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THE GENERALIZED DISTANCE SPECTRA OF THE $\mathcal{M}$-JOIN OF GRAPHS

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In this paper, we obtain the generalized distance spectra of the graphs constructed by 19 unary and 21 binary graph operations of $\mathcal{M}$-join type when the constituting graphs satisfy some conditions. Also, we deduce a result on the generalized distance spectrum of the double graph of a connected regular graph in the literature. As applications, we construct infinite families of generalized distance cospectral graphs. Also, we construct infinite families of distance (distance Laplacian, distance signless Laplacian) integral graphs.

1. Introduction

Throughout this paper, we consider only finite and simple graphs. Let G be a graph with vertex set $V(G)$. We denote $u \sim v$ ($u \nsim v$) if the vertices u and v are adjacent (non-adjacent) in G. For a vertex v in G, $N_G(v)$ denotes the set of all vertices in G which are adjacent with v. The *degree* of v in G is the cardinality of the set $N_G(v)$. If each vertex of G has the same degree r, then G is said to be *r-regular*. The complement of G, denoted by $\overline{G}$ is defined as the graph with vertex set $V(G)$ and two distinct vertices are adjacent in $\overline{G}$ if and only if they

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are non-adjacent in G. The union of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The union of k copies of G is denoted by kG. For $u, v \in V(G)$, the distance from u to v in G, denoted by $d_G(u,v)$ is defined as the length of a shortest path from u to v (if it exists); otherwise it is defined to be ∞. G is said to be connected if there exists a path between any two distinct vertices of G. The diameter of a connected graph G, denoted by $diam(G)$ is the maximum distance among the vertices of G. The complete graph and the cycle graph on n vertices are denoted by K_n and C_n respectively. $K_{m,n}$ denotes the complete bipartite graph having partition sizes m and n. I_n denotes the identity matrix of order n and $J_{n\times m}$ denotes the all one matrix of order $n \times m$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are the distinct eigenvalues of a matrix A with multiplicities $m_1, m_2, ..., m_k$, then we write the spectrum of A as $\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, ..., \lambda_k^{(m_k)}$. If $m_i = 1$, then we write λ_i instead of $\lambda_i^{(1)}$.

The following question arise naturally in spectral graph theory: 'to what extent one can describe the spectra of a given graph in terms of the spectra of some other graphs?' In literature, for answering this question, researchers used the graph operations as a tool to construct graphs from the given graphs. The adjacency (Laplacian, signless Laplacian, normalized Laplacian) spectra of graphs constructed by various graph operations have been extensively studied in the literature by several researchers. For instance, see [3, 6, 9, 13, 14] and the references therein.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is the $0-1$ matrix of size $n \times n$ whose rows and columns are indexed by $V(G)$ and for $i, j = 1, 2, \ldots, n$, the (i, j)-th entry of $A(G)$ is 1 if and only if $i \neq j$ and $v_i \sim v_j$. The *distance matrix* of a connected graph G, denoted by $D(G)$, is defined as the square matrix whose rows and columns are indexed by $V(G)$ and for $i, j = 1, 2, \ldots, n$, the (i, j)-th entry of $D(G)$ is $d_G(v_i, v_j)$ if $i \neq j$; 0 otherwise. The *transmission* of a vertex v of a connected graph G, denoted by $Tr_G(v)$, is defined as the sum of the distances from v to all other vertices of G. The *transmission matrix* of G, denoted by $Tr(G)$, is defined as the diagonal matrix $diag(Tr_G(v_1), Tr_G(v_2), \ldots, Tr_G(v_n))$. The *distance Laplacian matrix* $D^L(G)$ and the *distance signless Laplacian matrix* $D^Q(G)$ of G are defined as $Tr(G) - D(G)$ and $Tr(G) + D(G)$, respectively [2].

The *generalized distance matrix* [5] $D_\alpha(G)$ of G is defined by $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$ for $0 \leq \alpha \leq 1$. Notice that $D_0(G) = D(G)$, $D_{\frac{1}{2}}(G) = \frac{1}{2}D^Q(G)$, $D_1(G) = Tr(G)$ and $D_\alpha(G) - D_\beta(G) = (\alpha - \beta)D^L(G)$ for $0 \leq \alpha, \beta \leq 1$ with $\alpha \neq \beta$.

The spectrum of $A(G)$ (resp. $D(G)$, $D^L(G)$, $D^Q(G)$, $D_\alpha(G)$) is called the adjacency (resp. distance, distance Laplacian, distance signless Laplacian, generalized distance) spectrum of G. If the spectrum of $A(G)$ (resp. $D(G)$, $D^L(G)$,

$D^Q(G)$, $D_\alpha(G)$) contains only integers, then G is said to be an adjacency (resp. distance, distance Laplacian, distance signless Laplacian, generalized distance) integral graph. Two graphs are said to be adjacency (resp. distance, distance Laplacian, distance signless Laplacian, generalized distance) cospectral if they have same adjacency (resp. distance, distance Laplacian, distance signless Laplacian, generalized distance) spectrum. For more results on the distance (distance Laplacian, distance signless Laplacian and generalized distance) spectrum of graphs, we refer the reader to [1, 2, 4, 5, 7, 10, 12] and the references therein.

Let G and H be graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$ and let M be a $0-1$ matrix of size $n \times m$. The *M-join* of G and H [8], denoted by $G \vee_M H$, is the graph obtained by taking one copy of G and H, and joining the vertices u_i and v_j if and only if the (i, j)-th entry of M is 1 for $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$.

The definition of the M-join of two graphs is extended to a sequence of k graphs as follows [8]: Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ be a sequence of graphs with $|V(H_i)| = n_i$ for $i = 1, 2, \ldots, k$ and let $\mathcal{M} = (M_{12}, M_{13}, \ldots, M_{1k}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$, where M_{ij} is a $0-1$ matrix of size $n_i \times n_j$. The *$\mathcal{M}$-join of the graphs in $\mathcal{H}_k$*, denoted by $\bigvee_{\mathcal{M}} \mathcal{H}_k$, is the graph $\displaystyle\bigcup_{\substack{i,j=1,\\ i<j}}^{k} \left(H_i \vee_{M_{ij}} H_j\right)$.

The rest of this paper is arranged as follows: In Section 2, we determine the spectra with corresponding eigenvectors of two families block matrices of specific form. In Section 3, we determine the generalized distance spectra with their corresponding eigenvectors of the graphs constructed by various unary graph operations of $\mathcal{M}$-join type defined in [8] when the constituting graphs are regular, by using the results of Section 2. In Section 4, we define several binary graph operations using the unary graph operations considered in Section 3 and determine the generalized distance spectra of the graphs constructed by these binary graph operations when the constituting graphs are regular, by using the results of Section 2. In addition, we construct an infinite family of distance (distance Laplacian, distance signless Laplacian) integral graphs using the results of Sections 3 and 4. Also, we construct infinite pairs of generalized distance cospectral graphs by using the results of Sections 3 and 4. A part of this paper is a part of the dissertation [11].

