PROJECTIVELY COHEN-MACAULY SURFACES
OF SMALL DEGREE IN $\mathbb{P}^5$

MARINA MANCINI

In this paper we consider the nondegenerate projectively Cohen-Macaulay (p.C.M.) surfaces of small degree in $\mathbb{P}^5$. We determine those of degree $d \leq 9$ and all candidate rational surfaces as p.C.M. surfaces.

Introduction.

The problem of describing smooth embedded surfaces having particular properties, such as, for example, being projectively normal or projectively Cohen-Macaulay (p.C.M. for short), has been considered by many authors in the past (recall that such surface is p.C.M. if its homogeneous coordinate ring is Cohen-Macaulay).

Our aim in this paper is to determine all the nondegenerate p.C.M. surfaces of degree $d \leq 9$ in $\mathbb{P}^5_\mathbb{C} = \mathbb{P}^5$.

From our previous work (see [16]), we know that, if $g(H) = g$ is the sectional genus of a nondegenerate p.C.M. surface $X \subset \mathbb{P}^N$, then $N = d - g + 1 + p_a - h^1(\mathcal{O}_X(1)) = d - g + 1 + h^1(\mathcal{O}_H(1))$, where $p_a$ denotes the arithmetic genus of $X$, and that for the degree $d$ of $X$ we have the bounds

$$N - 1 \leq d \leq \binom{N}{2} + h^1(\mathcal{O}_H(1)).$$

Entrato in Redazione il 28 settembre 2000.
In particular, when \( p_a = 0 \), we have \( h^1(\mathcal{O}_H(1)) = p_a - h^2(\mathcal{O}_X(1)) = 0 \), hence

\[
N - 1 \leq d = N - g + 1 \leq \binom{N}{2}.
\]

We also recall that the irregularity \( q(X) = p_g - p_a \), of any p.C.M. surface is zero (e.g. see [16]).

So, in \( \mathbb{P}^5 \), we only have to consider surfaces of degree \( d \leq 10 + h^1(\mathcal{O}_H(1)) \) and sectional genus \( g = d - 4 + h^1(\mathcal{O}_H(1)) \).

All the nondegenerate p.C.M. surfaces \( X \subset \mathbb{P}^5 \) of degree \( d \leq 9 \) can be determined. Our results are summarized in Table 1.

### TABLE 1. Projectively C.M. surfaces in \( \mathbb{P}^5 \) of degree \( \leq 9 \)

<table>
<thead>
<tr>
<th>( d )</th>
<th>( g )</th>
<th>( p_a )</th>
<th>Structure of ( X )</th>
<th>( \mathcal{O}_X(H) = \mathcal{O}_X(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>Veronese Surface</td>
<td>( \mathcal{O}_X(3E_0 - E_1 - \ldots - E_4) )</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>Del Pezzo Surface, ( X_4 )</td>
<td>( \mathcal{O}_X(4E_0 - 2E_1 - E_2 - \ldots - E_5) )</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>0</td>
<td>Castelnuovo Surface, ( X_7 )</td>
<td>( \mathcal{O}_X(5E_0 - 3E_1 - E_2 - \ldots - E_6) )</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>0</td>
<td>Bordiga-White Surface, ( X_{10} )</td>
<td>( \mathcal{O}<em>X(5E_0 - 3E_1 - E_2 - \ldots - E</em>{10}) )</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>0</td>
<td>Veronese Surface, ( X_9 )</td>
<td>( \mathcal{O}_X(6E_0 - 2E_1 - \ldots - 2E_7 - E_8) )</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>1</td>
<td>K3 Surface</td>
<td>( \mathcal{O}<em>X(9E_0 - 3E_1 - \ldots - 2E</em>{10}) )</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>0</td>
<td>( X_9 )</td>
<td>( \mathcal{O}<em>X(5E_0 - 2E_1 - 2E_2 - E_3 - \ldots - E</em>{11}) )</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>0</td>
<td>( X_{10} )</td>
<td>( \mathcal{O}<em>X(6E_0 - 2E_1 - \ldots - 2E_6 - E_3 - \ldots - E</em>{10}) )</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>0</td>
<td>( X_{10} )</td>
<td>( \mathcal{O}<em>X(7E_0 - 2E_1 - \ldots - E</em>{10}) )</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>0</td>
<td>( X_{12} )</td>
<td>( \mathcal{O}<em>X(6E_0 - 2E_1 - \ldots - 2E_8 - E_6 - \ldots - E</em>{12}) )</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>0</td>
<td>( F^{10} : 0 \leq e \leq 2 )</td>
<td>( \mathcal{O}<em>X(4C_0 + (2e + 5)f - 2E_1 - \ldots - 2E_7 - E_8 - \ldots - E</em>{10}) )</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td></td>
<td>( Y_1 )</td>
<td></td>
</tr>
</tbody>
</table>
We use the following notations:

\[ d = \deg X, \ g(H) = g \text{ sectional genus of } X \]

\[-X_s: \text{ blowing-up of } \mathbb{P}^2 \text{ at } s \text{ generic points} \]

\[-\mathbb{P}_s: \text{ blowing-up of the rational ruled surface } \mathbb{P}_s \text{ at } s \text{ generic points} \]

\[-Y_s: \text{ blowing-up a K3 surface } Y \text{ at } s \text{ generic points}. \]

Since the maximum degree of a rational p.C.M. surface \( X \) in \( \mathbb{P}^N \) is \( d = \binom{N}{3} \), in order to complete the description of the rational p.C.M. surfaces in \( \mathbb{P}^3 \) it remains to consider the case \( d = 10 \).

Table 2, in Section 6, shows all the possible candidates as rational p.C.M. surfaces of degree \( d = 10 \); it still an open problem to check if all of them actually exist and which of them are p.C.M.

