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2-LOCAL) DERIVATIONS AND AUTOMORPHISMS AND BIDERIVATIONS OF COMPLEX ω-LIE ALGEBRAS A TITLE ON MULTIPLE ROWS

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The present paper is devoted to the description of local (2-local) derivations and automorphisms and biderivations on some complex ω-Lie algebras. Given a three-dimensional complex ω-Lie algebras $\mathfrak{g}$, we prove that every local $(2-$local) derivation and automorphisms on $\mathfrak{g}$ are derivations and automorphisms respectively if $\mathfrak{g}$ is not C_1. We give the matrix form of local derivations and automorphisms in the cases C_1. Also, we give a descriptions of biderivations of three dimensional complex ω-Lie algebras.

1. Introduction

The history of local mappings begins with the Gleason-Kahane-Zelazko theorem in [12] and [14], which is a fundamental contribution to the theory of Banach algebras. This theorem asserts that every unital linear functional F on a complex unital Banach algebra A, such that $F(a)$ belongs to the spectrum $\sigma(a)$ of a for every $a \in A$, is multiplicative.

Subsequently, in [13], the concept of local derivation is introduced, and it is proved that each continuous local derivation from a von Neumann algebra into its dual Banach bimodule is a derivation. Numerous new results regarding the

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description of local derivations and local automorphisms of algebras have been obtained; see, for example, [1, 2, 13, 18, 19]).

ω-Lie algebras are a natural generalization of Lie algebras. They were introduced by Nurowski in [16], motivated by the study of isoparametric hypersurfaces in Riemannian geometry. There have been extensive works on ω-Lie algebras (see [7–10, 17, 23] and references therein). In the cases of dimensions 1 and 2, there are no nontrivial ω-Lie algebras. The first example of a nontrivial 3-dimensional ω-Lie algebra was given by Nurowski [16], where the author provided a classification of 3-dimensional ω-Lie algebras over the field of real numbers under the action of the 3-dimensional orthogonal group.

On the other hand, the notion of biderivations has appeared in different areas. Maksa used biderivations to study real Hilbert space [15]. Vukman investigated symmetric biderivations in prime and semiprime rings [21]. The well-known result that every biderivation on a noncommutative prime ring A is of the form $\lambda[x,y]$ for some λ belonging to the extended centroid of A was discovered independently by Bresar et al [4], Skosyrskii [20], and Farkas and Letzter [11]. Biderivations were connected with noncommutative Jordan algebras by Skosyrskii and with Poisson algebras by Farkas and Letzter. Besides their wide applications, biderivations are interesting in their own right and have been introduced to Lie algebras [22], which have been studied by many authors recently. In particular, biderivations are closely related to the theory of commuting linear maps, which has a long and rich history. For the development of commuting maps and their applications, we refer to the survey [6]. It is worth mentioning that Bresar and Zhao considered a general but simple approach for describing biderivations and commuting linear maps on a Lie algebra L having their ranges in an L-module [5]. This approach covered most of the results in [3–6, 15, 18, 21] and inspires us to generalize their method to Hom-Lie algebras.

This paper is organized as follow. In Section 2, we recall some definitions and results needed for this study. In Section 3, we investigate local and 2local derivations on three-dimensional ω-Lie algebras. Section 4 is devoted to local and 2-local automorphisms on three-dimensional ω-Lie algebras. In the last section, we provide a description of biderivations on three-dimensional ω-Lie algebras.

2. Preliminaries

Definition 2.1. A vector space over $\mathbb{C}$ is called an ω-Lie algebra if there is a bilinear map $[,] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ and a skew-symmetric bilinear form $\omega : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ such that

1. $[x,y] = -[y,x]$,

2. $[[x,y],z]+[[y,z],x]+[[z,x],y]=\omega(x,y)z+\omega(y,z)x+\omega(z,x)y$,

for all $x,y,z\in\mathfrak{g}$.

Clearly, an ω-Lie algebra $\mathfrak{g}$ with $\omega=0$ is a Lie algebra, which is called a trivial ω-Lie algebra. Otherwise, $\mathfrak{g}$ is called a nontrivial ω-Lie algebra

Definition 2.2. Let $(\mathfrak{g},[,])$ be an ω-Lie algebra. A linear map $D:\mathfrak{g}\to\mathfrak{g}$ is called derivation, if

$$D([x,y])=[D(x),y]+[x,D(y)]$$

for all $x,y\in\mathfrak{g}$.