2. Spectra of two families of block matrices

Let A be a real symmetric matrix of order n having equal row sums r. Then A has n real eigenvalues, let them be $(r =)\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding orthogonal eigenvectors $(J_{n\times 1} =)X_1, X_2, \dots, X_n$. Now, we consider the matrix

$$M = \begin{bmatrix} a_1J_n + a_2A + a_3I_n & b_1J_n + b_2A + b_3I_n \\ b_1J_n + b_2A + b_3I_n & a_1J_n + a_2A + a_3I_n \end{bmatrix} \tag{1}$$

where $a_i, b_j \in \mathbb{R}$ for $i, j = 1,2,3$. Let R be the diagonal matrix of order $2n$, whose i-th diagonal entry is the i-th row sum of the matrix M, i.e.,

$$R = [(a_1+b_1)n + (a_2+b_2)r + a_3 + b_3]I_{2n}.$$

For $0 \le \alpha \le 1$, we define the matrix M_α by

$$M_\alpha = \alpha R + (1-\alpha)M. \tag{2}$$

Theorem 2.1. *The eigenvalues of M_α are given by*

(i) $\alpha[(a_1+b_1)n + (a_2+b_2)r - (a_2+b_2)\lambda_i] + (a_2+b_2)\lambda_i + a_3 + b_3$ *for* $i = 2,3,\dots,n$,

(ii) $\alpha[(a_1+b_1)n + (a_2+b_2)r - (a_2-b_2)\lambda_i + 2b_3] + (a_2-b_2)\lambda_i + a_3 - b_3$ *for* $i = 2,3,\dots,n$,

(iii) $(a_1+b_1)n + (a_2+b_2)r + a_3 + b_3$,

(iv) $2\alpha(b_1n + b_2r + b_3) + (a_1-b_1)n + (a_2-b_2)r + a_3 - b_3$.

Proof. Since for $i = 2,3,\dots,n$, X_i is orthogonal to $J_{n\times 1}$, $\begin{bmatrix} X_i^T & X_i^T \end{bmatrix}^T$ and $\begin{bmatrix} X_i^T & -X_i^T \end{bmatrix}^T$ are eigenvectors of M_α corresponding to the eigenvalues $\alpha[(a_1+b_1)n + (a_2+b_2)r - (a_2+b_2)\lambda_i] + (a_2+b_2)\lambda_i + a_3 + b_3$ and $\alpha[(a_1+b_1)n + (a_2+b_2)r - (a_2-b_2)\lambda_i + 2b_3] + (a_2-b_2)\lambda_i + a_3 - b_3$, respectively.

Till now, we have obtained $2n-2$ eigenvalues of M_α with corresponding eigenvectors. All these eigenvectors are orthogonal to $\begin{bmatrix} J_{1\times n} & 0_{1\times n} \end{bmatrix}^T$ and $\begin{bmatrix} 0_{1\times n} & J_{1\times n} \end{bmatrix}^T$. This implies that the remaining two eigenvectors of M_α are of the form $\begin{bmatrix} \alpha_1 J_{1\times n} & \alpha_2 J_{1\times n} \end{bmatrix}^T$ for some $(\alpha_1, \alpha_2) \ne (0,0)$. Let ν be an eigenvalue of M_α with corresponding one such eigenvector. Then we have,

$$M_\alpha \begin{bmatrix} \alpha_1 J_{n\times 1} \\ \alpha_2 J_{n\times 1} \end{bmatrix} = \nu \begin{bmatrix} \alpha_1 J_{n\times 1} \\ \alpha_2 J_{n\times 1} \end{bmatrix}. \tag{3}$$

It is equivalent to the system

$$[\mathcal{P} - \nu I_2] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4}$$

where

$$\mathcal{P}=\begin{bmatrix}\alpha(b_1n+b_2r+b_3)+a_1n+a_2r+a_3 & (1-\alpha)(b_1n+b_2r+b_3)\\ (1-\alpha)(b_1n+b_2r+b_3) & \alpha(b_1n+b_2r+b_3)+a_1n+a_2r+a_3\end{bmatrix}.$$

From equation (4), we have

$$\mathcal{P}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\nu\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}. \tag{5}$$

From equation (3) and (5), we have that if ν is an eigenvalue of $\mathcal{P}$ with corresponding eigenvector $\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}^T$, then ν is an eigenvalue of M_α with corresponding eigenvector $\begin{bmatrix}\alpha_1J_{1\times n} & \alpha_2J_{1\times n}\end{bmatrix}^T$. The eigenvalues of $\mathcal{P}$ are $(a_1+b_1)n+(a_2+b_2)r+a_3+b_3$ and $2\alpha(b_1n+b_2r+b_3)+(a_1-b_1)n+(a_2-b_2)r+a_3-b_3$ with corresponding eigenvectors $\begin{bmatrix}1 & 1\end{bmatrix}^T$ and $\begin{bmatrix}1 & -1\end{bmatrix}^T$, respectively. Hence the remaining two eigenvalues of the matrix M_α are $(a_1+b_1)n+(a_2+b_2)r+a_3+b_3$ and $2\alpha(b_1n+b_2r+b_3)+(a_1-b_1)n+(a_2-b_2)r+a_3-b_3$ with corresponding eigenvectors $\begin{bmatrix}J_{1\times n} & J_{1\times n}\end{bmatrix}^T$ and $\begin{bmatrix}J_{1\times n} & -J_{1\times n}\end{bmatrix}^T$, respectively. $\square$

For $i=1,2$, let A_i be a real symmetric matrix of order n_i having equal row sums r_i. Then A_i has n_i real eigenvalues; let them be $(r_i=)\lambda_{i1},\lambda_{i2},\ldots,\lambda_{in_i}$ with corresponding orthogonal eigenvectors $(J_{n_i\times 1}=)X_{i1},X_{i2},\ldots,X_{in_i}$ for $i=1,2$. Now, we consider the matrix M' given as follows.

$$M'=\begin{bmatrix}a_1J_{n_1}+a_2A_1+a_3I_{n_1} & b_1J_{n_1}+b_2A_1+b_3I_{n_1} & J_{n_1\times n_2}\\ b_1J_{n_1}+b_2A_1+b_3I_{n_1} & a_1J_{n_1}+a_2A_1+a_3I_{n_1} & J_{n_1\times n_2}\\ J_{n_2\times n_1} & J_{n_2\times n_1} & 2(J_{n_2}-I_{n_2})-A_2\end{bmatrix}, \tag{6}$$

where $a_i,b_i\in\mathbb{R}$ for $i=1,2,3$. Let R' denote the diagonal matrix of order $2n_1+n_2$ whose j-th diagonal entry is the j-th row sum of the matrix M', i.e.,

$$R'=\begin{bmatrix}[(a_1+b_1)n_1+n_2+(a_2+b_2)r_1+a_3+b_3]I_{2n_1} & 0_{2n_1\times n_2}\\ 0_{n_2\times 2n_1} & (2n_1+2n_2-r_2-2)I_{n_2}\end{bmatrix}.$$

For $0\leq\alpha\leq 1$, we define the matrix M'_α by

$$M'_\alpha=\alpha R'+(1-\alpha)M'. \tag{7}$$

Theorem 2.2. *The spectrum of the matrix M'_α is given by*

(i) $\alpha[(a_1+b_1)n_1+n_2+(a_2+b_2)r_1-(a_2+b_2)\lambda_{1j}]+(a_2+b_2)\lambda_{1j}+a_3+b_3$ *for* $j=2,3,\ldots,n_1$,

(ii) $\alpha[(a_1+b_1)n_1+n_2+(a_2+b_2)r_1-(a_2-b_2)\lambda_{1j}+2b_3]+(a_2-b_2)\lambda_{1j}+a_3-b_3$ *for* $j=2,3,\ldots,n_1$,

(iii) $\alpha(2n_1+2n_2-r_2+\lambda_{2k})-\lambda_{2k}-2$ *for* $k=2,3,\ldots,n_2$,

(iv) $\alpha[n_2+2(b_1n_1+b_2r_1+b_3)]+(a_1-b_1)n_1+(a_2-b_2)r_1+a_3-b_3$,

(v) $\frac{1}{2}[\sigma_1+\sigma_2+\chi\pm\sqrt{(\sigma_1-\sigma_2+\chi)^2-8(1-\alpha)^2n_1n_2}]$, *where* $\sigma_1=\alpha(b_1n_1+n_2+b_2r_1+b_3)+a_1n_1+a_2r_1+a_3$, $\sigma_2=2\alpha n_1+2(n_2-1)-r_2$ *and* $\chi=(1-\alpha)(b_1n_1+b_2r_1+b_3)$.