I am grateful to Prof. A. Gimigliano for some helpful talks.

1. **Background on the p.C.M. embeddings of blowing-ups of \( \mathbb{P}^2 \) at a finite set of distinct points.**

Let \( Z = (P_1, \ldots, P_s; m_1, \ldots, m_s) \) be, with \( m_1 \geq \ldots \geq m_s \), the 0-dimensional subscheme of \( \mathbb{P}^2_Z = \mathbb{P}^2 \) associated to the homogeneous ideal \( I_Z = p_1^{m_1} \cap \ldots \cap p_s^{m_s} \subset \mathbb{C}[x_0, x_1, x_2] \), where each \( p_i \) is a homogeneous prime ideal which corresponds to a point \( P_i \) of \( \mathbb{P}^2 \), \( i = 1, \ldots, s \).

If \( X_s \) is the blowing-up \( \mathbb{P}^2 \) at the distinct points \( P_1, \ldots, P_s \) of the support of \( Z \), we denote by \( E_1, \ldots, E_s \) the divisor classes on \( X_s \) which contain the exceptional divisor and by \( E_0 \) the divisor class on \( X_s \) of the strict transform of generic line of \( \mathbb{P}^2 \). It is well known that \( Pic X_s \cong \mathbb{Z}^{r+1} \) is freely generated by \( E_0, E_1, \ldots, E_s \) and that, if \( C \) is a plane curve of degree \( t \) with a singularity at \( P_i \) of multiplicity \( m_i \), \( i = 1, \ldots, s \), then the strict transform of \( C \) on \( X_s \) is an effective divisor in the divisor class of \( tE_0 - m_1E_1 - \ldots - m_sE_s \) (e.g. see [11]).

Now, let \( H_Z(t) \) be the Hilbert function of \( Z \); let \( \sigma(Z) = \min \{ t / \Delta H_Z(t) = 0 \} \), where \( \Delta H_Z(t) = H_Z(t) - H_Z(t - 1) \) is the first difference of \( H_Z(t) \), then we have:

\[ \sigma(Z) - 1 = \tau(Z) = \min \{ t / h^0(I_Z(t)) \cdot h^1(I_Z(t)) = 0 \}, \]

where \( I_Z \subseteq \mathcal{O}_{\mathbb{P}^2} \) denotes the ideal sheaf of \( Z \). Namely, \( \tau(Z) \) is the smallest integer \( t \) for which the linear system of all the plane curves of degree \( t \) passing through each \( P_i, i = 1, \ldots, s \), with multiplicity at least \( m_i \) is regular (e.g. see [7]).

If \( D_t = tE_0 - m_1E_1 - \ldots - m_sE_s \) is a divisor on \( X_s \) associated to the scheme \( Z \subset \mathbb{P}^2 \), then we have the following results (see [6]):
Proposition 1.1. \(D_t\) is very ample on \(X_t\) for every \(t \geq \sigma(Z)\) if, and only if, no line of \(\mathbb{P}^2\) has intersection of degree \(\geq \sigma(Z)\) with \(Z\).

Proposition 1.2. The very ample linear system \(|D_t|\) embeds \(X_t\) as a projectively Cohen-Macaulay surface for every \(t \geq \sigma(Z)\).

We also know (e.g. see [16]) that a necessary condition so that \(|D_t|\) embeds \(X_t\) as a projectively Cohen-Macaulay surface is that \(h^1(\mathcal{O}_{X_t}(D_t)) = 0\) and so that \(t \geq \tau(Z)\).

When \(D_t\) is very ample, we denote by \(V_{i,Z} \subset \mathbb{P}^N\) the image of the embedding \(\varphi_{i,Z} : X_t \rightarrow \mathbb{P}^N\), where \(N + 1 = h^0(\mathcal{O}_{X_t}(D_t)) = \binom{t+2}{2} - \deg Z\) (e.g. see [6]), which is determined by \(|D_t|\) on \(X_t\).

On the homogeneous ideal of the surface \(V_{i,Z} \subset \mathbb{P}^N\) know what follows:

Proposition 1.3. (See [6].) Let \(t \geq \sigma(Z) + 1\), then the homogeneous ideal of \(V_{i,Z} \subset \mathbb{P}^N\) is generated by forms of degree \(\leq 3\).

Proposition 1.4. (See [5].) Let \(t \geq \sigma(Z) + 1\), then the homogeneous ideal of \(V_{i,Z} \subset \mathbb{P}^N\) is generated by quadrics.

On the defining ideal of certain surfaces \(V_{i,Z} \subset \mathbb{P}^N\) we have more detailed information, namely we know that their generators can be given as minors of suitable matrices. In particular:

a) \(t = d, Z = (P_1, \ldots, P_s), s = \binom{d+1}{2}: \) for every \(d \geq 3\), the surface \(V_{d,Z}\) is called a **White Surface** in \(\mathbb{P}^d\). It has degree \(\binom{d}{2}\), sectional genus \(\binom{d-1}{2}\) and its ideal is generated by the \(3 \times 3\) minors of a \(3 \times d\) matrix of linear forms (see [8]);

b) \(t = d, Z = (P_1, \ldots, P_s), s = \binom{d+1}{2}: \) for every \(d \geq 3\), the surface \(V_{d+1,Z}\) is called a **Room Surface** in \(\mathbb{P}^{2d+2}\). It has degree \(\binom{d+2}{2}\) and sectional genus \(\binom{d}{2}\). Its ideal is generated by the \(2 \times 2\) minors of a \(3 \times (d + 1)\) matrix of linear forms (see [5]);