We write $gl(\mathfrak{g})$ for the general linear Lie algebra on $\mathfrak{g}$. Then the set $Der(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ forms a Lie subalgebra of $gl(\mathfrak{g})$, which is called the derivation algebra of $\mathfrak{g}$.

Definition 2.3. Let $(\mathfrak{g},[,])$ be an ω-Lie algebra. A derivation $D:\mathfrak{g}\to\mathfrak{g}$ of $\mathfrak{g}$ is called ω-derivation, if

$$\omega(D(x).y)+\omega(x,D(y))=0$$

for all $x,y\in\mathcal{A}$.

We write $Der_\omega(\mathfrak{g})$ for the set consisting of all ω-derivations of $\mathfrak{g}$. Clearly,$Der_\omega(\mathfrak{g})\subseteq Der(\mathfrak{g})$.

Let us initiate the categorization of 3-dimensional nontrivial ω-Lie algebras over $\mathbb{C}$.

Theorem 2.4. *(Chen-Liu-Zhang [7]). Let $\mathfrak{g}$ be a 3-dimensional nontrivial ω-Lie algebra over $\mathbb{C}$, then it must be isomorphic to one of the following algebras:*

(1) L_1: $[e_1,e_3]=0$, $[e_2,e_3]=e_3$, $[e_1,e_2]=e_2$ and $\omega(e_2,e_3)=\omega(e_1,e_3)=0$, $\omega(e_1,e_2)=1$.

(2) L_2: $[e_1,e_2]=0$, $[e_1,e_3]=e_2$, $[e_2,e_3]=e_3$ and $\omega(e_1,e_2)=0$, $\omega(e_1,e_3)=1$, $\omega(e_2,e_3)=0$.

(3) A_α: $[e_1,e_2]=e_1$, $[e_1,e_3]=e_1+e_2$, $[e_2,e_3]=e_3+\alpha e_1$ and $\omega(e_1,e_2)=\omega(e_1,e_3)=0$, $\omega(e_2,e_3)=-1$, where $\alpha\in\mathbb{C}$.

(4) B: $[e_1,e_2]=e_2$, $[e_1,e_3]=e_2+e_3$, $[e_2,e_3]=e_1$ and $\omega(e_1,e_2)=\omega(e_1,e_3)=0$, $\omega(e_2,e_3)=2$.

(5) C_α: $[e_1,e_2]=e_2$, $[e_1,e_3]=\alpha e_3$, $[e_2,e_3]=e_1$ and $\omega(e_1,e_2)=\omega(e_1,e_3)=0$, $\omega(e_2,e_3)=1+\alpha$, where $0,-1,\alpha\in\mathbb{C}$.

The exploration of local derivation of ω-Lie algebras $\mathfrak{g}$ relies heavily on a key theorem established by Chen Y. and al., as documented in their work [8]. This theorem serves as our principal instrument for delving into the intricacies of the derivations in question.

Theorem 2.5. *[8]*

1. $Der(L_1) = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & -a \\ 0 & 0 & b \end{pmatrix} \quad a,b \in \mathbb{C} \right\}.$

2. $Der(L_2) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} \quad a \in \mathbb{C} \right\}.$

3. $Der(A_\alpha) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & \\ \frac{a}{2} & a & 0 \end{pmatrix} \quad a \in \mathbb{C} \right\}.$

4. $Der(B) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix} \quad a \in \mathbb{C} \right\}.$

5. $Der(C_\alpha\,(\alpha \neq 1,0,-1)) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \quad a \in \mathbb{C} \right\}.$

6. $Der(C_1) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & c \\ 0 & b & -a \end{pmatrix} \quad a,b,c \in \mathbb{C} \right\}.$

Definition 2.6. Let $(\mathfrak{g},[,])$ be an ω-Lie algebra. A linear isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ is called an automorphism of $\mathfrak{g}$, if

$$\phi([x,y]) = [\phi(x),\phi(y)]$$

for all $x,y \in \mathfrak{g}$.

We write $Aut(\mathfrak{g})$ for set of all automorphisms of $\mathfrak{g}$.

The examination of local automorphisms of ω-Lie algebras $\mathfrak{g}$ heavily depends on a pivotal theorem established by Chen Y. et al., as detailed in their publication [8]. This theorem serves as our primary tool for delving into the complexities of the relevant automorphisms.