Proof. We have that X_{1j} is orthogonal to $J_{n_1\times 1}$ for $j=2,3,\ldots,n_1$ and X_{2k} is orthogonal to $J_{n_2\times 1}$ for $k=2,3,\ldots,n_2$. This implies that $\begin{bmatrix}X_{1j}{}^T & X_{1j}{}^T & 0_{1\times n_2}\end{bmatrix}^T$, $\begin{bmatrix}X_{1j}{}^T & -X_{1j}{}^T & 0_{1\times n_2}\end{bmatrix}^T$ and $\begin{bmatrix}0_{1\times 2n_1} & X_{2k}{}^T\end{bmatrix}^T$ are eigenvectors of M'_α corresponding to the eigenvalues $\alpha[(a_1+b_1)n_1+n_2+(a_2+b_2)r_1-(a_2+b_2)\lambda_{1j}]+[(a_2+b_2)\lambda_{1j}+a_3+b_3]$, $\alpha[(a_1+b_1)n_1+n_2+(a_2+b_2)r_1-(a_2-b_2)\lambda_{1j}+2b_3]+(a_2-b_2)\lambda_{1j}+a_3-b_3$ and $\alpha(2n_1+2n_2-r_2+\lambda_{2k})-(\lambda_{2k}+2)$, respectively.

All these eigenvectors are orthogonal to the vectors $\begin{bmatrix}J_{1\times n_1} & 0_{1\times(n_1+n_2)}\end{bmatrix}^T$, $\begin{bmatrix}0_{1\times n_1} & J_{1\times n_1} & 0_{1\times n_2}\end{bmatrix}^T$ and $\begin{bmatrix}0_{1\times 2n_1} & J_{1\times n_2}\end{bmatrix}^T$. This implies that the remaining three eigenvectors of M'_α are of the form $\begin{bmatrix}\alpha_1J_{1\times n_1} & \alpha_2J_{1\times n_1} & \alpha_3J_{1\times n_2}\end{bmatrix}^T$ for some $(\alpha_1,\alpha_2,\alpha_3)\neq(0,0,0)$. Let ν be an eigenvalue of M'_α with corresponding one such eigenvector. Then we have,

$$M'_\alpha\begin{bmatrix}\alpha_1J_{n_1\times 1}\\ \alpha_2J_{n_1\times 1}\\ \alpha_3J_{n_2\times 1}\end{bmatrix}=\nu\begin{bmatrix}\alpha_1J_{n_1\times 1}\\ \alpha_2J_{n_1\times 1}\\ \alpha_3J_{n_2\times 1}\end{bmatrix}. \tag{8}$$

It is equivalent to the system

$$[\mathcal{P}-\nu I_3]\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}^T=\begin{bmatrix}0 & 0 & 0\end{bmatrix}^T, \tag{9}$$

where

$$\mathcal{P}=\begin{bmatrix}\sigma_1 & \chi & (1-\alpha)n_2\\ \chi & \sigma_1 & (1-\alpha)n_2\\ (1-\alpha)n_1 & (1-\alpha)n_1 & \sigma_2\end{bmatrix},$$

$\sigma_1=\alpha(b_1n_1+n_2+b_2r_1+b_3)+a_1n_1+a_2r_1+a_3$, $\sigma_2=2\alpha n_1+2(n_2-1)-r_2$ and $\chi=(1-\alpha)(b_1n_1+b_2r_1+b_3)$.

From equation (9), we have

$$\mathcal{P}\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}^T=\nu\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}^T. \tag{10}$$

From equation (8) and (10), we have that if ν is an eigenvalue of $\mathcal{P}$ with corresponding eigenvector $\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}^T$, then ν is an eigenvalue of M'_α with corresponding eigenvector $\begin{bmatrix}\alpha_1J_{1\times n_1} & \alpha_2J_{1\times n_1} & \alpha_3J_{1\times n_2}\end{bmatrix}^T$.

The characteristic equation of $\mathcal{P}$ is

$$(x-\sigma_1+\chi)[x^2-(\sigma_1+\sigma_2+\chi)x+\sigma_2(\sigma_1+\chi)-2(1-\alpha)^2n_1n_2]=0.$$

So, $\sigma_1-\chi$ is an eigenvalue of $\mathcal{P}$ and it can be seen that $\begin{bmatrix}1 & -1 & 0\end{bmatrix}^T$ is its corresponding eigenvector. Thus $\sigma_1-\chi=\alpha[n_2+2(b_1n_1+b_2r_1+b_3)]+(a_1-b_1)n_1+(a_2-b_2)r_1+a_3-b_3$ is an eigenvalue of M'_α with corresponding eigenvector $\begin{bmatrix}J_{1\times n_1} & -J_{1\times n_1} & 0_{1\times n_2}\end{bmatrix}^T$ and the remaining two eigenvalues of M'_α are the roots of the equation

$$x^2-(\sigma_1+\sigma_2+\chi)x+\sigma_2(\sigma_1+\chi)-2(1-\alpha)^2n_1n_2=0,$$

which are given by $\frac{1}{2}[\sigma_1+\sigma_2+\chi\pm\sqrt{(\sigma_1-\sigma_2+\chi)^2-8(1-\alpha)^2n_1n_2}]$. This completes the proof. □

3. Generalized distance spectra of graphs constructed by some unary graph operations

In [8] several unary graph operations were defined as a M-join of two copies of a graph for a suitable $0-1$ matrix M. Some of them are given in Table 1, where G is a graph on n vertices.

S. No.	Description	Name of the unary graph operation
1.	$G\vee_{I_n}G$	Mirror graph of G
2.	$G\vee_{J_n-I_n}G$	VC-neighbourhood graph of G
3.	$G\vee_{A(G)}G$	double graph of G or N-neighbourhood graph of G
4.	$G\vee_{A(G)+I_n}G$	$\overline{N}$-neighbourhood graph of G
5.	$G\vee_{A(\overline{G})}G$	NC-neighbourhood graph of G
6.	$G\vee_{A(\overline{G})+I_n}G$	$\overline{N}C$-neighbourhood graph of G
7.	$\overline{G}\vee_{I_n}\overline{G}$	Mirror complement graph of G
8.	$\overline{G}\vee_{J_n}\overline{G}$	Join-neighbourhood-complement graph of G
9.	$\overline{G}\vee_{J_n-I_n}\overline{G}$	VC-neighbourhood-complement graph of G
10.	$\overline{G}\vee_{A(G)}\overline{G}$	N-neighbourhood complement graph of G
11.	$\overline{G}\vee_{A(G)+I_n}\overline{G}$	$\overline{N}$-neighbourhood-complement graph of G
12.	$\overline{G}\vee_{A(\overline{G})}\overline{G}$	NC-neighbourhood complement graph of G
13.	$\overline{G}\vee_{A(\overline{G})+I_n}\overline{G}$	$\overline{N}C$-neighbourhood-complement graph of G
14.	$\overline{K}_n\vee_{A(G)}\overline{K}_n$	Duplicate graph of G
15.	$\overline{K}_n\vee_{A(G)+I_n}\overline{K}_n$	$D\overline{N}$-graph of G
16.	$\overline{K}_n\vee_{A(\overline{G})}\overline{K}_n$	Complemented duplicate graph of G