c) \(t = d + 1, Z = (P_1, \ldots, P_s), s = \binom{d+1}{2} + k, \) with \(0 < k < d + 1\): for every \(d \geq 3\), the surface \(V_{d+1,Z}\) is called a **Veronesean Surface** in \(\mathbb{P}^{2d-k+2}\). It has degree \(\binom{d+2}{2} - k\) and sectional genus \(\binom{d}{2}\). Its ideal is given as follows: its generators are the entries of the matrix \(A \cdot B\), the \(2 \times 2\) minors of \(B\) and the \(3 \times d\) minors of \(A\), where \(B\) and \(A\) are two matrices of linear forms of order, respectively, \(3 \times (d - k + 1)\) and \(k \times 3\) (see [10]);

d) \(t = d, Z = (P_1, \ldots, P_s; d - 2, 1, \ldots, 1), s = 2d: \) for every \(d \geq 4\), the surface \(V_{d,Z}\) is called a **Bordiga-White Surface** in \(\mathbb{P}^d\). It has degree \(2d - 3\),
sectional genus \( d - 2 \) and its ideal is generated by the \( 2 \times 2 \) minors of a matrix of type
\[
\begin{pmatrix}
Y_{1,1} & Y_{1,2} & \cdots & Y_{1,d-2} & Q_1 \\
Y_{2,1} & Y_{2,2} & \cdots & Y_{2,d-2} & Q_2
\end{pmatrix},
\]
where the \( Y_{a,b} \) are linear forms, while \( Q_1 \) and \( Q_2 \) are quadratic forms (see [6]).

2. Some results on rational p.C.M. surfaces.

Let \( X \) be a rational p.C.M. surface in \( \mathbb{P}^N \) of degree \( d \), with \( n - 1 \leq d \leq \binom{N}{2} \); then \( X \) has sectional genus \( g = d + 1 - N + h^1(\mathcal{O}_X(1)) = d + 1 - N \) (see [16]). We recall that, in terms of the cohomology of the ideal sheaf of \( X \), \( I_X \), the fact that \( X \) is p.C.M. in \( \mathbb{P}^N \) can be expressed by the condition \( h^i(I_X(m)) = 0 \), for \( i = 1, 2 \) and for all \( m \geq 0 \).

In our previous work (see [16]) we showed that a rational surface \( X \subseteq \mathbb{P}^N \) of degree \( \binom{N}{2} \), sectional genus \( \binom{N-1}{2} \) and with \( h^1(\mathcal{O}_X(1)) = 0 \) is p.C.M. if, and only if, it is projectively normal.

Now we want to extend this result, namely we have:

**Proposition 2.1.** Let \( X \subseteq \mathbb{P}^N \) be a smooth surface of degree \( d = N + g - 1 \), sectional genus \( g \) and irregularity \( q = h^1(\mathcal{O}_X) = 0 \). If \( h^1(\mathcal{O}_X(1)) = 0 \), then \( X \) is p.C.M. if, and only if, it is projectively normal.

**Proof.** Let us suppose that \( X \subseteq \mathbb{P}^N \) is projectively normal, hence that \( h^i(I_X(m)) = 0 \), for all \( m \geq 0 \).

Since \( h^i(\mathcal{O}_\mathbb{P}(m)) = 0 \), for all \( 0 < i < N \) and \( m \geq 0 \), from the exact sequence
\[
0 \to I_X(m) \to \mathcal{O}_\mathbb{P}(m) \to \mathcal{O}_X(m) \to 0
\]
we deduce that \( h^1(\mathcal{O}_X(m)) = h^2(I_X(m)) \). We want to show that \( h^1(\mathcal{O}_X(m)) = 0 \), \( \forall m \geq 0 \). Consider the exact sequence
\[
0 \to \mathcal{O}_X(m - 1) \to \mathcal{O}_X(m) \to \mathcal{O}_H(m) \to 0,
\]
where \( H \) is a smooth hyperplane section of \( X \).

Since \( md = m(N + g - 1) > 2g - 2 \), \( \forall m \geq 2 \), we have \( h^1(\mathcal{O}_H(m)) = 0 \), \( \forall m \geq 2 \). Thus, by the above exact sequence, \( h^1(\mathcal{O}_X(m - 1)) = 0 \) implies \( h^1(\mathcal{O}_X(m)) = 0 \) for all \( m \geq 2 \), and so, since \( h^1(\mathcal{O}_X(1)) = 0 \) by hypothesis, we get what wanted. \( \square \)
**Proposition 2.2.** The homogeneous ideal of a rational p.C.M. surface $X \subseteq \mathbb{P}^N$ can always be generated by forms of degree $\leq 3$ and $h^0(I_X(2)) \neq 0$ except when $X$ has maximum degree $d = \binom{N}{2}$.

**Proof.** From [16] we know that the ideal $I_X$ of a rational p.C.M. surface $X \subseteq \mathbb{P}^N$ can always be generated by forms of degree $\leq 3$, and that only generators are cubics in the case in which $X$ has maximum degree $\binom{N}{2}$. So it remains to prove that, when $X$ has not maximum degree $\binom{N}{2}$, $I_X$ always contains quadratic forms.

Let $H$ be a smooth hyperplane section of $X$ and consider the exact sequence

$$0 \to \mathcal{O}_X(1) \to \mathcal{O}_X(2) \to \mathcal{O}_H(2) \to 0.$$ 

Since

$$h^0(\mathcal{O}_X(2)) = h^0(\mathcal{O}_X(1)) + h^0(\mathcal{O}_X(2)) = N + 1 + 2d + 1 - g = 3N + g,$$

we have:

$$0 = h^1(I_X(2)) = h^0(I_X(2)) - h^0(\mathcal{O}_{\mathbb{P}^N}(2)) + h^0(\mathcal{O}_X(2)) = h^0(I_X(2)) - \binom{N - 1}{2} + g.$$

Clearly, $h^0(I_X(2)) = 0$ only when $g = \binom{N - 1}{2}$, and this terminates the proof. \hfill \qed


The smooth surfaces of degree $d \leq 8$ in $\mathbb{P}^5$ have been completely described (see [12], [4]). In this section we determine which of them are p.C.M. (Table 1 in the introduction summarizes our results).