Theorem 2.7. *[8]*

1. $Aut(L_1) = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & -a \\ 0 & 0 & b \end{pmatrix} \quad 0 \neq b, a \in \mathbb{C} \right\}.$

2. $Aut(L_2) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} \quad 0 \neq a \in \mathbb{C} \right\}.$

3. $Aut(A_\alpha) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ \frac{a^2+a}{2} & a & 1 \end{pmatrix} \quad a \in \mathbb{C} \right\}.$

4. $Aut(B) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & b & a \end{pmatrix} \quad a^2 = 1, b \in \mathbb{C} \right\}.$

5. $Aut(C_\alpha\,(\alpha \neq 1,0,-1)) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} \quad 0 \neq a \in \mathbb{C} \right\}.$

6. $Aut(C_1) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & c \\ 0 & d & b \end{pmatrix} \quad ab - cd = 1, a,b,c,d \in \mathbb{C} \right\}.$

3. Local and 2-local derivations of 3−dimensional ω-Lie algebras

3.1. Local derivations

In this section, we study local derivation of 3-dimensional ω-Lie algebras. Detailed proofs are provided for the cases of L_1 and L_2, with the proofs for the remaining cases omitted due to the similarity of arguments.

Definition 3.1. Let $(\mathfrak{g}, [,])$ be an ω-Lie algebra. A linear map $\Delta : \mathfrak{g} \to \mathfrak{g}$ is called local derivation, if for any $x \in \mathfrak{g}$ there exist a derivation $D_x \in Der(\mathfrak{g})$ such that $\Delta(x) = D_x(x)$.

Let $LDer(\mathfrak{g})$ represent the set of all local derivations defined on $\mathfrak{g}$. It is easy to see that $Der(\mathfrak{g}) \subseteq LDer(\mathfrak{g})$. Initially, our focus is on the investigation of derivations within L_1, where L_1 is equipped with a basis $\{e_1, e_2, e_3\}$ and is defined according to Theorem 1.

Proposition 3.2. *Every local derivation of the algebra L_1 is a derivation.*

Proof. Let Δ be an arbitrary local derivation of L_1. By definition for all $x \in L_1$ there exists a derivation D_x on L_1 such that $\Delta(x) = D_x(x)$ By Theorem 2.5, the derivation D_x has the following matrix form:

$$A_x = \begin{pmatrix} 0 & 0 & a^x \\ 0 & 0 & -a^x \\ 0 & 0 & b^x \end{pmatrix}$$

Let A be the matrix of Δ then by choosing subsequently $x = e_1, e_2, e_3$, and using $\Delta(x) = D_x(x)$, i.e. $AX = A_x X$, where X is the vector corresponding to x, it is easy to see that

$$A = \begin{pmatrix} 0 & 0 & a^{e_3} \\ 0 & 0 & -a^{e_3} \\ 0 & 0 & b^{e_3} \end{pmatrix}$$

Hence, by Theorem 2.5, Δ is a derivation. This ends the proof. □

Proposition 3.3. *Every local derivation of the algebra L_2 is a derivation.*

Proof. Let Δ be an arbitrary local derivation of L_2. By definition for all $x \in L_2$ there exists a derivation D_x on L_1 such that $\Delta(x) = D_x(x)$ By Theorem 2.5, the derivation D_x has the following matrix form:

$$A_x = \begin{pmatrix} a^x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a^x \end{pmatrix}$$

Let A be the matrix of Δ then by choosing subsequently $x = e_1, e_2, e_3$, and using $\Delta(x) = D_x(x)$, i.e. $AX = A_x X$, where X is the vector corresponding to x, it is easy to see that

$$A = \begin{pmatrix} a^{e_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a^{e_3} \end{pmatrix}$$

Since, Δ is linear we have

$$a^{e_1+e_3} e_1 - a^{e_1+e_3} e_3 = \Delta(e_1 + e_3) = \Delta(e_1) + \Delta(e_3) = a^{e_1} e_1 - a^{e_3} e_3$$

Comparing coefficients at the basis element we obtain $a^{e_1} = a^{e_3}$.
Hence, by Theorem 2.5, Δ is a derivation on L_2. □

Analogous reasoning can be extended to the remaining 3-dimensional ω-Lie algebras. Consequently, we consolidate the outcome in the following theorem.