17.	$\overline{K}_n \vee_{A(\overline{G})+I_n} \overline{K}_n$	Closed duplicate graph of G
18.	$K_n \vee_{A(G)} K_n$	Fully complete duplicate graph of G
19.	$K_n \vee_{A(G)+I_n} K_n$	Fully complete $D\overline{N}$-graph of G
20.	$K_n \vee_{A(\overline{G})} K_n$	Fully complete complemented duplicate graph of G
21.	$K_n \vee_{A(\overline{G})+I_n} K_n$	Fully complete closed duplicate graph of G

Table 1: Some unary graph operations

In the following result, we obtain the generalized distance spectra of the graphs mentioned in Table 1 except those in S.Nos. 14^{th} and 16^{th}. Notice that the generalized distance spectrum of double graph of a connected regular graph G with $diam(G) \leq 2$ was determined in [5]. We show that this result can also be obtained from the following result.

Theorem 3.1. *Let G be an r-regular graph on n vertices with assumptions as in the 3^{rd} column of Table 2. In the 2^{nd} and 5^{th} columns of the same table, we give the graph constructed by each of the unary graph operation on G, and the generalized distance spectrum of the constructed graph, respectively.*

Proof. Let the eigenvalues of $A(G)$ be $(r=)\lambda_1, \lambda_2, \dots, \lambda_n$ with orthogonal eigenvectors $(J_{n\times 1}=)X_1, X_2, \dots, X_n$ respectively. By taking $M = A(G)$ and substituting the values a_i, b_j for $i, j = 1, 2, 3$ as given in Table 2 in (1) and (2), we get the distance matrix and the generalized distance matrix, respectively of the graph constructed by corresponding unary graph operation mentioned in that table. By substituting these values in Theorem 2.1, we obtain the generalized distance spectrum of the corresponding graph as mentioned in Table 2. □

S. No.	Graph	Additional assumptions	(a_1, a_2, a_3), (b_1, b_2, b_3)	Generalized distance spectrum
1.	$G \vee_{I_n} G$	G is connected with $diam(G) \leq 2$	$(2, -1, -2)$, $(3, -1, -2)$	$\alpha(5n-2r+2\lambda_i)-2\lambda_i-4$ for $i = 2, 3, \dots, n$; $[\alpha(5n-2r-4)]^{(n-1)}$, $5n-2r-4$, $2\alpha(3n-r-2)-n$
2.	$G \vee_{J_n-I_n} G$	$n \geq 3$, $G = \overline{K}_n$	$(2, 0, -2)$, $(1, 0, 2)$	$[\alpha(3n+4)-4]^{(n-1)}$, $(3n\alpha)^{(n-1)}$, $3n$, $2\alpha(n+2)+n-4$

3.	$G\vee_{J_n-I_n}G$	$n\geq 2$, $G\neq\overline{K}_n$	$(2,-1,-2)$, $(1,0,1)$	$\alpha(3n-r+\lambda_i+2)-\lambda_i-3$, $\alpha(3n-r+\lambda_i)-\lambda_i-1$ for $i=2,3,\ldots,n$, $3n-r-1$; $2\alpha(n+1)+n-r-3$
4.	$G\vee_{A(G)}G$	$n\geq 2$, G is connected, $diam(G)\leq 2$	$(2,-1,-2)$, $(2,-1,0)$	$\alpha(4n-2r+2\lambda_i)-2\lambda_i-2$ for $i=2,3,\ldots,n$; $[2\alpha(2n-r)-2]^{(n)}$, $4n-2r-2$ (See, [5])
5.	$G\vee_{A(G)+I_n}G$	G is connected with $diam(G)\leq 2$	$(2,-1,-2)$, $(2,-1,-1)$	$\alpha(4n-2r+2\lambda_i)-2\lambda_i-3$ for $i=2,3,\ldots,n$; $[\alpha(4n-2r-2)-1]^{(n-1)}$, $4n-2r-3$, $2\alpha(2n-r-1)-1$
6.	$G\vee_{A(\overline{G})}G$	$n\geq 3$, $G=\overline{K}_n$	$(2,0,-2)$, $(1,0,2)$	$[3\alpha n]^{(n-1)}$, $[\alpha(3n+4)-4]^{(n-1)}$, $3n$, $2\alpha(n+2)+n-4$
7.	$G\vee_{A(\overline{G})}G$	$G\neq\overline{K}_n$; for any two $u\sim v$, $N_G(u)\backslash N_G(v)\neq\varnothing$; for any two $u\nsim v$, $N_{\overline{G}}(u)\cap N_{\overline{G}}(v)\neq\varnothing$	$(2,-1,-2)$, $(1,1,2)$	$\alpha(3n+2\lambda_i+4)-2\lambda_i-4$ for $i=2,3,\ldots,n$; $[3\alpha n]^{(n-1)}$, $3n$, $2\alpha(n+r+2)+n-2r-4$
8.	$G\vee_{A(\overline{G})+I_n}G$	$-$	$(2,-1,-2)$, $(1,1,0)$	$\alpha(3n+2\lambda_i)-2\lambda_i-2$ for $i=2,3,\ldots,n$; $[3\alpha n-2]^{(n-1)}$, $3n-2$, $2\alpha(n+r)+n-2r-2$
9.	$\overline{G}\vee_{J_n-I_n}\overline{G}$	$n\geq 3$, $G=K_n$	$(2,0,-2)$, $(1,0,2)$	$[3\alpha n]^{(n-1)}$, $[\alpha(3n+4)-4]^{(n-1)}$, $3n$, $2\alpha(n+2)+n-4$
10.	$\overline{G}\vee_{J_n-I_n}\overline{G}$	$n\geq 2$, $G\neq K_n$	$(1,1,-1)$, $(1,0,1)$	$\alpha(2n+r-\lambda_i+2)+\lambda_i-2$, $\alpha(2n+r-\lambda_i)+\lambda_i$ for $i=2,3,\ldots,n$; $2n+r$, $2\alpha(n+1)+r-2$
11.	$\overline{G}\vee_{A(G)+I_n}\overline{G}$	$-$	$(1,1,-1)$, $(2,-1,-1)$	$\alpha(3n-2\lambda_i-2)+2\lambda_i$ for $i=2,3,\ldots,n$; $[3\alpha n-2]^{(n-1)}$, $3n-2$, $2\alpha(2n-r-1)-n+2r$