3.1. **Surfaces of degree $d = 4$.** In $\mathbb{P}^5$ the p.C.M. surfaces of degree $d = 4$ are either Veronese Surfaces or rational scrolls, which are well known to be p.C.M.

3.2. **Surfaces of degree $d = 5$.** The only p.C.M. surfaces of degree 5 in $\mathbb{P}^5$ are the Del Pezzo Surfaces.
3.3. **Surfaces of degree \( d = 6 \).** The possibilities for a smooth surface of degree 6 in \( \mathbb{P}^5 \) are described in [12] and are the following:

(i) An elliptic, scroll white \( e = 0 \) and \( g = 1 \);

(ii) A Castelnuovo Surface, with \( g = 2 \), defined by the embedding of \( X_7 \) in \( \mathbb{P}^5 \) via the very ample linear system \( |D_i| = |4E_0 - 2E_1 - E_2 - \ldots - E_7| \).

Since \( 5 = N \neq d - g + 1 + h^1(\mathcal{O}_H(1)) = 6 + H^1(\mathcal{O}_H(1)) \) (see [16]), the unique p.C.M. surface of degree \( d = 6 \) in \( \mathbb{P}^5 \) is the Castelnuovo Surface (see also [12]).

3.4. **Surfaces of degree \( d = 7 \).** The smooth surfaces of degree \( d = 7 \) in \( \mathbb{P}^5 \) are classified by Ionescu in [12] and they are described as follows.

If \( X \subseteq \mathbb{P}^5 \) is a smooth surface of degree \( d = 7 \), then it has sectional genus \( g(H) = 3 \) and it is one of the following rational surfaces:

(i) A blowing-up \( \pi \) of \( \mathbb{F}_e \), \( e = 0, 1, 2, 3 \), with center 9 points; \( H = \pi^*(H_e) - E_1 - \ldots - E_9 \), where \( H_e = 2C_0 + (4 + e)f \);

(ii) A blowing-up \( \pi \) of \( \mathbb{P}^2 \) with center 9 points, \( H = \pi^*(4L) - E_1 - \ldots - E_9 \);

(iii) A blowing-up of a point on a Del Pezzo double plane \( S \), i.e. on a double covering of \( \mathbb{P}^2 \) ramified along a smooth quartic, \( H = \pi^*(H_e) - E \).

The surface \( X \subseteq \mathbb{P}^5 \) of the case 3.4 (ii) is a p.C.M. surface, called (Veronesean Surface) (see Section 1). Hence we have to consider the surfaces of the cases (i) and (iii).

**A) The case 3.4 (i).**

Let us denote by \( \mathbb{F}_e \) the blowing-up of \( \mathbb{F}_e \) at 9 generic points and let us consider the exact sequence

\[
0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \to \mathcal{O}_X(1) \to 0.
\]

By [1], Theorem 4.1, the smooth rational surface \( X \subseteq \mathbb{P}^5 \) is projectively normal. Moreover, since \( d = 7 > 2g(H) - 2 = 4 \), we have \( h^1(\mathcal{O}_H(1)) = 0 \) and so \( h^1(\mathcal{O}_X(1)) = 0 \). By Proposition 2.1, this is enough to conclude that \( X \) is p.C.M. in \( \mathbb{P}^5 \).

In particular, when \( e = 0, 1 \), another description of the surface \( X \) can be given, using a plane model, as follows.

**a) \( e = 0 \):** \( \mathbb{F}_e \) is isomorphic to \( X_{10} \), the blowing-up of \( \mathbb{P}^2 \) at 10 generic points.

In fact, \( \mathbb{F}_0 \) is isomorphic to the Quadric Surface \( Q \subseteq \mathbb{P}^3 \) and \( Q \) is obtained from \( X_2 \) via the complete (not very ample) linear system \( |2E_0 - E_1 - E_2| \).
Since the line \( E_0 - E_1 - E_2 \) on \( X_2 \) is contracted to a point \( P \in Q \), we have that to blow-up \( Q \) at the point \( P \) and at other 8 generic points is equivalent to blow-up \( \mathbb{P}^2 \) at 10 generic points, as we said.

Taking \( C_0 = E_0 - E_1 \) and \( f = E_0 - E_2 \), to the very ample divisor \( H_0 = 2C_0 + 4f - E_1 - \ldots - E_9 \) on \( \mathbb{P}^9 \) corresponds the very ample divisor \( D_5 = 2(E_0 - E_1 + 4(E_0 - E_2) - (E_0 - E_1 - E_2) - E_3 - \ldots - E_{10} = 5E_0 - E_1 - 3E_2 - E_3 - \ldots - E_{10} \) on \( X_{10} \).

The embedding of \( X_{10} \) in \( \mathbb{P}^5 \) via the complete linear system \( |D_5| \) is a \textit{Bordiga-White Surface} (see Section 1).

\textbf{b) \( e = 1 \):} Since \( \mathbb{F}_1 \) is isomorphic to \( X_1 \), we have that \( \mathbb{F}_1^9 \) is isomorphic to \( X_{10} \).

