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ON DISCRETE HAHN'S THEOREM

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We give a discrete analogue of Hahn's theorem and a discrete extension for it; we show that an orthogonal polynomial sequence, whose $m-th$ associated sequence of $k-th$ "discrete derivative" sequence is orthogonal, is necessarily a $D_{-\omega}$-classical one, where D_ω is the divided-difference operator.

1. Introduction, preliminaries and first results

The Hahn's theorem is a main characterization of D-classical orthogonal polynomials, where D is the derivative operator. In [10], the authors give a new proof of Hahn's theorem [3, 4, 7, 11]. Also, they give an extension to Hahn's theorem.

In [5], the authors give a q-analogue of Hahn's theorem and an extension to it which characterizes the H_q-classical orthogonal polynomials, with H_q is the q-derivative operator [6].

In this paper, we present the discrete analogue of Hahn's theorem and a discrete extension for it that is by respecting the divided difference operator D_ω [1]. Our main results are Theorem 2.4 in section 2 and Theorem 3.2 in section 3. The proofs of these two theorems use new techniques based on duality.

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Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. For any form u, any polynomial g and any $a, \omega \in \mathbb{C}\backslash\{0\}$, $(b,c) \in \mathbb{C}^2$, let $\tau_b u$, $h_a u$, and $D_\omega u$ be the forms defined by duality

$$\langle \tau_b u, f\rangle = \langle u, \tau_{-b} f\rangle,\ \langle h_a u, f\rangle = \langle u, h_a f\rangle,$$

$$\langle gu, f\rangle = \langle u, gf\rangle,\ \langle \delta_c, f\rangle = f(c),\ f \in \mathcal{P},$$

where for all f in $\mathcal{P}$, [9]

$$(\tau_b f)(x) = f(x-b),\ (h_a f)(x) = f(ax).$$

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n,\ n \geq 0$ (polynomial sequence: PS) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by $\langle u_n, P_m\rangle := \delta_{n,m},\ n,m \geq 0$ [2, 9].

We call associated sequence of $\{P_n\}_{n\geq 0}$, the sequence $\{P_n^{(1)}\}_{n\geq 0}$ defined by [2, 9]

$$P_n^{(1)}(x) := \left\langle u_0, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x-\xi} \right\rangle, \quad n \geq 0.$$

Any polynomial $P_n^{(1)}$ is monic and $\deg P_n^{(1)} = n$.

Let us introduce the divided difference operator in $\mathcal{P}$ by [1]

$$(D_\omega f)(x) = \frac{f(x+\omega) - f(x)}{\omega}, \quad \omega \neq 0, \quad f \in \mathcal{P}. \tag{1}$$

By duality, we can define D_ω from $\mathcal{P}'$ to $\mathcal{P}'$ such that [1]

$$\langle D_\omega u, f\rangle = -\langle u, D_{-\omega} f\rangle,\ u \in \mathcal{P}',\ f \in \mathcal{P},\ \omega \in \mathbb{C}\backslash\{0\}. \tag{2}$$

In particular, this yields

$$(D_{-\omega} u)_n = \begin{cases} 0 & ,\ n = 0 \\ -\sum_{k=0}^{n-1} \binom{n}{k} \omega^{n-1-k} (u)_k & ,\ n \geq 1. \end{cases} \tag{3}$$

The dual sequence $\{u_n^{[1]}(\omega)\}_{n\geq 0}$ of $\{P_n^{[1]}(.;\omega)\}_{n\geq 0}$ where

$$P_n^{[1]}(x;\omega) := \frac{(D_{-\omega} P_{n+1})(x)}{n+1},\ n \geq 0,$$

is given by [1]

$$D_\omega(u_n^{[1]}(\omega)) = -(n+1) u_{n+1},\ n \geq 0. \tag{4}$$

More generally, we can define for $k \geq 1$ the sequence $\{P_n^{[k]}(.;\omega)\}_{n\geq 0}$ by

$$P_n^{[k]}(x;\omega) = \frac{(D_{-\omega}P_{n+1}^{[k-1]})(x;\omega)}{n+1},\ n \geq 0,$$

and we have

$$D_\omega(u_n^{[k]})(\omega) = -(n+1)u_{n+1}^{[k-1]}(\omega),\ n \geq 0,\ k \geq 1.$$

