LE MATEMATICHE
Vol. LXXIX (2024) – Issue I, pp. 93–111
doi: 10.4418/2024.79.1.5

AN APPROACH TO ZAREMBA'S CONJECTURE THROUGHOUT FIBONACCI SEQUENCES

K. AYADI - S. KALIA - M. SAADAOUI

According to Zaremba's conjecture, any integer is estimated to occur as the denominator of a finite continued fraction whose partial quotients are restricted by 5. This conjecture is proven true for infinity many integers, and we illustrate it for more additional ones that are related to Fibonacci sequences and its generalization.

1. Introduction

Zaremba's conjecture is an attractive open question in the theory of continued fractions. It's stated that for any positive integer q, there exists an integer p with $0 < p < q$ and $\gcd(p,q) = 1$ such that

$$\frac{p}{q} = [0, a_1, a_2, \ldots, a_s] \quad \text{with} \quad \max\{a_1, a_2, \ldots, a_s\} \leq 5\,.$$

In fact, this bound is conjectured to be replaced by 4 except for $q = 6, 54, 150$ ([12, A195901]) and replaced by 3 except 23 values, the largest one being $q = 6234$ ([2],[9, p.990]). Hensley [6] conjectured that for a large, prime denominator, a bound equal to 2 is sufficient. Ever since, further tightening this

Received on February 9, 2024

AMS 2010 Subject Classification: 11A55, 11D68
Keywords: Continued fraction, Zaremba's conjecture

bound has been an ongoing problem in the number theory community. Although great progress has been made in measure theory in recent years (see, e.g., [3, 8] and references therein), the essential problem remains unsolved. Zaremba's conjecture is connected with some questions about the theory of numerical integration. In recent years, there has been significant progress in relation to the Zaremba conjecture, associated with deep ideas about the growth of groups and the affine sieve. It has been proved by Niederreiter [9] that Zaremba's conjecture holds for special q of exponential type, for example, for those being powers of 2, 3 and 5, and the bound being 3. Yodphotong and Laohakosol [14] proved for q being powers of 6. By computations in [2], Zaremba's conjecture holds with bound 3 for $q = 2^n$ with $6 \leq n \leq 35$. In [1], the first author and Komatsu have proved Zarembas's conjecture for infinitely many integers including that for q^{2^n} for all $n \geq 0$ where $2 \leq q \leq 100$. On the other hand, it is well known that Zaremba's conjecture holds for the Fibonacci sequence $(F_n)_n$ and also for Fibonacci number $(F_n(a))_n$, for a fixed $2 \leq a \leq 5$ as consequence of the continued fraction of the ratio of two consecutive of these numbers, then, by the computation of the rational number representing $[0,\underbrace{b,\dots,b}_{m},\underbrace{a,\dots,a}_{n}] = [0,b^m,a^n]$, $[0,\underbrace{a,\dots,a}_{n},\underbrace{b,\dots,b}_{m},\underbrace{a,\dots,a}_{n}] = [0,a^n,b^m,a^n]$, the first author and Komatsu [1] have proved the following result.

Theorem 1.1. *For all $1 \leq a \leq 5$ and $1 \leq b \leq 5$, Zaremba conjecture is valid for*

1. $F_{n+1}(a)F_{m+1}(b) + F_n(a)F_m(b) \quad (n \geq 1, m \geq 1)$

2. $F_{n+1}(a)^2F_{m+1}(b) + F_n(a)^2F_{m-1}(b) + 2F_{n+1}(a)F_n(a)F_m(b) \quad (n \geq 1, m \geq 2)$ *and their 2^n-th powers except for $a = 5$ or $b = 5$.*

In [5], Dromta has given the continued fraction expansion of $\dfrac{F_n}{(F_{n+1}+1)}$ explicitly:

Theorem 1.2. *For $n, l \geq 1$ we have*

1. The continued fraction expansion of $\dfrac{F_n}{(F_{n+1}+1)}$ *is given by*

$$\frac{F_n}{(F_{n+1}+1)} = \begin{cases} [0,1^{\frac{n-1}{2}},5,4^{\frac{n-13}{6}}] & \textit{for } n = 12l+1, \\ [0,1^{\frac{n-3}{2}},2,3,4^{\frac{n-9}{6}}] & \textit{for } n = 12l+3, \\ [0,1^{\frac{n-1}{2}},5,4^{\frac{n-11}{6}},3] & \textit{for } n = 12l+5, \\ [0,1^{\frac{n-3}{2}},2,3,4^{\frac{n-13}{6}},5] & \textit{for } n = 12l+7, \\ [0,1^{\frac{n-1}{2}},5,4^{\frac{n-9}{6}}] & \textit{for } n = 12l+9, \\ [0,1^{\frac{n-3}{2}},2,3,4^{\frac{n-11}{6}},3] & \textit{for } n = 12l+11, \\ [0,1^{\frac{n}{2}-2},2] & \textit{for } n = 4l, \\ [0,1^{\frac{n}{2}-1},3] & \textit{for } n = 4l+2. \end{cases}$$

2. The continued fraction expansion of $\dfrac{F_n(2)}{(F_{n+1}(2)+1)}$ *is given by*

$$\frac{F_n(2)}{(F_{n+1}(2)+1)} = \begin{cases} [0,2^{\frac{n-1}{2}},4,2^{\frac{n-5}{2}},3] & \textit{for } n = 4l+1, \\ [0,2^{\frac{n-1}{2}},1,1,1,2^{\frac{n-5}{2}},3] & \textit{for } n = 4l+3. \end{cases} \tag{1}$$