Let us determine the very ample divisor \( D_t = tE_0 - mE_1 - E_2 - \ldots - E_{10} \) on \( X_{10} \) which corresponds to the divisor \( 2C_0 + 5f - E_1 - \ldots - E_9 \) on \( \mathbb{F}_1^9 \). The integers \( t, m > 0 \) are such that

\[
\begin{aligned}
&\binom{t+2}{2} - \binom{m+1}{2} - 10 = 5 \\
t^2 - m^2 - 9 = 7,
\end{aligned}
\]

from which we get:

\[
\begin{aligned}
t^2 + 3t + 2 - m^2 - m &= 30 \\
t^2 = m^2 + 16.
\end{aligned}
\]

Solving the equations we find \( t = 5 \) and \( m = 3 \).

Hence \( D_t = D_5 = 5E_0 - 3E_1 - E_2 - \ldots - E_{10} \), which is the same divisor we found in \( a \).

\textbf{B) The case 3.4 (iii).}

A Del Pezzo double plane \( S \) is defined by the embedding of the blowing-up \( X_7 \) of \( \mathbb{P}^2 \) at 7 generic points via the very ample linear system \( |6E_0 - 2E_1 - \ldots - 2E_7| \); it is a smooth surface of degree 8 in \( \mathbb{P}^6 \).

Hence our surface \( X \) is determined by the very ample linear system \( |6E_0 - 2E_1 - \ldots - 2E_7 - E_8| \) on \( X_8 \) (see also [15]).

From the work of Alzati, Bertolini and Besana (see[1]) we know that the surface \( X \) is projectively normal in \( \mathbb{P}^5 \) thus, since \( h^1(\mathcal{O}_X(6E_0 - 2E_1 - \ldots - 2E_7 - E_8)) = 0 \), \( X \) is p.C.M. in \( \mathbb{P}^5 \), by Proposition 2.1.

We summarize the above results as follows:

**Proposition 3.1.** If \( X \subseteq \mathbb{P}^5 \) is a p.C.M. surface of degree \( d = 7 \), then it has sectional genus \( g = 3 \) and it is one of the following rational surfaces:
1) A Bordiga-White Surface, obtained embedding $X_{10}$ in $\mathbb{P}^5$ via the linear system $|5E_0 - 3E_1 - E_2 - \ldots - E_{10}|$.

2) The embedding of $\mathbb{P}^6_e$, $e = 2, 3$, in $\mathbb{P}^5$ via the linear system $|2C_0 + (4 + e)f - E_1 - \ldots - E_9|$.

3) A Veronesean Surface, defined by the embedding of $X_9$ in $\mathbb{P}^5$ via the linear system $|4E_0 - E_1 - \ldots - E_9|$.

4) A blowing-up of a point on a Del Pezzo double plane $S$, i.e. $X_8$ embedded in $\mathbb{P}^3$ via the linear system $|6E_0 - 2E_1 - \ldots - 2E_7 - E_8|$.

3.5. Surfaces of degree $d = 8$. Since a p.C.M. surface $X \subseteq \mathbb{P}^5$ of degree $d = 8$ has sectional genus $g = 4 + h^1(\mathcal{O}_H(1))$ and since $g \leq 5$, by Castelnuovo’s bound, it is enough to consider the smooth surfaces of sectional genus $g = 4, 5$.

Their classification is known (see[4], [14] and [15]) and it is the following:

If $X \subseteq \mathbb{P}^5$ is a smooth surface of degree $d = 8$ and sectional genus $4 \leq g \leq 5$, then it is either a K3 Surface of sectional genus $g = 5$ or it is one of the following rational surfaces of sectional genus $g = 4$:

(i) A blowing-up $\pi$ of the quadric surface $Q \subseteq \mathbb{P}^3$ with center 10 generic points, $H = \pi^*(3H_Q) - E_1 - \ldots - E_{10}$;

(ii) A blowing-up $\pi$ of a cubic surface $S \subseteq \mathbb{P}^3$ with center 4 generic points, $H = \pi^*(2H_S) - E_1 - \ldots - E_4$;

(iii) A blowing-up $\pi$ of a Hirzebruch surface $\mathbb{F}_e$, $e \leq 4$, with center 12 generic points, $H = \pi^*(2C_0 + (5 + e)f) - E_1 - \ldots - E_{12}$.

K3 Surfaces in $\mathbb{P}^5$ of degree $d = 8$ and of sectional genus $g = 5$ are well know and are p.C.M. Hence it remains to prove that the surfaces $X \subseteq \mathbb{P}^5$ in cases (i), . . . , (iii) are p.C.M.

A) The case 3.5(i).

We recall that the quadric surface $Q \subseteq \mathbb{P}^3$ can be defined as the image of the morphism $X_2 \to \mathbb{P}^3$, where $X_2$ is the blowing-up $\mathbb{P}^2$ at 2 points, determined by the complete linear system $|2E_0 - E_1 - E_2|$ (see 3.4 (i)a)).

Thus to blow-up $Q$ at 10 generic points is equivalent to blow-up $\mathbb{P}^2$ at 11 generic points.

So $H_Q = 2E_0 - E_1 - E_2$, while $H = \pi^*(3H_Q) - (E_0 - E_1 - E_2 - E_3 - \ldots - E_{11}) = (6E_0 - 3E_1 - 3E_2) - (E_0 - E_1 - E_2) - E_3 - \ldots - E_{11} = 5E_0 - 2E_1 - 2E_2 - E_3 - \ldots - E_{11}$ is a very ample divisor on $X_{11}$ which defines a Bordiga-White Surface in $\mathbb{P}^5$ which is p.C.M. (see Section 1).
B) The case 3.5(ii).

The cubic surface $S \subseteq \mathbb{P}^3$ is defined by the embedding of $X_6$ in $\mathbb{P}^3$ via the very ample linear system $|3E_0 - E_1 - \ldots - E_6|$.