Likewise, the dual sequence $\{\widetilde{u}_n\}_{n\geq 0}$ of $\{\widetilde{P}_n\}_{n\geq 0}$ with

$$\widetilde{P}_n(x) = a^{-n}(h_a \circ \tau_{-b})P_n(x),\ n \geq 0, (a,b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C},$$

is given by

$$\widetilde{u}_n = a^n(h_{a^{-1}} \circ \tau_{-b})u_n,\ n \geq 0. \tag{5}$$

In the sequel, we shall need the following formulas [1, 8]:

Lemma 1.1. *For any $f \in \mathcal{P}$, $g \in \mathcal{P}$ and $u \in \mathcal{P}'$, we have*

$$D_\omega(fu) = (\tau_{-\omega}f)(D_\omega u) + (D_\omega f)u, \tag{6}$$

$$\tau_b \circ D_\omega f = D_\omega \circ \tau_b f,\ \tau_b \circ D_\omega u = D_\omega \circ \tau_b u, \tag{7}$$

$$D_\omega \circ D_{-\omega} = D_{-\omega} \circ D_\omega \text{ in } \mathcal{P} \text{ and in } \mathcal{P}', \tag{8}$$

The form u is called regular if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ such that [2, 9]

$$< u, P_nP_m >= r_n\delta_{n,m},\ n,m \geq 0;\ r_n \neq 0,\ n \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. We call it an orthogonal sequence (OPS for short) whose any polynomial can be supposed monic (MOPS for short). Necessarily, $u = \lambda u_0,\ \lambda \neq 0$. In this case, we have $u_n = r_n^{-1}P_nu_0,\ n \geq 0$. Also, the MOPS $\{P_n\}_{n\geq 0}$ fulfils the standard three-term recurrence relation (TTRR for short) [2, 9]

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0\ , \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0, \end{cases}$$

where

$$\beta_n = \frac{\langle u, xP_n^2 \rangle}{r_n} \quad , \quad \gamma_{n+1} = \frac{r_{n+1}}{r_n} \neq 0,\ n \geq 0.$$

In the sequel, a regular form u will be supposed normalized that is to say $(u)_0 = 1$. Thus, $u = u_0$.

Let Φ monic and Ψ be two polynomials, deg $\Phi = t \geq 0$, deg $\Psi = p \geq 1$. We suppose that the pair (Φ, Ψ) is admissible, ie, when $p = t-1$, writing $\Psi(x) = a_p x^p + ...$, then $a_p \neq n+1,\ n \in \mathbb{N}$.

Definition 1.2. [1, 8] A form u is called D_ω-semiclassical when it is regular and satisfies the equation

$$D_\omega(\Phi u) + \Psi u = 0, \tag{9}$$

where the pair (Φ, Ψ) is admissible. The corresponding MOPS $\{P_n\}_{n\geq 0}$ is called $D_{-\omega}$-semiclassical.

Remark 1.3. if u is D_ω-semiclassical, the class of u, denoted s is defined by [1, 8]

$$s := \min\Big(\max(\deg(\Phi) - 2, \deg(\Psi) - 1)\Big) \geq 0,$$

where the minimum is taken over all pairs (Φ, Ψ) satisfying (9). When $s = 0$ that is to say the D_ω-classical case, this one is well described in [1].

2. Discrete Hahn's theorem

First some lemmas.

Lemma 2.1. *[1] For any $g \in \mathcal{P}$ and $u \in \mathcal{P}'$, we have*

$$D^n_\omega((\tau_{n\omega} g)u) = \sum_{\nu=0}^{n} \binom{n}{\nu} (D^\nu_{-\omega} g)(D^{n-\nu}_\omega u),\ n \geq 0. \tag{10}$$

Lemma 2.2. *Let u be a D_ω- semiclassical form satisfying*

$$D_\omega(\Phi_1 u) + \Psi_1 u = 0, \tag{11}$$

and

$$D_\omega(\Phi_2 u) + \Psi_2 u = 0, \tag{12}$$

where $\Phi_1, \Psi_1, \Phi_2, \Psi_2$ are polynomials, Φ_1, Φ_2 monic, $\deg \Psi_1 \geq 1, \deg \Psi_2 \geq 1$. *Denoting* $s_1 = \max(\deg \Phi_1 - 2, \deg \Psi_1 - 1), s_2 = \max(\deg \Phi_2 - 2, \deg \Psi_2 - 1)$. *Let us denote by Φ the highest common factor of Φ_1 and Φ_2. Then, there exists a polynomial Ψ,* $\deg \Psi \geq 1$ *such that*

$$D_\omega(\Phi u) + \Psi u = 0, \tag{13}$$

with

$$\max(\deg \Phi - 2, \deg \Psi - 1) = s_1 - \deg \Phi_1 + \deg \Phi = s_2 - \deg \Phi_2 + \deg \Phi. \tag{14}$$