3. The continued fraction expansion of $\dfrac{F_n(4)}{(F_{n+1}(4)+1)}$ *is given by*

$$\frac{F_n(4)}{(F_{n+1}(4)+1)} = \begin{cases} [0,4^{\frac{n-1}{2}},5,1,(1,1,1)^{\frac{n-3}{2}},2] & \textit{for } n = 4l+1, \\ [0,4^{\frac{n-1}{2}},3,2,(1,1,1)^{\frac{n-3}{2}},2] & \textit{for } n = 4l+3. \end{cases} \tag{2}$$

According to this theorem, Zaremba's conjecture holds for $F_{n+1}+1$, $F_{n+1}(2)+1$ and $F_{n+1}(4)+1$ for n satisfying the congruence listed in the theorem.

In this paper, Zaremba's conjecture will be examined in relation to the Fibonacci sequences as complementary results to the previous two theorems. Section 2 contains a background on continued fractions of real numbers and Fibonacci sequences. In Section 3, we introduce three lemmas needed to prove our results. The fourth and fifth Sections contain, respectively, our main results and their proofs.

2. Continued fraction, Fibonacci sequence and Properties

A continued fraction expansion is an expression of the shape:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

which we represent using a flat notation to save space by

$$[a_0, a_1, \cdots, a_n, \cdots],$$

where $(a_i)_{i\geq 1}$ are positive integers and $a_0 \in \mathbb{Z}$. Every continued fraction converges to a real number, and the equality of two such continued fraction implies the equality of the corresponding partial quotients, except for the ambiguity in the last digits in the terminating case due to $n = (n-1) + 1/1$.
The n^{th} convergent of the continued fraction $[a_0, a_1, \cdots, a_n, \cdots]$ is given by

$$\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n],$$

where the numerators p_n and denominators q_n are given in terms of the coefficients according to the matrix identity

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

and both p_n and q_n are obtained recursively via the same linear three-term recurrence relation, that is

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} & \text{for } n \geq 1 \\ q_n = a_n q_{n-1} + q_{n-2} & \text{for } n \geq 1 \end{cases} \tag{3}$$

with the initial values

$$p_{-1} = 1, \quad p_0 = a_0, \quad q_{-1} = 0, \quad q_0 = 1.$$

The following proposition collect basic properties needed throughout.

Proposition 2.1. *For $n \geq 1$, we have*

1. $\dfrac{\alpha p_n + p_{n-1}}{\alpha q_n + q_{n-1}} = [a_0, a_1, \cdots, a_n, \alpha]$, *where* $\alpha \in \mathbb{R}^*$.

2. $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.

3. $[a_n, a_{n-1}, \ldots, a_1] = \dfrac{q_n}{q_{n-1}}$.

The reader is directed to [10] or [13] for detailed information, proofs and further results on continued fractions of a real number.
From the equality (1) of this we deduce that, if $p_n/q_n = [a_0, a_1, a_2, \ldots, a_n]$ and $r_m/s_m = [b_0, b_1, \ldots, b_m]$, then

$$[a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m] = \frac{p_{n-1}s_m + p_n r_m}{q_{n-1}s_m + q_n r_m}. \tag{4}$$

We need also to recall some properties of continued fraction which are detailed in [11]. For $\beta \in \mathbb{Q}^*$ and $a, b \in \mathbb{N}^*$ we have

$$[a, -b, -\beta] = [a-1, 1, b-1, \beta]. \tag{5}$$

$$[\ldots, a, 0, b, \ldots] = [\ldots, a+b, \ldots]. \tag{6}$$

Our work is based on the Fibonacci number sequences, which are described by:

$$F_n = \begin{cases} 0 \text{ if } n = 0 \\ 1 \text{ if } n = 1 \\ F_{n+1} = F_n + F_{n-1} \quad \text{for all } n \geq 1. \end{cases} \tag{7}$$

The sequence's initial few values are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ ([12, A000045]).
The Fibonacci polynomials are a polynomial sequence that can be considered a generalization of the Fibonacci numbers. These polynomials are defined by the recurrence relation:

$$F_n(a) = \begin{cases} 0 \text{ if } n = 0 \\ 1 \text{ if } n = 1 \\ F_{n+1}(a) = aF_n(a) + F_{n-1}(a) \quad \text{for all } n \geq 1. \end{cases} \tag{8}$$

Clearly $F_n(1) = F_n$. The first few Fibonacci polynomials are:
$F_0(a) = 0,\ F_1(a) = 1,\ F_2(a) = a,\ F_3(a) = a^2 + 1,\ F_4(a) = a^3 + 2a,\ F_5(a) = a^4 + 3a^2 + 1,\ F_6(a) = a^5 + 4a^3 + 3a$ ...
The sequence $(L_n)_n$ of Lucas numbers is defined in accordance with the same principle as Fibonacci numbers: $L_n = L_{n-1} + L_{n-2}$. Here, however, the starting

values are $L_0 = 2$ and $L_2 = 1$. There are countless connections between the Fibonacci and Lucas numbers, Fibonacci and Lucas polynomials. The last ones are denoted by $(L_n(a))_n$, ([12, A114525]).