12.	$K_n \vee_{A(G)} K_n$	$G \neq \overline{K}_n$	$(1,0,-1)$, $(2,-1,0)$	$\alpha(3n-r+\lambda_i)-\lambda_i-1$, $\alpha(3n-r-\lambda_i)+\lambda_i-1$ for $i=2,3,\ldots,n$; $3n-r-1$, $2\alpha(2n-r)-n+r-1$
13.	$\overline{K}_n \vee_{A(G)+I_n} \overline{K}_n$	G is connected with $diam(G) \leq 2$	$(2,0,-2)$, $(3,-2,-2)$	$\alpha(5n-2r+2\lambda_i)-2\lambda_i-4$, $\alpha(5n-2r-2\lambda_i-4)+2\lambda_i$ for $i=2,3,\ldots,n$; $5n-2r-4$, $2\alpha(3n-2r-2)-n+2r$
14.	$K_n \vee_{A(G)+I_n} K_n$	–	$(1,0,-1)$, $(2,-1,-1)$	$\alpha(3n-r-\lambda_i)-\lambda_i-2$, $\alpha(3n-r-\lambda_i-2)+\lambda_i$ for $i=2,3,\ldots,n$; $3n-r-2$, $2\alpha(2n-r-1)-n+r$
15.	$K_n \vee_{A(\overline{G})} K_n$	$G \neq K_n$	$(1,0,-1)$, $(1,1,1)$	$\alpha(2n+r+\lambda_i+2)-\lambda_i-2$, $\alpha(2n+r-\lambda_i)+\lambda_i$ for $i=2,3,\ldots,n$; $2n+r$, $2\alpha(n+r+1)-r-2$
16.	$K_n \vee_{A(\overline{G})+I_n} K_n$	–	$(1,0,-1)$, $(1,1,0)$	$\alpha(2n+r-\lambda_i)+\lambda_i-1$, $\alpha(2n+r+\lambda_i)-\lambda_i-1$ for $i=2,3,\ldots,n$; $2n+r-1$, $2\alpha(n+r)-r-1$
17.	$\overline{G} \vee_{I_n} \overline{G}$	for any two $u \sim v$, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \varnothing$	$(1,1,-1)$, $(2,1,-1)$	$\alpha(3n+2r-2\lambda_i)+2\lambda_i-2$ for $i=2,3,\ldots,n$; $[\alpha(3n+2r-2)]^{(n-1)}$, $3n+2r-2$, $2\alpha(2n+r-1)-n$
18.	$\overline{G} \vee_{J_n} \overline{G}$	–	$(1,1,-1)$, $(1,0,0)$	$[\alpha(2n+r-\lambda_i)+\lambda_i-1]^{(2)}$ for $i=2,3,\ldots,n$; $2n+r-1$, $2\alpha n+r-1$
19.	$\overline{G} \vee_{A(G)} \overline{G}$	$n \geq 3$, $G = K_n$	$(2,0,-2)$, $(1,0,2)$	$[3\alpha n]^{(n-1)}$, $[\alpha(3n+4)-4]^{(n-1)}$, $3n$, $2\alpha(n+2)+n-4$

20.	$\overline{G} \vee_{A(G)} \overline{G}$	$G \neq K_n$; for any two $u \sim v$, $N_G(u) \cap N_G(v) \neq \varnothing$; for any two $u \nsim v$, $N_{\overline{G}}(u) \backslash N_{\overline{G}}(v) \neq \varnothing$	$(1,1,-1)$, $(2,-1,1)$	$[3\alpha n]^{(n-1)}$, $[\alpha(3n-2\lambda_i+2)+2\lambda_i-2]$ for $i=2,3,\ldots,n$; $3n$, $2\alpha(2n-r+1)-n+2r-2$
21.	$\overline{G} \vee_{A(\overline{G})} \overline{G}$	$n \geq 2$; for any two $u \sim v$, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \varnothing$	$(1,1,-1)$, $(1,1,1)$	$2\alpha(n+r-\lambda_i)+2\lambda_i$ for $i=2,3,\ldots,n$; $[2\alpha(n+r+1)-2]^{(n)}$, $2(n+r)$
22.	$\overline{G} \vee_{A(\overline{G})+I_n} \overline{G}$	for any two $u \sim v$, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \varnothing$	$(1,1,-1)$, $(1,1,0)$	$\alpha(2n+2r-2\lambda_i)+2\lambda_i-1$ for $i=2,3,\ldots,n$; $[2\alpha(n+r)-1]^{(n)}$, $2n+2r-1$
23.	$\overline{K}_n \vee_{A(\overline{G})+I_n} \overline{K}_n$	for any two $u \sim v$, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \varnothing$	$(2,0,-2)$, $(1,2,0)$	$\alpha(3n+2r-2\lambda_i)+2\lambda_i-2$, $\alpha(3n+2r+2\lambda_i)-2\lambda_i-2$ for $i=2,3,\ldots,n$; $3n+2r-2$, $2\alpha(n+2r)+n-2r-2$

Table 2: Generalized distance spectra of graphs constructed by the unary graph operations given in Table 1

In the following result, we construct infinite pairs of generalized distance cospectral graphs by using Theorem 3.1.

Corollary 3.2. *Let G_1 and G_2 be two r regular adjacency cospectral graphs on n vertices with same assumptions from the 3^{rd} column of Table 2. Then the graphs constructed from G_1 and G_2 by the corresponding graph operation mentioned in the same table are generalized distance cospectral graphs. In particular, they are distance (distance Laplacian, distance signless Laplacian) cospectral graphs.*

As a consequence of Theorem 3.1, in the following result, we show that infinite families of distance (distance Laplacian, distance signless Laplacian) integral graphs can be constructed by using the unary graph operations in Table 2 and regular adjacency integral graphs with some additional assumptions.

Corollary 3.3. *Let G be an r-regular graph on n vertices with assumptions as in the 3^{rd} column of Table 2. Then the graph constructed from G by the corresponding graph operation mentioned in the same table is distance (distance Laplacian, distance signless Laplacian) integral if and only if G is adjacency integral.*

Proof. Let G' be the graph constructed from G by the corresponding unary graph operation mentioned in Table 2. By substituting $\alpha = 0$ in the generalized distance spectra of G' given in the same table, it can be seen that the eigenvalues of $D(G')$ are integers if and only if the eigenvalues of $A(G)$ are integers.

Since G' is t-transmission regular, it follows that the distance Laplacian and the distance signless Laplacian matrices of G' are $tI - D(G')$ and $tI + D(G')$ respectively. Hence G' is distance Laplacian (distance signless Laplacian) integral if and only if G' is distance integral. So the proof follows. □

4. Generalized distance spectra of graphs constructed by some binary graph operations

Let G_1 and G_2 be graphs on n_1 and n_2 vertices respectively. Let $H \in \{G_1, \overline{G}_1, K_{n_1}, \overline{K}_{n_1}\}$ and let H_1 and H_2 denote two copies of H. Take $\mathcal{H}_3 = (H_1, H_2, G_2)$. In Table 3, we define some binary graph operations, which can be viewed as a $\mathcal{M}$-join of graphs in $\mathcal{H}_3$, where $\mathcal{M} = (M_{12}, M_{13}, M_{23})$ with $M_{13} = M_{23} = J_{n_1 \times n_2}$. The choice of H and M_{12} for the corresponding graph operation is mentioned in the same table.