Theorem 3.4.
1. *$LDer(L_1) = Der(L_1)$.*
2. *$LDer(L_2) = Der(L_2)$.*
3. *$LDer(A_\alpha) = Der(A_\alpha)$, $\alpha \in \mathbb{C}$.*
4. *$LDer(B) = Der(B)$.*
5. *$LDer(C_\alpha) = Der(C_\alpha)$, $\alpha \in \mathbb{C} \setminus \{1, 0, -1\}$.*

6. $Der(C_1) \subseteq LDer(C_1) \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{C} \right\}$

We summarize a characterization on local derivations of 3-dimensional ω-Lie algebras as following Table

$\mathfrak{g}$	$LDer(\mathfrak{g})$	$LDer(\mathfrak{g}) = Der(\mathfrak{g})$
L_1	$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & -a \\ 0 & 0 & b \end{pmatrix} \quad a,b \in \mathbb{C}$	True
L_2	$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} \quad a \in \mathbb{C}$	True
$A_\alpha \quad \alpha \in \mathbb{C}$	$\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ \frac{a}{2} & a & 0 \end{pmatrix} \quad a \in \mathbb{C}$	True
B	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix} \quad a \in \mathbb{C}$	True
$C_\alpha \ \ (\alpha \in \mathbb{C} - \{1,0,-1\})$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad a \in \mathbb{C}$	True
C_1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{pmatrix} \quad a,b,c,d \in \mathbb{C}$	False

Table 1: Local derivations of $3-$dimensional ω-Lie algebras $\mathfrak{g}$.

3.2. 2-Local derivations

In this section, we establish the proof that any 2-local derivation of 3-dimensional ω-Lie algebras is indeed a derivation. Detailed proofs are provided for the cases of L_1 and L_2, with the proofs for the remaining cases omitted due to the similarity of arguments.

Definition 3.5. Let $(\mathfrak{g}, [,])$ be an ω-Lie algebra. A (not necessary linear) map $\Delta : \mathfrak{g} \to \mathfrak{g}$ is called 2-local derivation, if for any $x, y \in \mathfrak{g}$ there exist a derivation $D_{x,y} \in Der(\mathfrak{g})$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

Theorem 3.6. *Every 2-local derivation of the ω-Lie algebras L_1, L_2, A_α, B and C_α ($\alpha \neq 1, 0, -1$) is a derivation.*

Proof. We shall establish the theorem for L_1; the remaining cases can be demonstrated analogously.
Let Δ be n arbitrary 2-local derivation of L_1. Then by definition, for every element $x) \in L_1$ there exist element $a^{x,e_3}, b^{x,e_3} \in \mathbb{C}$ such that

$$A_{x,e_3} = \begin{pmatrix} 0 & 0 & a^{x,e_3} \\ 0 & 0 & b^{x,e_3} \\ 0 & 0 & -a^{x,e_3} \end{pmatrix}$$

$\Delta(x) = A_{x,e_3}\bar{x}$, where $\bar{x} = (x_1, x_2, x_3)^t$ is the vector corresponding to x, and $\Delta(e_3) = A_{x,e_3}\bar{e_1} = (a^{x,e_3}, b^{x,e_3}, -a^{x,e_3})^t$.
Since $\Delta(e_3) = D_{x,e_3}(e_3) = D_{y,e_3}(e_3)$, we have

$$\Delta(e_3) = (a^{x,e_3}, b^{x,e_3}, -a^{x,e_3})^t = (a^{y,e_3}, b^{y,e_3}, -a^{y,e_3})^t$$

for each pair x, y elements of L_1. Hence, $a^{x,e_3} = a^{y,e_3}$ and $b^{x,e_3} = b^{y,e_3}$. Therefore $\Delta(x) = A_{y,e_3}\bar{x}$ for any $x \in L_1$ and the matrix of Δ does not depend on x. Thus, by Theorem 2.5, Δ is a derivation. $\square$

4. Local and 2-local automorphisms of 3−dimensional ω-Lie algebras

4.1. Local automorphisms

In this section, we establish the proof that every local automorphism of 3-dimensional ω-Lie algebras is indeed an automorphism. We provide detailed proofs specifically for the cases of L_1 and L_2, opting to omit the proofs for the remaining cases due to the similarity of arguments.

Definition 4.1. Let $(\mathfrak{g}, [,])$ be an ω-Lie algebra. A linear isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ is called an local automorphism of $\mathfrak{g}$, if for any $x \in \mathfrak{g}$ there exists an automorphism ϕ_x such that $\phi(x) = \phi_x(x)$. for all $x \in \mathfrak{g}$.