We derive some properties and recurrence relations of Fibonacci polynomials, see [7]. Let $a \geq 1$ be an integer, then for all integers $m, n \geq 1$

$$F_{n+1}(a)F_{n-1}(a) - F_n^2(a) = (-1)^n \qquad \text{(Cassini's identity)} \tag{9}$$

$$F_{n+1}(a) + F_{n-1}(a) = L_n(a) \tag{10}$$

$$F_{2n}(a) = F_{n+1}^2(a) - F_{n-1}^2(a) = (2F_{n+1}(a) - F_n(a))F_n(a) \tag{11}$$

$$F_{2n-1}(a) = F_n^2(a) + F_{n-1}^2(a). \tag{12}$$

3. Primarily Lemmas

The following lemmas are keys to our main results:

Lemma 3.1. *Let for a positive integer n,* $p_n/q_n = [0, a_1, a_2, \ldots, a_n] = [0, \overrightarrow{X_n}]$. *Let y be a nonzero integer. Then*

1. $[0, \overrightarrow{X_n}, y, \overrightarrow{X_n}] = \dfrac{yp_nq_n + p_n^2 + q_np_{n-1}}{yq_n^2 + q_n(p_n + q_{n-1})}$.

2. $[0, \overrightarrow{X_n}, y, -\overrightarrow{X_n}] = \dfrac{yp_nq_n - p_n^2 + q_np_{n-1}}{yq_n^2 + q_n(q_{n-1} - p_n)}$.

3. $[0, \overrightarrow{X_n}, y, \overleftarrow{X_n}] = \dfrac{yq_np_n + p_nq_{n-1} + q_np_{n-1}}{yq_n^2 + 2q_nq_{n-1}}$.

Proof. By the identity (1) and (3) of the Proposition 2.1:

1.

$$\begin{aligned} [0, \overrightarrow{X_n}, y, \overrightarrow{X_n}] &= [0, \overrightarrow{X_n}, y + p_n/q_n] \\ &= \frac{p_n(y + p_n/q_n) + p_{n-1}}{q_n(y + p_n/q_n) + q_{n-1}} \\ &= \frac{yp_nq_n + p_n^2 + q_np_{n-1}}{yq_n^2 + q_n(p_n + q_{n-1})}. \end{aligned}$$

2.

$$\begin{aligned}[0,\overrightarrow{X_n},y,-\overrightarrow{X_n}] &= [0,\overrightarrow{X_n},y-p_n/q_n] \\ &= \frac{p_n(y-p_n/q_n)+p_{n-1}}{q_n(y-p_n/q_n)+q_{n-1}} \\ &= \frac{yp_nq_n-p_n^2+q_np_{n-1}}{yq_n^2-q_np_n+q_nq_{n-1}}.\end{aligned}$$

3.

$$\begin{aligned}[0,\overrightarrow{X_n},y,\overleftarrow{X_n}] &= [0,\overrightarrow{X_n},y+q_{n-1}/q_n] \\ &= \frac{p_n(y+q_{n-1}/q_n)+p_{n-1}}{q_n(y+q_{n-1}/q_n)+q_{n-1}} \\ &= \frac{yq_np_n+p_nq_{n-1}+q_np_{n-1}}{yq_n^2+2q_nq_{n-1}}.\end{aligned}$$

□

Lemma 3.2. *Let n, a, b, c, d, x and y be positive integers, then*

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\underbrace{\cdots}_{n-4\text{ 1's}}\cfrac{1}{1+\cfrac{1}{\frac{ax+cy}{bx+dy}}}}}} = \frac{(aF_n+bF_{n-1})x+(cF_n+dF_{n-1})y}{(aF_{n+1}+bF_n)x+(cF_{n+1}+dF_n)y}.$$

Proof. The continued fraction expansion

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\underbrace{\cdots}_{n-4\text{ 1's}}\cfrac{1}{1+\cfrac{1}{\frac{a}{b}}}}}}$$

with n 1's in total and where a and b are constants, can be evaluated from the bottom up. By absorbing a few of the ones into the fraction at the bottom, we get

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\underbrace{\cdots}_{n-4\text{ 1's}}\cfrac{1}{1+\cfrac{1}{\frac{a}{b}}}}}} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\underbrace{\cdots}_{n-5\text{ 1's}}\cfrac{1}{1+\cfrac{1}{\frac{a+b}{a}}}}}}$$

$$= \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\underbrace{\cdots}_{n-6\text{ 1's}}\cfrac{1}{1+\cfrac{1}{\frac{2a+b}{a+b}}}}}}$$

We can see that a Fibonacci-like sequence f_n is forming, with $f_0 = b, f_1 = a$ and the fraction at the bottom is a ratio between consecutive terms of this sequence. This Fibonacci-like sequence can be reduced to a linear combination of F_n and F_{n+1}, since any two beginning values can be expressed and the Fibonacci recurrence will be satisfied by a linear combination. Since $F_0 = 0$ and $F_1 = 1$, the coefficient of F_{n+1} must be b. Then, since $F_1 = 1$ and $F_2 = 1$ and the coefficient of F_{n+1} is b, the coefficient of F_n must be $a-b$. Therefore, our Fibonacci-like sequence is $bF_{n+1} + (a-b)F_n$ and the original expression is equal to