S. No.	Name of the graphs	H	M_{12}
1.	Mirror graph of G_1 join G_2	G_1	I_{n_1}
2.	VC-neighbourhood graph of G_1 join G_2	G_1	$J_{n_1} - I_{n_1}$
3.	N-neighbourhood graph of G_1 join G_2	G_1	$A(G_1)$
4.	$\overline{N}$-neighbourhood graph of G_1 join G_2	G_1	$A(G_1) + I_{n_1}$
5.	NC-neighbourhood graph of G_1 join G_2	G_1	$J_{n_1} - A(G_1) - I_{n_1}$
6.	$\overline{N}C$-neighbourhood graph of G_1 join G_2	G_1	$J_{n_1} - A(G_1)$
7.	Mirror-complement graph of G_1 join G_2	$\overline{G}_1$	I_{n_1}
8.	Join-neighbourhood-complement graph of G_1 join G_2	$\overline{G}_1$	J_{n_1}
9.	VC-neighbourhood-complement graph of G_1 join G_2	$\overline{G}_1$	$J_{n_1} - I_{n_1}$
10.	N-neighbourhood-complement graph of G_1 join G_2	$\overline{G}_1$	$A(G_1)$

11.	$\overline{N}$-neighbourhood-complement graph of G_1 join G_2	$\overline{G}_1$	$A(G_1)+I_{n_1}$
12.	NC-neighbourhood-complement graph of G_1 join G_2	$\overline{G}_1$	$J_{n_1}-A(G_1)-I_{n_1}$
13.	$\overline{N}C$-neighbourhood-complement graph of G_1 join G_2	$\overline{G}_1$	$J_{n_1}-A(G_1)$
14.	Duplicate graph of G_1 join G_2	$\overline{K}_{n_1}$	$A(G_1)$
15.	$D\overline{N}$ graph of G_1 join G_2	$\overline{K}_{n_1}$	$A(G_1)+I_{n_1}$
16.	Complemented duplicate graph of G_1 join G_2	$\overline{K}_{n_1}$	$J_{n_1}-A(G_1)-I_{n_1}$
17.	Closed duplicate graph of G_1 join G_2	$\overline{K}_{n_1}$	$J_{n_1}-A(G_1)$
18.	Fully complete duplicate graph of G_1 join G_2	K_{n_1}	$A(G_1)$
19.	Fully complete $D\overline{N}$ graph of G_1 join G_2	K_{n_1}	$A(G_1)+I_{n_1}$
20.	Fully complete complemented duplicate graph of G_1 join G_2	K_{n_1}	$J_{n_1}-A(G_1)-I_{n_1}$
21.	Fully complete closed duplicate graph of G_1 join G_2	K_{n_1}	$J_{n_1}-A(G_1)$

Table 3: Some binary graph operations defined as $\mathcal{M}$-join of graphs

Theorem 4.1. *For $i=1,2$ let G_i be an r_i-regular graph on n_i vertices and let the eigenvalues of $A(G_i)$ be $(r_i=)\lambda_{i1},\lambda_{i2},\ldots,\lambda_{in_i}$. Then the graph constructed from G_1 and G_2 by each of the graph operation mentioned in the 2^{nd} column of Table 4 has the n_2-1 generalized distance eigenvalues $\alpha(2n_1+2n_2-r_2+\lambda_{2k})-\lambda_{2k}+2$ for $k=2,3,\ldots,n_2$ and the remaining eigenvalues are given in the 4^{th} column of the same table.*

Proof. Let $(J_{n_i\times 1}=)X_{i1},X_{i2},\ldots,X_{in_i}$ be orthogonal eigenvectors corresponding to the eigenvalues $(r_i=)\lambda_{i1},\lambda_{i2},\ldots,\lambda_{in_i}$ of $A(G_i)$. Taking $A_1=A(G_1)$, $A_2=A(G_2)$ and substituting the values a_i,b_j for $i,j=1,2,3$ as given in Table 4, in (6) and (7), we get the distance matrix and the generalized distance matrix respectively of the graph constructed by corresponding graph operation mentioned in the same table. By substituting these values in Theorem 2.2, we obtain the generalized distance spectrum of the corresponding graph which includes the n_2-1 eigenvalues $\alpha(2n_1+2n_2-r_2+\lambda_{2k})-\lambda_{2k}+2$ for $k=2,3,\ldots,n_2$ and the remaining eigenvalues are given in Table 4. $\square$

S. No.	Graphs	(a_1,a_2,a_3), (b_1,b_2,b_3)	Remaining generalized distance eigenvalues
1.	Mirror graph of G_1 join G_2	$(2,-1,-2)$, $(2,0,-1)$	$\alpha(4n_1+n_2-r_1+\lambda_{1j})-\lambda_{1j}-3$, $\alpha(4n_1+n_2-r_1+\lambda_{1j}-2)-\lambda_{1j}-1$ for $j=2,3,\dots,n_1$; $\alpha(n_2+4n_1-2)-r_1-1$, $\frac{1}{2}\{(\alpha+2)(2n_1+n_2)-r_1-r_2-5\pm[((\alpha+2)(2n_1+n_2)-r_1-r_2-5)^2-4(\alpha n_2+4n_1-r_1-3)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
2.	VC-neighbourhood graph of G_1 join G_2	$(2,-1,-2)$, $(1,0,1)$	$\alpha(3n_1+n_2-r_1+\lambda_{1j})-\lambda_{1j}-1$, $\alpha(3n_1+n_2-r_1+\lambda_{1j}+2)-\lambda_{1j}-3$ for $j=2,3,\dots,n_1$; $\alpha(n_2+2n_1+2)+n_1-r_1-3$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2-r_1-r_2-3\pm[(\alpha(2n_1+n_2)+3n_1+2n_2-r_1-r_2-3)^2-4(\alpha n_2+3n_1-r_1-1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
3.	N-neighbourhood graph of G_1 join G_2	$(2,-1,-2)$, $(2,-1,0)$	$\alpha(4n_1+n_2-2r_1+2\lambda_{1j})-2\lambda_{1j}-2$ for $j=2,3,\dots,n_1$; $[\alpha(4n_1+n_2-2r_1)-2]^{(n_1-1)}$, $\alpha(n_2+4n_1-2r_1)-2$, $\frac{1}{2}\{(\alpha+2)(2n_1+n_2)-2r_1-r_2-4\pm[((\alpha+2)(2n_1+n_2)-2r_1-r_2-4)^2-4(\alpha n_2+4n_1-2r_1-2)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
4.	$\overline{N}$-neighbourhood graph of G_1 join G_2	$(2,-1,-2)$, $(2,-1,-1)$	$\alpha(4n_1+n_2-2r_1+2\lambda_{1j})-2\lambda_{1j}-3$ for $j=2,3,\dots,n_1$; $[\alpha(4n_1+n_2-2r_1-2)-1]^{(n_1-1)}$, $\alpha(n_2+4n_1-2r_1-2)-1$, $\frac{1}{2}\{(\alpha+2)(2n_1+n_2)-2r_1-r_2-5\pm[((\alpha+2)(2n_1+n_2)-2r_1-r_2-5)^2-4(\alpha n_2+4n_1-2r_1-3)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$