We write $LAut(\mathfrak{g})$ for set of all automorphisms of $\mathfrak{g}$.

Proposition 4.2. *Every local automorphism of the ω-Lie algebra L_1 is an automorphism.*

Proof. Let ϕ be an arbitrary automorphism of L_1. By definition for all $x \in L_1$ there exists an automorphism ϕ_x on L_1 such that $\phi(x) = \phi_x(x)$.
Using theorem 2.7 and applying the similar arguments used above we can assume the local automorphism ϕ on L_1 has the following form

$$\begin{pmatrix} 1 & 0 & a^{e_3} \\ 0 & 1 & -a^{e_3} \\ 0 & 0 & b^{e_3} \end{pmatrix}$$

which implies that ϕ is an automorphism from theorem 2.7. $\square$

Proposition 4.3. *Every local automorphism of the ω-Lie algebra L_2 is an automorphism.*

Proof. Let ϕ be an arbitrary automorphism of L_2. By definition for all $x \in L_2$ there exists an automorphism ϕ_x on L_2 such that $\phi(x) = \phi_x(x)$.
Using theorem 2.7 and applying the similar arguments used above we can assume the local automorphism ϕ on L_2 has the following form

$$A = \begin{pmatrix} a^{e_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a^{e_3}} \end{pmatrix}$$

Using the fact ϕ is linear we get $a^{e_1} = a^{e_3}$, which implies that ϕ is an automorphism on L_2. $\square$

By applying a similar line of reasoning, we can generalize our findings to the remaining 3-dimensional ω-Lie algebras. As a result, we encapsulate the results in the following theorem.

Theorem 4.4.
1. *$LAut(L_1) = Der(L_1)$.*
2. *$LAut(L_2) = Der(L_2)$.*
3. *$LAut(A_\alpha) = Der(A_\alpha)$, $\alpha \in \mathbb{C}$.*
4. *$LAut(B) = Der(B)$.*
5. *$LAut(C_\alpha) = Der(C_\alpha)$, $\alpha \in \mathbb{C} - \{1, 0, -1\}$.*
6. *$Aut(C_1) \subseteq LAut(C_1) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & c \\ 0 & d & b \end{pmatrix}, \quad a, b, c, d \in \mathbb{C} \right\}$*

We may summarize a characterization of local automorphisms for 3-dimensional ω-Lie algebras in the following table:

$\mathfrak{g}$	$LDer(\mathfrak{g})$	$LDer(\mathfrak{g}) = Der(\mathfrak{g})$
L_1	$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & -a \\ 0 & 0 & b \end{pmatrix} \quad 0 \neq ba \in \mathbb{C}$	True
L_2	$\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} \quad 0 \neq a \in \mathbb{C}$	True
$A_\alpha \quad \alpha \in \mathbb{C}$	$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ \frac{a^2+a}{2} & a & 1 \end{pmatrix} \quad a \in \mathbb{C}$	True
B	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & b & a \end{pmatrix} \quad a^2 = 1, b \in \mathbb{C}$	True
$C_\alpha \quad (\alpha \in \mathbb{C} - \{1, 0, -1\})$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} \quad 0 \neq a \in \mathbb{C}$	True
C_1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & c \\ 0 & d & b \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}$	False

Table 2: Local automorphisms of 3−dimensional ω-Lie algebras $\mathfrak{g}$.

4.2. 2-Local automorphisms

In this section, we establish the proof that any 2-local automorphisms of 3-dimensional ω-Lie algebras is indeed an automorphism. Detailed proofs are provided for the cases of L_1 and L_2, with the proofs for the remaining cases omitted due to the similarity of arguments.

Definition 4.5. Let $(\mathfrak{g}, [,])$ be an ω-Lie algebra. A (not necessary linear) map $\phi : \mathfrak{g} \to \mathfrak{g}$ is called 2-local derivation, if for any $x, y \in \mathfrak{g}$ there exist a derivation $\phi_{x,y} \in Aut(\mathfrak{g})$ such that $\phi(x) = \phi_{x,y}(x)$ and $\phi(y) = \phi_{x,y}(y)$.

