.$$

It is easy to show $(AT - TA)x_i = x_i, i = 1,3$. This is a contradiction.

Case 2 If A is not injective then we obtain $a = 0$. Hence $(A - \lambda I)A = 0$. It follows that $R(A - \lambda I) \subset N(A)$.

Suppose that $\dim R(A - \lambda I) \geq 2$. There exist $x_i, y_i \in X, i = 1,2$ such that $(A - \lambda I)x_i = y_i$ and $y_1, y_2 \in N(A)$ are linearly independent vectors. Hence $Ax_i \neq \lambda x_i$. Since $y_1, y_2 \in N(A)$ we can obtain $\{x_1, y_1, y_2\}$ and $\{x_2, y_1, y_2\}$ are linearly independent vectors.

Now, we choose $T \in B(X)$ to be an operator satisfying

$$Tx_i = 0, \quad and \quad TAx_i = -x_i, \quad i = 1,2.$$

One can get $(AT - TA)(x_i) = x_i$ for $i = 1,2$. Note that from linearity y_1 and y_2 we obtain x_1 and x_2 are linearly independent vectors. This is a contradiction. Therefore, we have $\dim R(A - \lambda I) \leq 1$ and so $A \in F_1(X) + \mathbb{C}I$.

□

3. Proof of Main Theorem

Clearly, we only need to prove the necessary implication.

Step 1. For every operator $R \in B(X)$, we have $\phi(R) \in F_1(X) + \mathbb{C}I$ if and only if $R \in F_1(X) + \mathbb{C}I$.

If $\phi(R) \in F_1(X) + \mathbb{C}I$, then for all $T \in B(X)$ we have

$$\dim F(RT - TR) = \dim F(\phi(R)\phi(T) - \phi(T)\phi(R)) \leq 1.$$

By Lemma 2.3 we have $R \in F_1(X) + \mathbb{C}I$.
Conversely, if dim $F(RT - TR) \leq 1$, then using Lemma 2.3 and surjectivity of ϕ we have

$$\phi(R) \in F_1(X) + \mathbb{C}I.$$

Step 2. For every $A \in N_1(X)$ there exist scalars $\lambda_A, \gamma_A \in \mathbb{C}$ such that $\phi(A) = \gamma_A A + \lambda_A I$.

Let $A = z \otimes h \in N_1(X)$. From Step 1, there exist $z' \in X$, $h' \in X^*$ and $\lambda \in \mathbb{C}$ such that $\phi(z \otimes h) = z' \otimes h' + \lambda I$. Firstly, we show that z and z' are linearly dependent. Assume that z and z' are linearly independent. There exists $x \notin N(h)$ such that x and z' are linearly independent. We can choose $f \in X^*$, such that $f(x) = 0 = f(z')$ and $f(z)h(x) = 1$. From step 1, there exist $y \in X$, $g \in X^*$ and $\gamma \in \mathbb{C}$ such that $\phi(x \otimes f) = y \otimes g + \gamma I$. We have

$$\begin{aligned} span\{z\} &= F((z \otimes h)(x \otimes f) - (x \otimes f)(z \otimes h)) \\ &= F(\phi(z \otimes h)\phi(x \otimes f) - \phi(x \otimes f)\phi(z \otimes h) \\ &= F((z' \otimes h')(y \otimes g) - (y \otimes g)(z' \otimes h')). \end{aligned}$$

Hence we have

$$g(z)h'(y)z' - h'(z)g(z')y = z. \tag{2}$$

Also,

$$\begin{aligned} span\{x\} &= F((x \otimes f)(z \otimes h) - (z \otimes h)(x \otimes f)) \\ &= F(\phi(x \otimes f)\phi(z \otimes h) - \phi(z \otimes h)\phi(x \otimes f)) \\ &= F(y \otimes g z' \otimes h' - z' \otimes h' y \otimes g), \end{aligned}$$

which follows

$$h'(x)g(z')y - g(x)h'(y)z' = x. \tag{3}$$

By acting f on both direction equation (3) we obtain

$$h'(x)g(z')f(y) = 0. \tag{4}$$

It follows $h'(x)g(z') = 0$ or $f(y) = 0$.

If $h'(x)g(z') = 0$, then from (3) we have x and z' are linearly dependent which is a contradiction.

If $f(y) = 0$, then by acting f on both direction equation (2) we obtain $f(z) = 0$ which is a contradiction. Therefore, z and z' are linearly dependent.

Without loss of generality, we can write $\phi(z\otimes h) = z\otimes h^{'} + \lambda I$.