Thus a smooth hyperplane section of the surface $X \subseteq \mathbb{P}^5$ is a divisor of type $H = \pi^*(2H_3) - E_7 - \ldots - E_{10} = 6E_0 - 2E_1 - \ldots - 2E_6 - E_7 - \ldots - E_{10} = 6E_0 - E$, where $E = 2E_1 + \ldots + 2E_6 + E_7 + \ldots + E_{10}$.

Hence we can denote by $X_{10}$ the blowing-up of the cubic surface $S \subseteq \mathbb{P}^3$ at 4 generic points.

Since $h^1(\mathcal{O}_S(1)) = h^1(\mathcal{O}_{X_{10}}(6E_0 - E)) = 0$ and the surface $X$ is projectively normal in $\mathbb{P}^5$ (see [1]), then $X$ is p.C.M., by Proposition 2.1.

C) The case 3.5(iii).

The embedding $X$ of $\mathbb{P}^{12}$, the blowing-up of $\mathbb{P}_e$ at 12 generic points, in $\mathbb{P}^5$ via the very ample linear system $|2C_0 + (5 + e)f - E_1 - \ldots - E_{12}|$, with $e \leq 4$, is not projectively normal, by [1; Theorem 5.4]. Thus, clearly, it is not p.C.M. too.

The following proposition summarizes what we have seen above.

**Proposition 3.2.** $X \subseteq \mathbb{P}^5$ is a p.C.M. surface of degree $d = 8$, then $X$ is either a $K3$ Surface of sectional genus $g = 5$ or a rational surface of sectional genus $g = 4$. In this case $X$ is one of the following:

1) A Bordiga-White Surface, defined by the embedding of $X_{11}$ in $\mathbb{P}^5$ via the linear system $|5E_0 - 2E_1 - 2E_2 - E_3 - \ldots - E_{11}|$;

2) The embedding of $X_{10}$, the blowing-up a cubic surface $S \subseteq \mathbb{P}^3$ at 4 generic points, in $\mathbb{P}^5$ via the linear system $|6E_0 - 2E_1 - \ldots - 2E_6 - E_7 - \ldots - E_{10}|$.


In order to complete the description of the rational p.C.M. surfaces in $\mathbb{P}^5$, it would remain to consider the rational surfaces of degree $d = 9, 10$. Here we consider the case $d = 9$.

Let $X$ be a rational p.C.M. surface in $\mathbb{P}^5$ of degree $d = 9$, then its sectional genus has to be $g(H) = g = 5$ (see the introduction).

On the other hand, if $X \subseteq \mathbb{P}^5$ ia a smooth rational surface of degree $d = 9$ and sectional genus $g = 5$, consider the exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \to \mathcal{O}_X(1) \to 0.$$
Since \( d = 9 > 2g - 2 = 8 \), we have \( h^1(\mathcal{O}_H(1)) = 0 \), hence \( h^1(\mathcal{O}_H(1)) = 0 \). This implies, by Proposition 2.1, that the surface \( X \subseteq \mathbb{P}^5 \) is p.C.M. if, and only if, it is projectively normal.

On the projective normality of smooth surfaces of degree 9 and sectional genus 5 in \( \mathbb{P}^5 \) we have the following result:

**Theorem 4.1.** (See [2; Theorem 1.1].) Let \( S \) be a smooth surface embedded by the complete linear system associated with a very ample line bundle \( L \) as a surface of degree \( d = 9 \) and sectional genus \( g = 5 \) in \( \mathbb{P}^5 \). Assume \((S, L)\) is not a scroll over a curve. Then \((S, L)\) fails to be projectively normal if and only if it is a rational conic bundle such as \((S, L) = (\mathbb{P}^5_{e}, 2C_0 + (6+e)f - E_1 - \ldots - E_5), 0 \leq e \leq 5\).

Note that, if \((S, L)\) is a scroll over a curve, in order to be p.C.M. it must be a rational scroll in \( \mathbb{P}^5 \) (see [16]). But there exist no values of \( b > e > 0 \) such that the very ample linear system \([C_0 + bf]\) determines an embedding of \( \mathbb{P}^5_{e} \) in \( \mathbb{P}^N = \mathbb{P}^5 \) of degree \( d = 9 \).

In fact \( d = -e + 2b = 9 \), while \( N = 2(b + 1) - e - 1 = 5 \), from which we get the equations \(-e + 2b = 9 \) and \(-e + 2b = 4 \), which give no solutions.

Thus any smooth rational surface \( X \subseteq \mathbb{P}^5 \) with \( d = 9 \) and \( g = 5 \), different from a rational conic bundle as in Theorem 4.1, is projectively normal and so p.C.M.. Such surfaces have been classified by E. L. Livorni in [15] and we list them as follows:

Let \( X \) be a smooth rational surface and \( L \) a very ample line bundle on \( X \) such that \( L^2 = 9, h^0(L) = 6, g(X, L) = g = 5 \). Then \( X \) is one of the following:

(i) \((X_{10}, 7E_0 - 2E_1 - \ldots - 2E_{10})\);
(ii) \((X_{12}, 6E_0 - 2E_1 - \ldots - 2E_5 - E_6 - \ldots - E_{12})\);
(iii) \((\mathbb{P}^5_{e}, 2C_0 + (6+e)f - E_1 \ldots - E_{15}), 0 \leq e \leq 5\);
(iv) \((\mathbb{P}^5_{e}, 4C_0 + (2e + 5)f - 2E_1 \ldots - 2E_7 - E_8 - \ldots - E_{10}), 0 \leq e \leq 2\);
(v) \((\mathbb{P}^5_{e}, 3C_0 + 5f - E_1 \ldots - E_{12})\).

It has been shown that there exist no surfaces as in (v) (e.g. see [2]), while Theorem 4.1 gives us that only case (iii) is not p.C.M.. So we can conclude that the only rational p.C.M. surfaces in \( \mathbb{P}^5 \) of degree \( d = 9 \) are the ones in Table 1, if they exist. In order to check that they actually do, see [15] for cases (i), (iv) and [3] for case (ii).