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\underbrace{\cdots}_{n-4\text{ 1's}}\cfrac{1}{1+\cfrac{1}{\frac{aF_1+bF_0}{aF_0+bF_{-1}}}}}}} = \frac{aF_n + bF_{n-1}}{aF_{n+1} + bF_n}.$$

since there are n 1's. Similarly, the continued fraction expansion

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\underbrace{\cdots}_{n-4\text{ 1's}}\cfrac{1}{1+\cfrac{1}{\frac{ax+cy}{bx+dy}}}}}}$$

where a, b, c, and d were constants, is equal to

$$\frac{(aF_n + bF_{n-1})x + (cF_n + dF_{n-1})y}{(aF_{n+1} + bF_n)x + (cF_{n+1} + dF_n)y}$$

since each variable's form its own Fibonacci-like sequence of coefficients. □

Lemma 3.3. *Let n be an odd integer. Let m, a and b be positive integers. Then*

$$\cfrac{1}{L_n + \cfrac{1}{L_n + \cfrac{1}{L_n + \underbrace{\cdots}_{m-4\ L_n\text{'s}} \cfrac{1}{L_n + \cfrac{1}{\frac{a}{b}}}}}} = \frac{aF_{nm} + bF_{nm-n}}{aF_{nm+n} + bF_{nm}}.$$

Proof. For the continued fraction expansion

$$\cfrac{1}{L_n + \cfrac{1}{L_n + \cfrac{1}{L_n + \underbrace{\cdots}_{m-4\ L_n\text{'s}} \cfrac{1}{L_n + \cfrac{1}{\frac{a}{b}}}}}}$$

where n is odd, rather than ending up with a Fibonacci-like sequence, we end up with a sequence with a similar recursion, $f_{i+2} = L_n f_{i+1} + f_i$. We can solve this recursion through generating functions. If we say $f(x) = \sum_{i=0}^{\infty} f_i x^i$ and look for f_i when $f_0 = 0$ and $f_1 = 1$, then

$$\begin{aligned} f(x) = L_n x f(x) + x^2 f(x) + x \quad &\Longrightarrow \quad (1 - L_n x - x^2) f(x) = x \\ &\Longrightarrow \quad f(x) = \frac{x}{1 - L_n x - x^2}. \end{aligned}$$

Solving the quadratic on the bottom, we get

$$\begin{aligned} \frac{L_n \pm \sqrt{L_n^2 + 4}}{-2} &= -\frac{L_n \pm \sqrt{(2F_{n+1} - F_n)^2 + 4}}{2} \\ &= -\frac{L_n \pm \sqrt{4F_{n+1}^2 - 4F_{n+1}F_n + F_n^2 + 4}}{2} \\ &= -\frac{L_n \pm \sqrt{4(F_{n+1}^2 - F_{n+1}F_n - F_n^2) + 5F_n^2 + 4}}{2} \\ &= -\frac{L_n \pm \sqrt{4(F_{n+1}F_{n-1} - F_n^2) + 5F_n^2 + 4}}{2}. \end{aligned}$$

By Cassini's identity (9) and the fact that n is odd, we get

$$-\frac{L_n \pm \sqrt{-4+5F_n^2+4}}{2} = \frac{\pm F_n\sqrt{5}-L_n}{2}.$$

As we will see later in the proof for the powers of the golden ratio, this is $-\phi^n$ and $\frac{1}{\phi^n}$. So, we get the original generating function is equal to

$$\frac{x}{(x+\phi^n)(\frac{1}{\phi^n}-x)} = \frac{\frac{\frac{1}{\phi^n}}{\frac{1}{\phi^n}-x}-\frac{\phi^n}{\phi^n+x}}{F_n\sqrt{5}} = \frac{\frac{1}{1-\phi^n x}-\frac{1}{1+\frac{x}{\phi^n}}}{F_n\sqrt{5}} = \frac{\sum_{i=0}^{\infty}(\phi^n x)^i - \sum_{i=0}^{\infty}(-\frac{x}{\phi^n})^i}{F_n\sqrt{5}}$$
$$= \sum_{i=0}^{\infty}\frac{\phi^{ni}-(-\frac{1}{\phi^n})^i}{F_n\sqrt{5}}x^i.$$

Remembering that n is odd, we get that $f_i = \frac{\phi^{ni}-(-\frac{1}{\phi})^{ni}}{F_n\sqrt{5}} = \frac{F_{ni}}{F_n}$. We see, then that F_{ni} and therefore $F_{n(i+1)}$ satisfy the original recurrence, so as we did with the first pattern, we can express any series as a linear combination of these two. So,[1]

$$\cfrac{1}{L_n+\cfrac{1}{L_n+\cfrac{1}{L_n+\underbrace{\cdots}_{m-4\ L_n\text{'s}}\cfrac{1}{L_n+\cfrac{1}{\frac{a}{b}}}}}} = \cfrac{1}{L_n+\cfrac{1}{L_n+\cfrac{1}{L_n+\underbrace{\cdots}_{m-4\ L_n\text{'s}}\cfrac{1}{L_n+\cfrac{1}{\frac{\frac{a}{F_n}F_{n*1}+\frac{b}{F_n}F_{n*0}}{\frac{a}{F_n}F_{n*0}+\frac{b}{F_n}F_{n*-1}}}}}}}$$
$$= \frac{1}{\frac{\frac{a}{F_n}F_{nm+n}+\frac{b}{F_n}F_{nm}}{\frac{a}{F_n}F_{nm}+\frac{b}{F_n}F_{nm-n}}} = \frac{aF_{nm}+bF_{nm-n}}{aF_{nm+n}+bF_{nm}}.$$

As with the previous pattern, we could apply this to a fraction with two variables in the same way. □

4. Main Results

We begin by results on sum, square, product and cube of Fibonacci sequences as complementary result to Theorem 1.1.

Theorem 4.1. *Zaremba's conjecture holds for*

[1] $F_{-n} = (-1)^{n+1}F_n$

1. $F_{n+1}(a)F_{n+1}(b) - F_n(a)F_n(b)$, $1 \leq a, b \leq 5$ *and* $n \geq 1$.