5.	NC-neighbourhood graph of G_1 join G_2	$(2,-1,-2)$, $(1,1,1)$	$\alpha(3n_1+n_2+2\lambda_{1j}+2)-2\lambda_{1j}-3$ for $j=2,3,\ldots,n_1$; $[\alpha(3n_1+n_2)-1]^{(n_1-1)}$, $\alpha(n_2+2n_1+2r_1+2)+n_1-2r_1-3$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2-r_2-3\pm[(\alpha(2n_1+n_2)+3n_1+2n_2-r_2-3)^2-4(\alpha n_2+3n_1-1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
6.	$\overline{N}C$-neighbourhood graph of G_1 join G_2	$(2,-1,-2)$, $(1,1,0)$	$\alpha(3n_1+n_2+2\lambda_{1j})-2\lambda_{1j}-2$ for $j=2,3,\ldots,n_1$; $[\alpha(3n_1+n_2)-2]^{(n_1-1)}$, $\alpha(n_2+2n_1+2r_1)+n_1-2r_1-2$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2-r_2-4\pm[(\alpha(2n_1+n_2)+3n_1+2n_2-r_2-4)^2-4(\alpha n_2+3n_1-2)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
7.	Mirror-complement graph of G_1 join G_2	$(1,1,-1)$, $(2,0,-1)$	$\alpha(3n_1+n_2+r_1-\lambda_{1j})+\lambda_{1j}-2$, $\alpha(3n_1+n_2+r_1-\lambda_{1j}-2)+\lambda_{1j}$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+4n_1-2)-n_1+r_1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2+r_1-r_2-4\pm[(\alpha(2n_1+n_2)+3n_1+2n_2+r_1-r_2-4)^2-4(\alpha n_2+3n_1+r_1-2)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
8.	Join-neighbourhood-complement graph of G_1 join G_2	$(1,1,-1)$, $(1,0,0)$	$\alpha(2n_1+n_2+r_1-\lambda_{1j})+\lambda_{1j}-1$, $\alpha(2n_1+n_2+r_1-\lambda_{1j})+\lambda_{1j}-1$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+2n_1)+r_1-1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+2n_1+2n_2+r_1-r_2-3\pm[(\alpha(2n_1+n_2)+2n_1+2n_2+r_1-r_2-3)^2-4(\alpha n_2+2n_1+r_1-1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$

9.	VC-neighbourhood-complement graph of G_1 join G_2	$(1,1,-1)$, $(1,0,1)$	$\alpha(2n_1+n_2+r_1-\lambda_{1j})+\lambda_{1j}$, $\alpha(2n_1+n_2+r_1-\lambda_{1j}+2)+\lambda_{1j}-2$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+2n_1+2)+r_1-2$, $\frac{1}{2}\{\alpha(2n_1+n_2)+2n_1+2n_2+r_1-r_2-2\pm[(\alpha(2n_1+n_2)+2n_1+2n_2+r_1-r_2-2)^2-4(\alpha n_2+2n_1+r_1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
10.	N-neighbourhood-complement graph of G_1 join G_2	$(1,1,-1)$, $(2,-1,0)$	$\alpha(3n_1+n_2-2\lambda_{1j})+2\lambda_{1j}-1$ for $j=2,3,\ldots,n_1$; $[\alpha(3n_1+n_2)-1]^{(n_1-1)}$, $\alpha(n_2+4n_1-2r_1)-n_1+2r_1-1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2-r_2-3\pm[(\alpha(2n_1+n_2)+3n_1+2n_2-r_2-3)^2-4(\alpha n_2+3n_1-1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
11.	$\overline{N}$-neighbourhood-complement graph of G_1 join G_2	$(1,1,-1)$, $(2,-1,-1)$	$\alpha(3n_1+n_2-2\lambda_{1j}-2)+2\lambda_{1j}$ for $j=2,3,\ldots,n_1$; $[\alpha(3n_1+n_2)-2]^{(n_1-1)}$, $\alpha(n_2+4n_1-2r_1-2)-n_1+2r_1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2-r_2-4\pm[(\alpha(2n_1+n_2)+3n_1+2n_2-r_2-4)^2-4(\alpha n_2+3n_1-2)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
12.	NC-neighbourhood-complement graph of G_1 join G_2	$(1,1,-1)$, $(1,1,1)$	$\alpha(2n_1+n_2+2r_1-2\lambda_{1j})+2\lambda_{1j}$ for $j=2,3,\ldots,n_1$; $[\alpha(2n_1+n_2+2r_1+2)-2]^{(n_1-1)}$, $\alpha(n_2+2n_1+2r_1+2)-2$, $\frac{1}{2}\{\alpha(2n_1+n_2)+2n_1+2n_2+2r_1-r_2-2\pm[(\alpha(2n_1+n_2)+2n_1+2n_2+2r_1-r_2-2)^2-4(\alpha n_2+2n_1+2r_1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$

13.	$\overline{N}C$-neighbourhood-complement graph of G_1 join G_2	$(1,1,-1)$, $(1,1,0)$	$\alpha(2n_1+n_2+2r_1-2\lambda_{1j})+2\lambda_{1j}-1$ for $j=2,3,\ldots,n_1$; $[\alpha(2n_1+n_2+2r_1)-1]^{(n_1-1)}$, $\alpha(n_2+2n_1+2r_1)-1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+2n_1+2n_2+2r_1-r_2-3\pm[(\alpha(2n_1+n_2)+2n_1+2n_2+2r_1-r_2-3)^2-4(\alpha n_2+2n_1+2r_1-1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
14.	Duplicate graph of G_1 join G_2	$(2,0,0)$, $(2,-1,0)$	$\alpha(4n_1+n_2-r_1+\lambda_{1j})-\lambda_{1j}$, $\alpha(4n_1+n_2-r_1-\lambda_{1j})+\lambda_{1j}$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+4n_1-2r_1)-r_1$, $\frac{1}{2}\{(\alpha+2)(2n_1+n_2)-r_1-r_2-2\pm[((\alpha+2)(2n_1+n_2)-r_1-r_2-2)^2-4(\alpha n_2+4n_1-r_1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
15.	$D\overline{N}$ graph of G_1 join G_2	$(2,0,0)$, $(2,-1,-1)$	$\alpha(4n_1+n_2-r_1+\lambda_{1j})-\lambda_{1j}-1$, $\alpha(4n_1+n_2-r_1-\lambda_{1j}-2)+\lambda_{1j}+1$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+4n_1-2r_1-2)+r_1+1$, $\frac{1}{2}\{(\alpha+2)(2n_1+n_2)-r_1-r_2-3\pm[((\alpha+2)(2n_1+n_2)-r_1-r_2-3)^2-4(\alpha n_2+4n_1-r_1-1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
16.	Complemented duplicate graph of G_1 join G_2	$(2,0,0)$, $(1,1,1)$	$\alpha(3n_1+n_2+r_1-\lambda_{1j})+\lambda_{1j}+1$, $\alpha(3n_1+n_2+r_1+\lambda_{1j}+2)-\lambda_{1j}-1$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+2n_1+2r_1+2)+n_1-r_1-1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2+r_1-r_2-1\pm[(\alpha(2n_1+n_2)+3n_1+2n_2+r_1-r_2-1)^2-4(\alpha n_2+3n_1+r_1+1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$