Suppose that \( X \) is a nonrational p.C.M. surface in \( \mathbb{P}^5 \) of degree \( d = 9 \), then \( g = 5 + h^1(\mathcal{O}_H(1)) \leq 7 \).

All the smooth surfaces of sectional genus \( g \leq 7 \) whose minimal model is a surface with nonnegative Kodaira dimension have been classified in [15], from where we have the following:

**Fact.** Let \( X \subseteq \mathbb{P}^5 \) be a nonrational smooth surface of degree \( d = 9 \), sectional genus \( 5 \leq g \leq 7 \), arithmetic genus \( p_a \) and geometric genus \( p_g \). Then we have the following cases:

(i) \( g = 6, q = h^1(\mathcal{O}_X) = 0, p_a = p_g = 1 \), \( X \) is the blowing-up at one point of a K3 Surface;

(ii) \( g = 7, q = h^1(\mathcal{O}_X) = 0, p_a = p_g = 2 \), \( X \) is an Elliptic Surface.

By [2, Theorem 1.1], the surfaces \( X \) listed above are both projectively normal.

**Proposition 5.1.** Let \( d, g \in \mathbb{Z} \) be such that \( d > g - 1 \). Let \( X \subseteq \mathbb{P}^N \) be a smooth surface of degree \( d = N + g - 1 - h^1(\mathcal{O}_H(1)) \), sectional genus \( g(H) = g \) and irregularity \( q = h^1(\mathcal{O}_X) = 0 \). If \( h^1(\mathcal{O}_X(1)) = 0 \), then \( X \) is p.C.M. if, and only if it is projectively normal.

**Proof.** Let \( X \subseteq \mathbb{P}^N \) be projectively normal. We want to prove that, when \( h^1(\mathcal{O}_X(1)) = 0 \), the surface \( X \) is P.C.M., i.e. that \( h^2(\mathcal{I}_X(m)) = h^1(\mathcal{O}_X(m)) = 0 \), for all \( m \geq 2 \).

Applying the Riemann-Roch Theorem on the smooth hyperplane section \( H \) of \( X \), since \( d > g - 1 \), by hypothesis, we get

\[
N - h^1(\mathcal{O}_H(1)) = h^0(\mathcal{O}_H(1)) - h^1(\mathcal{O}_H(1)) = d - g + 1 > 0,
\]

hence \( N > h^1(\mathcal{O}_H(1)) \). So we have:

\[
md = m(N + g - 1 - h^1(\mathcal{O}_H(1))) = m(N - h^1(\mathcal{O}_H(1))) + m(g - 1) > 2(g - 1),
\]

\( \forall m \geq 2 \), from which we deduce that \( h^1(\mathcal{O}_H(m)) = 0, \forall m \geq 2 \).

Consider the exact sequence

\[
0 \to \mathcal{O}_X(m-1) \to \mathcal{O}_X(m) \to \mathcal{O}_H(m) \to 0.
\]

Since \( h^1(\mathcal{O}_X(1)) = 0 \) and \( h^1(\mathcal{O}_H(m)) = 0, \forall m \geq 2 \), we have that \( h^1(\mathcal{O}_X(m-1)) = 0 \) implies \( h^1(\mathcal{O}_X(m)) = 0, \forall m \geq 2 \), and this is what we required. \( \square \)

Now, let us consider the two projectively normal surfaces \( X \subseteq \mathbb{P}^5 \) quoted in the Fact above. Since \( d = 9 > g - 1 \), by Proposition 5.1, the surfaces \( X \) will also be p.C.M. in \( \mathbb{P}^5 \) if \( h^1(\mathcal{O}_X(H)) = h^1(\mathcal{O}_X(1)) = 0 \).

This is what we show in the following proposition.
Proposition 5.2. A nonrational smooth surface \(X \subseteq \mathbb{P}^5\) of degree 9 is p.C.M.

Proof. Let \(X \subseteq \mathbb{P}^5\) be as above.

Consider the exact sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(1) \rightarrow \mathcal{O}_H(1) \rightarrow 0,
\]

where \(H\) is a smooth hyperplane section of \(X\) of genus \(g\).

Since \(h^1(\mathcal{O}_X) = 0\), \(h^1(\mathcal{O}_H(1)) = h^0(\mathcal{O}_H(1)) = d - 1 + g = g - 5\) and, in our two cases, \(g - 5 = p_g = h^2(\mathcal{O}_X)\), we have \(h^1(\mathcal{O}_X(1)) = h^2(\mathcal{O}_X(1))\).

By the Serre Duality Theorem, \(h^2(\mathcal{O}_X(1)) = h^2(\mathcal{O}_X(H)) = 0\) if, and only if, \(h^0(\mathcal{O}_X(K_X - H)) = 0\).

Consider \((K_X - H).H = K_X.H - H^2 = 2g - 2 - 2d = 2g - 20\), which is \(< 0\) when \(g = 6\) or \(7\), hence \(h^0(\mathcal{O}_X(K_X - H)) = 0\).

So \(h^1(\mathcal{O}_X(1)) = h^2(\mathcal{O}_X(1)) = 0\) and, by what we have seen above, this is enough to conclude that the surfaces \(X \subseteq \mathbb{P}^5\) are p.C.M. \(\square\)


In [16] we showed that the maximum degree of a rational p.C.M. surface \(X \subseteq \mathbb{P}^N\) is \(d = \binom{N}{3}\).

There are known rational p.C.M. surfaces which attain the maximum degree, namely the White Surfaces (see Section 1).