2. $yF_{n+1}(a)^2 + 2F_n(a)F_{n+1}(a)$, $1 \leq y \leq 5$, $1 \leq a \leq 5$ *and* $n \geq 1$.

3. $yF^2_{n+1}(a)$, $1 \leq y \leq 6$, $1 \leq a \leq 5$ *and* $n \geq 1$.

Corollary 4.2. *Zaremba's conjecture holds for:*

$$(\sum_{j=0}^{[n/2]} \binom{n-j}{j} b^{n-2j})F_{n+1} \pm (\sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} b^{n-2j-1})F_n,$$

for all $n \geq 1$ *and for* $1 \leq b \leq 5$.

Corollary 4.3. *Zaremba's conjecture holds for*

i) $2F_{n+2}F_{n+1}$,

ii) $F_{n+1}F_{n+4}$,

iii) $2F_{n+1}F_{n+3}$,

iv) $F_{n+1}(F_{n+3} + L_{n+2})$,

for all $n \geq 1$.

Proposition 4.4. *Let b and c be two integers such that* $1 \leq b \leq 5$ *and* $1 \leq c \leq 5$ *and* $a = bc + 1$. *Then*

$$aF_{n+1} + bF_n$$

satisfies Zaremba's conjecture for all $n \geq 1$.

Theorem 4.5. *For* $n \geq 3$, *Zaremba's conjecture holds for:*

1. F^3_{3n+2}

2. F^3_{3n+4}

The following two theorems are complementary to Dromta's result: Theorem 1.2.

Theorem 4.6. *Zaremba's conjecture holds for*
$F_{n+1} + 1$, $F_{n+1}(2) + 1$ *and* $F_{n+1}(4) + 1$ *for all* $n \geq 1$.

Theorem 4.7. *Zaremba's conjecture holds for*

1. $(F_{n+1}(a) + 1)^2 + (F_n(a) + 1)^2$ *for* $a = 1$, $a = 2$ *and* $a = 4$ *and* $n \geq 1$.

2. *$F_{2n+1}(a)+2F_{n+1}(a)+F_n(a)+1$ for*

   (a) *$a=1$ and $n\in\{12l+1,12l+3,12l+5,12l+7,12l+9,12l+11\}$,*

   (b) *$a=2,4$ and $n\in\{4l+1,4l+3\}$.*

3. *$y(F_{n+1}(a)+1)^2+2(F_n(a)+1)(F_{n+1}(a)+1)$ with $1\le y\le 5$, for $a=1$, $a=2$ and $a=4$ and $n\ge 1$.*

4. *$y(F_{n+1}(a)+1)^2+(F_{n+1}(a)+1)(2F_n(a)+1)$ with $1\le y\le 5$, for*

   (a) *$a=1$ and $n\in\{12l+1,12l+3,12l+5,12l+7,12l+9,12l+11\}$,*

   (b) *$a=2,4$ and $n\in\{4l+1,4l+3\}$.*

5. *$(F_{n+1}(a)+1)(yF_{n+1}(a)+y+1)$ with $1\le y\le 5$, for*

   (a) *$a=1$ and $n\in\{12l+1,12l+3,12l+5,12l+7,12l+9,12l+11\}$,*

   (b) *$a=2,4$ and $n\in\{4l+1,4l+3\}$.*

5. Proofs of the main results

We denote for $n\ge 1$ and for a given positive integer a

$$\frac{F_n(a)}{F_{n+1}(a)}=[0,a_1,a_2,\cdots,a_n]=[0,a^n]=[0,X_n],$$

and

$$-\frac{F_n(a)}{F_{n+1}(a)}=[0,(-a)^n]=[0,-X_n].$$

Proof. Theorem 4.1

(1). We apply the identity (4) to $[0,a^n,(-b)^n]$ we obtain

$$[0,a^n,(-b)^n]=\frac{F_n(a)F_{n+1}(b)-F_{n-1}(a)F_n(b)}{F_{n+1}(a)F_{n+1}(b)-F_n(a)F_n(b)}\,. \tag{13}$$

We note that, for rendering negative partial quotients positive, we shall use (5), which gives that

$$[0,a^n,(-b)^n]=[0,a^{n-1},a-1,1,b-1,1,b^{n-1}].$$

For $a=1$, the property (6) gives:

$$\begin{aligned}[0,1^n,(-b)^n] &= [0,1^{n-1},0,1,b-1,1,b^{n-1}]\\ &= [0,1^{n-2},2,b-1,1,b^{n-1}].\end{aligned}$$

For $b=1$:

$$\begin{aligned}[0,a^n,(-1)^n] &= [0,a^{n-1},a-1,1,0,1,b^{n-1}] \\ &= [0,a^{n-2},a-1,2,1^{n-1}].\end{aligned}$$

(2). We apply the identity (1) of Lemma 3.1 for $p_n = F_n(a)$ and $q_n = F_{n+1}(a)$.

(3). We apply the identity (2) of Lemma 3.1 for $p_n = F_n(a)$ and $q_n = F_{n+1}(a)$.