17.	Closed duplicate graph of G_1 join G_2	$(2,0,0)$, $(1,1,0)$	$\alpha(3n_1+n_2+r_1-\lambda_{1j})+\lambda_{1j}$, $\alpha(3n_1+n_2+r_1+\lambda_{1j})-\lambda_{1j}$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+2n_1+2r_1)+n_1-r_1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2+r_1-r_2-2\pm[(\alpha(2n_1+n_2)+3n_1+2n_2+r_1-r_2-2)^2-4(\alpha n_2+3n_1+r_1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
18.	Fully complete duplicate graph of G_1 join G_2	$(1,0,-1)$, $(2,-1,0)$	$\alpha(3n_1+n_2-r_1+\lambda_{1j})-\lambda_{1j}-1$, $\alpha(3n_1+n_2-r_1-\lambda_{1j})+\lambda_{1j}-1$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+4n_1-2r_1)-n_1+r_1-1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2-r_1-r_2-3\pm[(\alpha(2n_1+n_2)+3n_1+2n_2-r_1-r_2-3)^2-4(\alpha n_2+3n_1-r_1-1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
19.	Fully complete $D\overline{N}$ graph of G_1 join G_2	$(1,0,-1)$, $(2,-1,-1)$	$\alpha(3n_1+n_2-r_1+\lambda_{1j})-\lambda_{1j}-2$, $\alpha(3n_1+n_2-r_1-\lambda_{1j}-2)+\lambda_{1j}$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+4n_1-2r_1-2)-n_1+r_1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+3n_1+2n_2-r_1-r_2-4\pm[(\alpha(2n_1+n_2)+3n_1+2n_2-r_1-r_2-4)^2-4(\alpha n_2+3n_1-r_1-2)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$
20.	Fully complete complemented duplicate graph of G_1 join G_2	$(1,0,-1)$, $(1,1,1)$	$\alpha(2n_1+n_2+r_1-\lambda_{1j})+\lambda_{1j}$, $\alpha(2n_1+n_2+r_1+\lambda_{1j}+2)-\lambda_{1j}-2$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+2n_1+2r_1+2)-r_1-2$, $\frac{1}{2}\{\alpha(2n_1+n_2)+2n_1+2n_2+r_1-r_2-2\pm[(\alpha(2n_1+n_2)+2n_1+2n_2+r_1-r_2-2)^2-4(\alpha n_2+2n_1+r_1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$

21.	Fully complete closed duplicate graph of G_1 join G_2	$(1,0,-1)$, $(1,1,0)$	$\alpha(2n_1+n_2+r_1-\lambda_{1j})+\lambda_{1j}-1$, $\alpha(2n_1+n_2+r_1+\lambda_{1j})-\lambda_{1j}-1$ for $j=2,3,\ldots,n_1$; $\alpha(n_2+2n_1+2r_1)-r_1-1$, $\frac{1}{2}\{\alpha(2n_1+n_2)+2n_1+2n_2+r_1-r_2-3\pm[(\alpha(2n_1+n_2)+2n_1+2n_2+r_1-r_2-3)^2-4(\alpha n_2+2n_1+r_1-1)(2\alpha n_1+2n_2-r_2-2)-8(1-\alpha)^2n_1n_2]^{\frac{1}{2}}\}$

Table 4: Remaining generalized distance eigenvalues of graphs constructed by the graph operations defined in Table 3

In the following result, we construct infinite pairs of generalized distance cospectral graphs by using Theorem 4.1.

Corollary 4.2. *For $i=1,2$, let G_i and H_i be two r_i-regular cospectral graphs on n_i vertices. The graphs constructed by using the pairs of graphs (G_1,G_2) and (H_1,H_2) by the same graph operation mentioned in Table 4 are generalized distance cospectral. In particular, they are distance (distance Laplacian, distance signless Laplacian) cospectral.*

As a consequence of Theorem 4.1, in the following result we show that the binary graph operations in Table 4 with some additional assumptions can be used to construct infinite families of distance (distance Laplacian, distance signless Laplacian) integral graphs.

Corollary 4.3. *For $i=1,2$ let G_i be an r_i-regular graph on n_i vertices. The graphs constructed from G_1 and G_2 by the graph operations mentioned in Table 4 are distance (distance Laplacian, distance signless Laplacian) integral if and only if G_1 and G_2 are adjacency integral and $[(a_1+b_1)n_1+2n_2+(a_2+b_2)r_1-r_2+a_3+b_3-2]^2-4[(a_1+b_1)n_1+(a_2+b_2)r_1+a_3+b_3](2n_2-r_2-2)-8n_1n_2$ is a perfect square for the corresponding values of a_1,a_2,a_3,b_1,b_2,b_3 mentioned in the same table.*

Proof. Let G_1 and G_2 be two graphs. Let G' be the graph constructed from G_1 and G_2 by a binary graph operation mentioned in Table 4. Since $D(G')$ and $Tr(G')$ have common eigenvectors, it follows that G' is distance Laplacian (distance signless Laplacian) integral if and only if G' is distance integral.

S. No.s of graph operations in Table 4	n_1	n_2	r_1	r_2	G_1	G_2
1	$2t$	$10t$	$2t-1$	0	K_{2t}	$\overline{K}_{10t}$
2,3	$2t$	$24t$	$2t-1$	$24t-2$	K_{2t}	$\overline{12tK_2}$
4	$2t$	$24t$	$2t-2$	$24t-3$	$\overline{tK_2}$	$\overline{8tK_3}$
5,9,10	$3t$	$2t$	$3t-1$	$2t-1$	K_{3t}	K_{2t}
6,7,11	$2t$	$9t$	0	0	$\overline{K}_{2t}$	$\overline{K}_{9t}$
8,13	$2t$	$24t$	0	$24t-1$	$\overline{K}_{2t}$	K_{24t}
12,20	$2t$	$24t$	0	$24t-2$	$\overline{K}_{2t}$	$\overline{12tK_2}$
14	$2t$	$30t$	0	$30t-2$	$\overline{K}_{2t}$	$\overline{15tK_2}$
15	$3t$	$2t$	$3t-1$	$2t-2$	K_{3t}	$\overline{tK_2}$
16	$2t$	$2t$	$2t-1$	$2t-2$	K_{2t}	$\overline{tK_2}$
17	$3t$	$2t$	0	$2t-2$	$\overline{K}_{3t}$	$\overline{tK_2}$
18	$2t$	$9t$	1	0	tK_2	$\overline{K}_{9t}$
19	$6t$	$2t$	$6t-2$	$2t-2$	$\overline{3tK_2}$	$\overline{tK_2}$
21	$6t$	$2t$	1	$2t-2$	$3tK_2$	$\overline{tK_2}$

Table 5: Existence of infinite family of distance integral graphs constructed from the graph operations in Table 4

Now, we take G_i to be an r_i-regular graph on n_i vertices for $i = 1,2$ and construct the graph G' as mentioned above. Then, by substituting $\alpha = 0$ in the generalized distance spectrum of G' given in Theorem 4.1, it can be seen that the eigenvalues of $D(G')$ are integers if and only if G_1 and G_2 are adjacency integral and the roots of the equation

$$x^2-[(a_1+b_1)n_1+2n_2+(a_2+b_2)r_1-r_2+a_3+b_3-2]x+[(a_1+b_1)n_1+(a_2+b_2)r_1+a_3+b_3](2n_2-r_2-2)-2n_1n_2=0$$

are integers for the corresponding values of a_1,a_2,a_3,b_1,b_2,b_3 mentioned in Table 4. The roots of the above equation are integers if and only if

$$[(a_1+b_1)n_1+2n_2+(a_2+b_2)r_1-r_2+a_3+b_3-2]^2-4[(a_1+b_1)n_1+(a_2+b_2)r_1+a_3+b_3](2n_2-r_2-2)-8n_1n_2$$

is a perfect square. The existence of positive integers n_1,n_2,r_1,r_2 for which the above number is a perfect square and the existence of a r_i-regular adjacency integral graph G_i on n_i vertices for $i = 1,2$ are given in Table 5, where t is any positive integer. This shows the existence of an infinite family of distance (distance Laplacian, distance signless Laplacian) integral graphs constructed from the graph operations in Table 4. □

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