In \(\mathbb{P}^5\) the candidate rational surfaces as p.C.M. surfaces of maximum degree \(d = 10\) are described in the following table (see [15] for a classification of rational surfaces of degree 10).

The existence of the surface is known in case (vi) (White Surface) and in case (v), see [15], while in cases (i), (iii), (vii) and (viii) we can consider the following theorem.

Theorem 6.1. (See [3], Theorem 2.1]). Let \(P_1, \ldots, P_r, R_1, \ldots, R_n\) be general points on \(\mathbb{P}^2\), with \(r \geq 1\). Define \(X_{r,n}\) as the blowing-up of \(\mathbb{P}^2\) along these points, \(\pi_{r,n}\) the corresponding projection map, and \(E_1, \ldots, E_r, F_1, \ldots, F_n\) the exceptional divisor corresponding resp. to the points \(P_1, \ldots, P_r, R_1, \ldots, R_n\). Let \(l_1, \ldots, l_r\) be integers, with \(l_1 \geq \ldots \geq l_r \geq 2\). Suppose \(m, r\) and \(l_i\) are such that there exists a “good” curve of degree \(m - 1\); and either \(l_1 \leq 3\) and \(4m \geq l_1 + l_2 + \ldots + l_r + 9\) or \(l_1 > 3\) and \(4m \geq 2l_1 + l_2 + \ldots + l_r + 10\). Then the sheaf \(\mathcal{L} = \pi_{r,n}^*(\mathcal{O}_{\mathbb{P}^2}(m)) \otimes (-l_1E_1 - \ldots - l_rE_r - F_1 - \ldots - F_n)\) is very ample on \(X_{r,n}\) for all \(n \leq \frac{m(m+3) - l_1(l_1+1) - \ldots - l_r(l_r+1)}{2} - 5\).
TABLE 2. Rational p.C.M. surfaces in $\mathbb{P}^5$ of degree 10

<table>
<thead>
<tr>
<th>$(X_s, D_i) - (\mathbb{P}_r^3, D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(i)$ $(X_{14}, 6E_0 - 2E_1 - \ldots - 2E_4 - E_5 - \ldots - E_{14})$</td>
</tr>
<tr>
<td>$(ii)$ $(X_{12}, 9E_0 - 3E_1 - \ldots - 3E_7 - 2E_8 - E_9 - \ldots - E_{12})$</td>
</tr>
<tr>
<td>$(iii)$ $(X_{12}, 7E_0 - 2E_1 - \ldots - 2E_9 - E_{10} - \ldots - E_{12})$</td>
</tr>
<tr>
<td>$(iv)$ $(X_{11}, 9E_0 - 3E_1 - \ldots - 3E_6 - 2E_7 - \ldots - 2E_{10} - E_{11})$</td>
</tr>
<tr>
<td>$(v)$ $(X_{10}, 10E_0 - 3E_1 - \ldots - 3E_9)$</td>
</tr>
<tr>
<td>$(vi)$ $(X_{15}, 5E_0 - E_1 - \ldots - E_{15})$</td>
</tr>
<tr>
<td>$(vii)$ $(X_{15}, 6E_0 - 3E_1 - 2E_2 - E_3 - \ldots - E_{15})(\text{equiv. } \mathbb{P}<em>0^{14}, 3C_0 + 4f - E_1 - \ldots - E</em>{14})$</td>
</tr>
<tr>
<td>$(viii)$ $(X_{11}, 8E_0 - 3E_1 - 3E_2 - 2E_3 - \ldots - 2E_{11})(\text{equiv. } \mathbb{P}<em>0^{10}, 5C_0 + 5f - 2E_1 - \ldots - 2E</em>{10})$</td>
</tr>
<tr>
<td>$(ix)$ $(\mathbb{P}<em>e^{14}, 3C_0 + 7f - E_1 - \ldots - E</em>{14})$</td>
</tr>
<tr>
<td>$(x)$ $(\mathbb{P}<em>e^{12}, 4C_0 + (2e + 5)f - 2E_1 - \ldots - 2E_6 - E_7 - \ldots - E</em>{12}; 0 \leq e \leq 2)$</td>
</tr>
<tr>
<td>$(xi)$ $(\mathbb{P}<em>e^{11}, 4C_0 + (2e + 6)f - 2E_1 - \ldots - 2E_9 - E</em>{10} - E_{11}; 0 \leq e \leq 2)$</td>
</tr>
</tbody>
</table>

By Theorem 6.1, in order to have that $D_i$ is very ample on $X_s$, it is enough to show that there exists a “good” plane curve of degree $(t - 1)$, i.e. a curve having, as its only singularities, $r$ multiple points at the $P_i^j$s, $i = 1, \ldots, r$, of multiplicity $= I_i$, respectively, and such that its strict transform on $X_s$ is smooth.

Since there are plane curves of degree 5 and 6 with, respectively, 4 and 9 nodes (e.g. see [9; Proposition 1.1]) and there are plane curves of degree 5 (resp. 7) with 1 triple point and 1 node (resp. 2 triple points and 9 nodes) (e.g. see Section 2 in [17]), we deduce that the surfaces in cases (i), (iii), (vii) and (viii) really exist, as required.

The problem of the existence of the surfaces in cases (ii), (iv), (ix), (x), (xi) remains open.

Except for the White Surface (case (vi)), it is still unknown if the surfaces in Table 2 are really p.C.M.

We recall that (see [16; Proposition 3.5]) a sufficient condition to have that the surfaces in Table 2 are p.C.M. is that their ideal contains no quadric.

REFERENCES


degree nine, Geometria Dedicata, 74 (1999), pp. 1–21.


Dipartimento di Matematica,
Università di Bologna,
p.za S. Donato, 5
40127 Bologna (ITALY)
e-mail: mancinnm@libero.it