Note that according to (5) we have

$$[0,\overrightarrow{X_n},y,-\overrightarrow{X_n}] = [0,a^n,y,-a,(-a)^{n-1}] = [0,a^n,y-1,1,a-1,a^{n-1}].$$

This why we added in the hypothesis $y=6$.
For $a=1$, from (6) we get

$$[0,a^n,y-1,1,a-1,a^{n-1}] = [0,a^n,y-1,1,0,1,1^{n-2}] = [0,1^n,y-1,2,1^{n-2}].$$

□

Proof. Corollary 4.2

It is well known that $F_n(b) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} b^{n-2j-1}$, then the result is obtained directly from (1) of the Theorems 1.1 and 4.1 by taking $a=1$.

□

Proof. Corollary 4.3

This result is deduced from the identity (2) of the Theorem 4.1.
i) It suffice to take $a=1$ and $y=2$. Then

$$\begin{aligned}2F_{n+1}^2 + 2F_nF_{n+1} &= 2F_{n+1}(F_{n+1}+F_n) \\ &= 2F_{n+1}F_{n+2}.\end{aligned}$$

ii) It suffice to take $a=1$ and $y=3$. Then,

$$\begin{aligned}3F_{n+1}^2 + 2F_nF_{n+1} &= 2F_{n+2}F_{n+1} + F_{n+1}^2 \\ &= F_{n+1}(2F_{n+2}+F_{n+1}) = F_{n+1}F_{n+4}.\end{aligned}$$

iii) It suffice to take $a=1$ and $y=4$. Then,

$$\begin{aligned}4F_{n+1}^2 + 2F_nF_{n+1} &= F_{n+1}F_{n+4} + F_{n+1}^2 = F_{n+1}(F_{n+4}+F_{n+1}) \\ &= 2F_{n+1}F_{n+3}.\end{aligned}$$

iv) It suffice to take $a = 1$ and $y = 5$. Then, by the property (10):

$$\begin{aligned} 5F_{n+1}^2 + 2F_nF_{n+1} &= 2F_{n+1}F_{n+3} + F_{n+1}^2 = F_{n+1}(2F_{n+3} + F_{n+1}) \\ &= F_{n+1}(F_{n+3} + L_{n+2}). \end{aligned}$$

□

Proof. Proposition 4.4

A direct application of lemma 3.2 with $x = 1$ and $y = 0$. □

Proof. Theorem 4.5

Recall the following property of continued fractions:

$$[a_0, a_1, \cdots, a_n] = [a_0, a_1, \cdots, a_{n-1} + \frac{1}{a_n}]. \tag{14}$$

(1) We will convert the continued fraction expansion

$$[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},1,1,1,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},2,2,1,1,\overbrace{\cdots}^{3n-5\text{ 1's}},1]$$

to the corresponding rational number. Clearly, this expansion is equal to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},1,1,1,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},2,2,\frac{F_{3n-1}}{F_{3n-2}}\right].$$

Then, by a simple calculation using (14), the last expansion is equivalent to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},1,1,1,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},\frac{5F_{3n-1}+2F_{3n-2}}{2F_{3n-1}+F_{3n-2}}\right],$$

which equal to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},1,1,1,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},4+\frac{F_{3n-1}-2F_{3n+1}}{F_{3n+1}}\right]$$

or also

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},1,1,1,4,4,\overbrace{\cdots}^{n-2\text{ 4's}},\frac{-F_{3n+1}}{F_{3n+2}}\right]. \tag{15}$$

By applying the Lemma 3.3 with $n = 3$, so that $L_3 = 4$, the expansion (15) becomes

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},1,1,1,\frac{F_{3n+1}F_{3n+3}-F_{3n+2}F_{3n}}{F_{3n+1}F_{3n}-F_{3n+2}F_{3n-3}}\right]$$

which, by Lemma 3.2, is equal to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},\frac{3F_{3n+1}F_{3n+3}-3F_{3n+2}F_{3n}+2F_{3n+1}F_{3n}-2F_{3n+2}F_{3n-3}}{2F_{3n+1}F_{3n+3}-2F_{3n+2}F_{3n}+F_{3n+1}F_{3n}-F_{3n+2}F_{3n-3}}\right].$$

By using the Fibonacci recurrence (7), this expansion lead to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},\frac{3F_{3n+1}F_{3n+3}-2F_{3n+2}F_{3n+3}+5F_{3n+2}F_{3n}+2F_{3n+1}F_{3n}}{2F_{3n+1}F_{3n+3}-F_{3n+2}F_{3n+3}+2F_{3n+2}F_{3n}+F_{3n+1}F_{3n}}\right]$$

which equal to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},\frac{F_{3n-1}F_{3n+3}-F_{3n+1}F_{3n}+2F_{3n+4}F_{3n}}{F_{3n-1}F_{3n+3}+F_{3n+4}F_{3n}}\right].$$

Hence, by Lemma 3.3, we get that the last expansion is equal to the rational number

$$\frac{F_{3n-1}F_{3n+3}F_{3n}-F_{3n+1}F_{3n}^2+2F_{3n+4}F_{3n}^2+F_{3n-1}F_{3n+3}F_{3n-3}+F_{3n+4}F_{3n}F_{3n-3}}{F_{3n-1}F_{3n+3}^2-F_{3n+1}F_{3n}F_{3n+3}+2F_{3n+4}F_{3n}F_{3n+3}+F_{3n-1}F_{3n+3}F_{3n}+F_{3n+4}F_{3n}^2}.$$

Some computations, using only the Fibonacci recurrence (7), simplify this rational number to

$$\frac{F_{3n+1}^3}{F_{3n+2}^3}.$$

Then

$$\frac{F_{3n+1}^3}{F_{3n+2}^3}=[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},1,1,1,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},2,2,1,1,\overbrace{\cdots}^{3n-5\text{ 1's}},1],$$

yielding the desired result.