Theorem 4.6. *Every 2-local automorphism of the ω-Lie algebras L_1, L_2, A_α, B and C_α ($\alpha \neq 1, 0, -1$) is an automorphism.*

Proof. We shall establish the theorem for L_1; the remaining cases can be demonstrated analogously.
Let Δ be n arbitrary 2-local automorphism of L_1. Then by definition, for every element $x) \in L_1$ there exist element $a^{x,e_3}, b^{x,e_3} \in \mathbb{C}$ such that

$$A_{x,e_3} = \begin{pmatrix} 1 & 0 & a^{x,e_3} \\ 0 & 1 & b^{x,e_3} \\ 0 & 0 & -a^{x,e_3} \end{pmatrix}$$

$\phi(x) = A_{x,e_3}\bar{x}$, where $\bar{x} = (x_1, x_2, x_3)^t$ is the vector corresponding to x, and $\phi(e_3) = A_{x,e_3}\bar{e_1} = (a^{x,e_3}, b^{x,e_3}, -a^{x,e_3})^t$.
Since $\phi(e_3) = \phi_{x,e_3}(e_3) = \phi_{y,e_3}(e_3)$, we have

$$\phi(e_3) = (a^{x,e_3}, b^{x,e_3}, -a^{x,e_3})^t = (a^{y,e_3}, b^{y,e_3}, -a^{y,e_3})^t$$

for each pair x, y elements of L_1. Hence, $a^{x,e_3} = a^{y,e_3}$ and $b^{x,e_3} = b^{y,e_3}$. Therefore $\phi(x) = A_{y,e_3}\bar{x}$ for any $x \in L_1$ and the matrix of ϕ does not depend on x. Thus, by Theorem 2.7, ϕ is an automorphism. □

5. Biderivations of ω-Lie algebras

In this section, we provide characterizations of biderivations for 3-dimensional ω-Lie algebras. We furnish detailed proofs for the cases of L_1 and L_2, while omitting the proofs for the remaining cases due to the similarity of arguments.

We commence by revisiting the definition of a biderivation in the context of an arbitrary Lie algebra.

Definition 5.1. Let $(\mathfrak{g}, [,])$ be an arbitrary algebra. A bilinear map $\delta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called a biderivation on $\mathfrak{g}$ if

$$\delta([x,y],z) = [x, \delta(y,z)] + [\delta(x,z), y],$$

$$\delta(x,[y,z]) = [y, \delta(x,z)] + [\delta(x,y), z],$$

for all $x, y, z \in \mathfrak{g}$. Denote by $BDer(\mathfrak{g})$ the set of all biderivations on $\mathfrak{g}$ which is clearly a vector space.

Let $BDer(\mathfrak{g})$ represent the collection of all biderivations defined on $\mathfrak{g}$. It is evident that this set forms a vector space.

A biderivation $\delta \in BDer(\mathfrak{g})$ is termed symmetric if $\delta(x,y) = \delta(y,x)$ holds for all $x, y \in \mathfrak{g}$; conversely, it is designated as skew-symmetric if $\delta(x,y) = -\delta(y,x)$ for all $x, y \in \mathfrak{g}$. We denote the subspaces of all symmetric biderivations and all skew-symmetric biderivations on $\mathfrak{g}$ as $BDer_+(\mathfrak{g})$ and $BDer_-(\mathfrak{g})$, respectively.

For any $\delta \in BDer(\mathfrak{g})$, we define two bilinear maps by

$$\delta^+(x,y) = \frac{1}{2}(\delta(x,y) + \delta(y,x)), \quad \delta^-(x,y) = \frac{1}{2}(\delta(x,y) - \delta(y,x))$$

It is easy to see $\delta^+ \in BDer_+(\mathfrak{g})$ and $\delta^- \in BDer_-(\mathfrak{g})$. Since $\delta = \delta^+ + \delta^-$ it follows that

$$BDer(\mathfrak{g}) = BDer_+(\mathfrak{g}) \oplus BDer_-(\mathfrak{g})$$

To characterize $BDer(\mathfrak{g})$, we only need to characterize $BDer_+(\mathfrak{g})$ and $BDer_-(\mathfrak{g})$.