Now, we show h and $h^{'}$ are linearly dependent. If not, there exist $x \in X$ such that $h(x) \neq 0$ and $h^{'}(x) = 0$. Hence, we can choose $f \in X^{*}$ such that $f(x) = 0$ and $f(z)h(x) = 1$. As above there exist $f^{'} \in X^{*}$ and $\gamma \in \mathbb{C}$ such that $\phi(x\otimes f) = y\otimes f^{'} + \gamma I$. We have

$$\begin{aligned} span\{z\} &= F((z\otimes h)(x\otimes f) - (x\otimes f)(z\otimes h)) \\ &= F(\phi(z\otimes h)\phi(x\otimes f) - \phi(x\otimes f)\phi(z\otimes h)) \\ &= F((z\otimes h^{'})(y\otimes f^{'}) - (y\otimes f^{'})(z\otimes h^{'})) \\ &= \{0\}. \end{aligned}$$

That is a contradiction. Therefore, there exist two scalar $\gamma_A, \lambda_A \in \mathbb{C}$ such that $\phi(A) = \gamma_A A + \lambda_A I$.

Step 3. There exist a nonzero scalar $\gamma \in \mathbb{C}$ with $\gamma^2 = 1$ and a map $\lambda : B(X) \to \mathbb{C}$ such that $\phi(A) = \gamma A + \lambda(A)I$ for every $A \in P_1(X)$.
Let $A = x\otimes f \in P_1(X)$ and $N = z\otimes h \in N_1(X)$. By Step 1 and Step 2 there exist $y \in X$, $g \in X^{*}$ and $\eta \in \mathbb{C}$ such that $\phi(x\otimes f) = y\otimes g + \lambda_A I$ and $\phi(z\otimes h) = \gamma_N z\otimes h + \lambda_N I$. Assume that x and y are linearly independent.
We continue to prove this Step in the following two cases.
Case 1. f, g are linearly independent.

Then we can choose $z \in Kerf \setminus Kerg$ and $h \in X^{*}$ such that $\gamma_A g(z)h(y) = 1$ and $h(z) = 0$. We can conclude

$$\begin{aligned} \{0\} &= F((z\otimes h)(x\otimes f) - (x\otimes f)(z\otimes h)) \\ &= F(\phi(z\otimes h)\phi(x\otimes f) - \phi(x\otimes f)\phi(z\otimes h)) \\ &= F(\gamma_A((z\otimes h)(y\otimes g) - (y\otimes g)(z\otimes h)) \\ &= span\{z\}. \end{aligned}$$

That is a contradiction.
Case 2. f, g are linearly dependent. We can choose $z \notin Kerf$ and define $h \in X^{*}$ such that $\gamma_A g(z)h(y) = -1$, $f(z)h(x) = 1$ and $h(z) = 0$, where, $A = z\otimes h$. Then

$$z \in F((z\otimes h)(x\otimes f) - (x\otimes f)(z\otimes h)).$$

But

$$z \notin F((z\otimes h)(y\otimes g) - (y\otimes g)(z\otimes h)).$$

That is a contradiction. Therefore, without of we can write $\phi(x\otimes f) = x\otimes g + \lambda_A I$.

Now, suppose that g and f are linearly independent. There exists $z \in X$ such that $h(z) = 0 = g(z)$ and $f(z)h(x) = 1$. Hence, we can write

$$\begin{aligned} span\{z\} &= F((z \otimes h)(x \otimes f) - (x \otimes f)(z \otimes h)) \\ &= F(\phi(z \otimes h)\phi(x \otimes f) - \phi(x \otimes f)\phi(z \otimes h)) \\ &= F((z \otimes h)(x \otimes g) - (x \otimes g)(z \otimes h)) \\ &= \{0\}. \end{aligned}$$

That is a contradiction. Therefore, $\phi(A) = \gamma_A A + \lambda I$.
Let $A \in P_1(X)$, in fact A is a non-scalar operator and $A \neq 0$. By Lemma 2.1 there exists $B \in P_1(X)$ such that $F(AB - BA) \neq \{0\}$. We can conclude

$$\begin{aligned} F(AB - BA) &= F(\phi(A)\phi(B) - \phi(B)\phi(A)) \\ &= F(\gamma_A \gamma_B (AB - BA)). \end{aligned}$$

It follows $\gamma_A \gamma_B = 1$.
For $A \in P_1(X)$ we have

$$\begin{aligned} \{0\} &= F(AI - IA) \\ &= F(\phi(A)\phi(I) - \phi(I)\phi(A)) \\ &= F(\gamma_A (A\phi(I) - \phi(I)A)) \\ &= F((A(\gamma_A)\phi(I) - (\gamma_A \phi(I))A)). \end{aligned}$$

which follows $\gamma_A \phi(I) = \beta I$. Because this equality holds for every $A \in P_1(X)$, we have $\gamma_A = cte$ and so we obtain $\phi(A) = \gamma A + \lambda_A I$ with $\gamma^2 = 1$.

Step 4. ϕ takes the desired form.
From Step 3 we have

$$\begin{aligned} F(AT - TA) &= F(\phi(A)\phi(T) - \phi(T)\phi(A)) \\ &= F(\gamma\phi(A)T - \gamma T\phi(A)), \end{aligned}$$

for every $T \in P_1(X)$ and $A \in B(X)$. This identity together with Lemma 2.2 proved that there exists a map $\tau : B(X) \to \mathbb{C}$ such that $\phi(A) = \gamma A + \tau(A)I$ for all $A \in B(X)$ with $\gamma^2 = 1$.
This completes the proof.

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