(2) We will now convert the continued fraction expansion

$$[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},3,2,3,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},2,2,1,1,\overbrace{\cdots}^{3n-3\text{ 1's}},1]$$

to the corresponding rational number. According to the proof of (1) this expansion is equal to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},3,2,3,\frac{F_{3n+3}^2-F_{3n+4}F_{3n}}{F_{3n+3}F_{3n}-F_{3n+4}F_{3n-3}}\right]. \tag{16}$$

Applying the equality (14), the continued fraction (16) is equal to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},\frac{24F_{3n+3}^2+4F_{3n+4}F_{3n}+7F_{3n+3}F_{3n}-7F_{3n+4}F_{3n+3}}{7F_{3n+3}^2+F_{3n+4}F_{3n}+2F_{3n+3}F_{3n}-2F_{3n+4}F_{3n+3}}\right].$$

By using the Fibonacci recurrence (7), this expansion lead to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},\frac{17F_{3n+3}^2+4F_{3n+4}F_{3n}+7F_{3n+3}F_{3n}-7F_{3n+2}F_{3n+3}}{5F_{3n+3}^2+F_{3n+4}F_{3n}+2F_{3n+3}F_{3n}-2F_{3n+2}F_{3n+3}}\right].$$

The last expansion is equivalent to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},\frac{17F_{3n+3}^2+4F_{3n+4}F_{3n}-7F_{3n+1}F_{3n+3}}{5F_{3n+3}^2+F_{3n+4}F_{3n}-2F_{3n+1}F_{3n+3}}\right]$$

which equal to

$$\left[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},\frac{3F_{3n+5}F_{3n+3}+8F_{3n+4}F_{3n+2}}{F_{3n+5}F_{3n+3}+2F_{3n+4}F_{3n+2}}\right]. \tag{17}$$

Hence, by Lemma 3.3, we get that (17) is equal to the rational number

$$\frac{3F_{3n+5}F_{3n+3}F_{3n}+8F_{3n+4}F_{3n+2}F_{3n}+F_{3n+5}F_{3n+3}F_{3n-3}+2F_{3n+4}F_{3n+2}F_{3n-3}}{3F_{3n+5}F_{3n+3}^2+8F_{3n+4}F_{3n+2}F_{3n+3}+F_{3n+5}F_{3n+3}F_{3n}+2F_{3n+4}F_{3n+2}F_{3n}}$$

After some computations, using always the Fibonacci recurrence (7), simplify this rational number to

$$\frac{F_{3n+5}F_{3n+3}^2-F_{3n+3}^2F_{3n+2}}{F_{3n+4}^2F_{3n+6}-F_{3n+4}^2F_{3n+3}}$$

which equal to

$$\frac{F_{3n+3}^3}{F_{3n+4}^3}.$$

Then we get

$$\frac{F_{3n+3}^3}{F_{3n+4}^3}=[0,4,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},3,2,3,4,4,\overbrace{\cdots}^{n-3\text{ 4's}},2,2,1,1,\overbrace{\cdots}^{3n-3\text{ 1's}},1],$$

achieving the desired result. □

Proof. Theorem 4.6

Our proof is based on the property (3) of Proposition 2.1. We use the identities in the Theorem 1.2.

We have for $n=12l+1$, $\dfrac{F_n}{(F_{n+1}+1)}=[0,1^{\frac{n-1}{2}},5,4^{\frac{n-13}{6}}]$. Then

$$\frac{(F_{n+1}+1)}{(F_n+1)}=[4^{\frac{n-13}{6}},5,1^{\frac{n-1}{2}}]=[4^{2(l-1)},5,1^{6l}].$$

Hence

$$\begin{aligned}\frac{(F_{12l+2}+1)}{(F_{12l+1}+1)}-4 &= \frac{F_{12l+2}-4F_{12l+1}-3}{(F_{12l+1}+1)}\\ &= [0,4^{2l-3},5,1^{6l}].\end{aligned}$$

This gives that $F_{n+1}+1$ satisfies Zaremba's conjecture for $n=12l$.
With the same method we prove

$$\frac{F_{12l+4}-4F_{12l+3}-3}{(F_{12l+3}+1)}=[0,4^{2l-2},3,2,1^{6l}],$$

$$\frac{F_{12l+6}-3F_{12l+5}-2}{(F_{12l+5}+1)}=[0,4^{2l-1},5,1^{6l+2}],$$

$$\frac{F_{12l+8}-5F_{12l+7}-4}{(F_{12l+7}+1)}=[0,4^{2l-1},3,2,1^{6l+2}],$$

$$\frac{F_{12l+10}-4F_{12l+9}-3}{(F_{12l+9}+1)}=[0,4^{2l-1},5,1^{6l+4}],$$

$$\frac{F_{12l+12}-3F_{12l+11}-2}{(F_{12l+11}+1)}=[0,4^{2l},3,2,1^{6l+4}].$$

Hence Zaremba's conjecture holds for
$F_{n+1}+1$ if $n\in\{12l,12l+2,12l+4,12l+6,12l+8,12l+10\}$.
Further we have

$$\frac{F_{4l+2}(2)-3F_{4l+1}(2)-2}{(F_{4l+1}(2)+1)}=[0,2^{2l-2},4,2^{2l}],$$

$$\frac{F_{4l+4}(2)-3F_{4l+3}(2)-2}{(F_{4l+3}(2)+1)}=[0,2^{2l-1},1,1,1,2^{2l+1}],$$

$$\frac{F_{4l+2}(4)-2F_{4l+1}(4)-1}{(F_{4l+1}(4)+1)}=[0,(1,1,1)^{2l-1},1,5,4^{2l}],$$

$$\frac{F_{4l+4}(4)-2F_{4l+3}(4)-1}{(F_{4l+3}(4)+1)}=[0,(1,1,1)^{2l},2,3,4^{2l+1}].$$