Now, let δ be a biderivation on ω-Lie algebra $\mathfrak{g}$ and $x, y \in \mathfrak{g}$, such that $x = \sum_{i=1}^{3} x_i e_i$ and $y = \sum_{i=1}^{3} y_i e_i$. Then, by the bilinearity of δ, we obtain,

$$\delta(x,y) = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j \delta(e_i, e_j) = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j \delta_{e_i}(e_j). \tag{1}$$

Theorem 5.2. *Let $(\mathfrak{g}, [,])$ be a three dimensional ω-Lie algebra over $\mathbb{C}$. Then, δ is a skew-symmetric biderivation of $\mathfrak{g}$ if and only if $\delta(x,y) = 0$, for all $x, y \in \mathfrak{g}$.*

Proof. We shall establish the theorem for L_1; the remaining cases can be demonstrated analogously. Let δ be an arbitrary skew-symmetric biderivation on L_1. By Theorem 2.5, the matrix D_{e_i} of δ_{e_i}, for $i = 1,2,3$ is of the form

$$D_{e_i} = \begin{pmatrix} 0 & 0 & a^{e_i} \\ 0 & 0 & -a^{e_i} \\ 0 & 0 & b^{e_i} \end{pmatrix}$$

Since δ is skew-symmetric, then, $\delta(e_3, e_3) = 0$. Therefore,

$$D_{e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In other hand, $\delta(e_i, e_3) = -\delta(e_3, e_i)$ for $i = 1,2$. Then,

$$D_{e_i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1,2,3.$$

Hence, $\delta = 0$, which ends the proof. $\square$

Theorem 5.3. *Every symmetric biderivation of three dimensional ω-Lie algebras L_2, A_α, B and C_α ($\alpha \neq 1, 0, -1$) is the zero map.*

Proof. The proof of the theorem will be presented for L_2; similar demonstrations can be carried out for the remaining cases.

Let δ be a symmetric derivation on L_2 Similarly to above the matrix the matrix D_{e_i} of δ_{e_i}, for $i = 1,2,3$ is of the form

$$D_{e_i} = \begin{pmatrix} a^{e_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a^{e_i} \end{pmatrix}$$

The equality $\delta(e_i, e_2) = \delta(e_2, e_i)$ for $i = 1,3$ we deduce

$$D_{e_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the identities $\delta(e_1, e_3) = \delta(e_3, e_1)$ we obtain

$$D_{e_1} = D_{e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, $\delta = 0$. $\square$

Theorem 5.4. *δ is a symmetric biderivation on L_1 if and only if there exist $a, b \in \mathbb{C}$ such that*

$$D_{e_i} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & -a \\ 0 & 0 & b \end{pmatrix} \quad i = 1,2,3.$$

Proof. Given δ be an arbitrary symmetric biderivation on L_1. By Theorem 2.5, the matrix D_{e_i} of δ_{e_i}, for $i = 1,2,3$ is of the form

$$D_{e_i} = \begin{pmatrix} 0 & 0 & a^{e_i} \\ 0 & 0 & -a^{e_i} \\ 0 & 0 & b^{e_i} \end{pmatrix}$$

Since, δ is symmetric then the equalities $\delta(e_1, e_3) = \delta(e_3, e_1)$ implies that

$$a^{e_1} = a^{e_3} \quad \textit{and} \quad b^{e_1} = b^{e_3}. \tag{2}$$

From the equalities $\delta(e_2, e_3) = \delta(e_3, e_)$ we get that

$$a^{e_2} = a^{e_3} \quad \textit{and} \quad b^{e_2} = b^{e_3}. \tag{3}$$

Which ends the proof by setting $a = a^{e_i}$ and $b = b^{e_i}$ for $i = 1,2,3$. $\square$

Theorem 5.5. *δ is a symmetric biderivation on C_1 if and only if there exist $a,b,c,d \in \mathbb{C}$ such that*

$$D_{e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad D_{e_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -d & a \\ 0 & b & d \end{pmatrix} D_{e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & c \\ 0 & d & -a \end{pmatrix}.$$

Proof. For an arbitrary symmetric biderivation δ on C_1. By Theorem 2.5, the matrix D_{e_i} of δ_{e_i}, for $i = 1,2,3$ is of the form

$$D_{e_i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^{e_i} & c^{e_i} \\ 0 & b^{e_i} & -a^{e_i} \end{pmatrix}$$

Using the equalty $\delta(e_1,e_i)$ for $i = 2,3$ we deduce

$$D_{e_i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the identities $\delta(e_2,e_3) = \delta(e_3,e_2)$ we deduce

$$c^{e_2} = a^{e_3} \quad and - a^{e_2} = b^{e_3}$$

Set $a = a^{e_3} = c^{e_2}$ and $d = -a^{e_2} = b^{e_3}$, which ends the proof. □

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