Hence Zaremba's conjecture holds for
$F_{n+1}(2)+1$ and $F_{n+1}(4)+1$ if $n\in\{4l,4l+2\}$. So we obtain the desired result. □

Proof. Theorem 4.7

1. According to identity (4), if $p_n/q_n = [0, a_1, a_2, \ldots, a_n]$ then
$$[0, a_1, a_2, \ldots, a_n, a_n, \ldots, a_2, a_1] = \frac{p_n q_n + p_{n-1} q_{n-1}}{q_n^2 + q_{n-1}^2}.$$
The result follows by taking $\frac{p_n}{q_n} = \frac{\cdots}{F_{n+1}(a)+1}$. We obtain that Zaremba's conjecture holds for $(F_{n+1}(a)+1)^2 + (F_n(a)+1)^2$.

2. According to identity (4), if $p_n/q_n = [0, a_1, a_2, \ldots, a_n]$ then
$$[0, a_1, a_2, \ldots, a_n, a_1, a_2, \ldots, a_n] = \frac{p_n q_n + p_{n-1} p_n}{q_n^2 + q_{n-1} p_n}.$$
The result follows by taking $\frac{p_n}{q_n} = \frac{F_n(a)}{F_{n+1}(a)+1}$. We obtain that Zaremba's conjecture holds for $(F_{n+1}(a)+1)^2 + (F_n(a)+1)F_n(a)$, which is equal to $F_{2n+1}(a) + 2F_{n+1}(a) + F_n(a) + 1$.

3. We apply identity (3) of Lemma 3.1 with $\frac{p_n}{q_n} = \frac{\cdots}{F_{n+1}(a)+1}$.

4. We apply identity (1) of Lemma 3.1 with $\frac{p_n}{q_n} = \frac{F_n(a)}{F_{n+1}(a)+1}$.

5. We apply identity (2) of Lemma 3.1 with $\frac{p_n}{q_n} = \frac{F_n(a)}{F_{n+1}(a)+1}$.

□

Acknowledgement The authors thank the referee for his helpful comments and suggestions.

REFERENCES

[1] Kh. Ayadi and T. Komatsu, *Continued fraction expansions towards Zaremba's conjecture*, to appear in Experimental Mathematics.

[2] I. Borosh and H. Niederreiter, *Optimal multipliers for pseudo-random number generation by the linear congruential method*, BIT **23** (1983), 65–74.

[3] J. Bourgain and A. Kontorovich, *On Zaremba's conjecture*, Ann. of Math. (2) **180** (2014), No. 1, 137–196.

[4] P. F. Byrd, *Expansion of analytic functions in polynomials associated with Fibonacci numbers*, Fibonacci Quart. **1** (1963), no. 1, 16–29.

[5] M. Drmota, *Fibonacci Numbers and Continued Fraction Expansions*, In: Bergum, G.E., Philippou, A.N., Horadam, A.F. (eds) Applications of Fibonacci Numbers (1993), 185–197. Springer, Dordrecht.

[6] D. Hensley, *A polynomial time algorithm for the Hausdorff dimension of continued fraction Cantor sets*, J. Number Theory **58** (1996), No. 1, 9–45.

[7] T. Koshy, *Fibonacci and Lucas numbers with applications. Vol. 2*, Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, NJ, 2019.

[8] N. G. Moshchevitin and I. D. Shkredov, *On a modular form of Zaremba's conjecture*, Pacific J. Math. **309** (2020), No. 1, 195–211.

[9] H. Niederreiter, *Dyadic fractions with small partial quotients*, Monatsh. Math. **101** (1986) 309–315.

[10] O. Perron, *Die Lehre von den Kettenbruchen*, 2nd ed. Chelsea Publishing Co., New York, N. Y., 1950.

[11] A. J. van der Poorten and J. Shallit, *A specialised continued fraction*, Can. J. Math. **45** (1993), No. 5, 1067–1079.

[12] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available at oeis.org. (2023).

[13] H. S. Wall, *Analytic theory of continued fractions*, Chelsea Publ., New York, 1948.

[14] M. Yodphotong and V. Laohakosol, *Proofs of Zaremba's conjecture for powers of* 6, Proceedings of the International Conference on Algebra and its Applications (ICAA 2002) (Bangkok), 278–282, Chulalongkorn Univ., Bangkok, 2002.

[15] S. K. Zaremba, *La méthode des 'bons treillis' pour le calcul des intégrales multiples*, pp. 39–119: In Applications of number theory to numerical analysis (Montreal, 1971), edited by S. K. Zaremba, Academic, New York, 1972.

K. AYADI
Sfax University,
Higher Institute of Industrial Management of Sfax,
Faculty of Sciences, Lab AGTS, Tunisia
e-mail: khalil.ayadi@isgis.usf.tn

S. KALIA
University of Minnesota,
School of Physics and Astronomy, Minnesota, USA
e-mail: skalia618@gmail.com

M. SAADAOUI
Sfax University,
Faculty of Sciences, Lab AGTS, Tunisia.
e-mail: mahersaadaoui716@gmail.com