Proof. Let Φ be the highest common factor of Φ_1 and Φ_2. Then, there exist two coprime polynomials $\check{\Phi}_1$ and $\check{\Phi}_2$ such that

$$\Phi_1 = \Phi\check{\Phi}_1 \text{ and } \Phi_2 = \Phi\check{\Phi}_2. \tag{15}$$

Taking into account (6), equations (11), (12) become

$$(\tau_{-\omega}\check{\Phi}_1)D_\omega(\Phi u) + \left\{\Psi_1 + \Phi(D_\omega\check{\Phi}_1)\right\} u = 0, \tag{16}$$

$$(\tau_{-\omega}\check{\Phi}_2)D_\omega(\Phi u) + \left\{\Psi_2 + \Phi D_\omega\check{\Phi}_2\right\} u = 0. \tag{17}$$

The operation $(\tau_{-\omega}\check{\Phi}_2)\times$ (16) $-(\tau_{-\omega}\check{\Phi}_1)\times$ (17) gives
$\{(\tau_{-\omega}\check{\Phi}_2)(\Psi_1 + \Phi(D_\omega\check{\Phi}_1)) - (\tau_{-\omega}\check{\Phi}_1)(\Psi_2 + \Phi(D_\omega\check{\Phi}_2))\}u = 0.$
From regularity of u, we get

$$(\tau_{-\omega}\check{\Phi}_2)(\Psi_1 + \Phi D_\omega\check{\Phi}_1) = (\tau_{-\omega}\check{\Phi}_1)(\Psi_2 + \Phi D_\omega\check{\Phi}_2). \tag{18}$$

Thus, there exists a polynomial Ψ such that

$$\begin{cases} \Psi_1 + \Phi D_\omega\check{\Phi}_1 = \Psi(\tau_{-\omega}\check{\Phi}_1), \\ \Psi_2 + \Phi D_\omega\check{\Phi}_2 = \Psi(\tau_{-\omega}\check{\Phi}_2). \end{cases} \tag{19}$$

Then, formulas (11), (12) become
$(\tau_{-\omega}\check{\Phi}_i)\{D_\omega(\Phi u) + \Psi u\} = 0,\ i \in \{1,2\}.$
Writing $\check{\Phi}_i(x) = \prod_{k=1}^{l_i}(x - c_{i,k})^{\alpha_{i,k}}$, $i \in \{1,2\}$, which yields
$D_\omega(\Phi u) + \Psi u = \sum_{k=1}^{l_1} \beta_{1,k}\delta_{c_{1,k}}^{(\alpha_1,k)} = \sum_{k=1}^{l_2} \beta_{2,k}\delta_{c_{2,k}}^{(\alpha_2,k)}.$
But the polynomials $\check{\Phi}_1$ and $\check{\Phi}_2$ have no common zero, which allows (13). From (15) and (19), it is easy to prove (14). □

Lemma 2.3. *Let $\{P_n\}_{n\geq 0}$ be a $D_{-\omega}$- semiclassical sequence, orthogonal with respect to u_0. Suppose that u_0 fulfils the two equations*

$$\begin{cases} D_\omega(\Phi_1 u_0) + \Psi_1 u_0 = 0 \\ D_\omega(\Phi_2 u_0) + \Psi_2 u_0 = 0. \end{cases} \tag{20}$$

and there exist an integer $m \geq 0$ and four polynomials E, F, G, H such that

$$\begin{cases} \Phi_1(x) = E(x)P_{m+1}(x) + F(x)P_m(x) \\ \Phi_2(x) = G(x)P_{m+1}(x) + H(x)P_m(x). \end{cases} \tag{21}$$

Let Δ the determinant of the system (21)

$$\Delta(x) = \begin{vmatrix} E(x) & F(x) \\ G(x) & H(x) \end{vmatrix}. \tag{22}$$

Then if one of the following conditions is fulfilled, the form u_0 is D_ω-classique:

a) $\exists i = 1,2$ *such that* $\deg\Psi_i \leq \deg\Phi_i - 1$ *and* $\deg\Delta = 2$.

b) $\exists i = 1,2$ *such that* $\deg\Psi_i \leq \deg\Phi_i$ *and* $\deg\Delta = 1$.

c) $\exists i = 1,2$ *such that* $\deg\Psi_i \leq \deg\Phi_i + 1$ *and* $\deg\Delta = 0$.

Proof. From (21), we have

$$\Delta(x)P_{m+1}(x) = \begin{vmatrix} \Phi_1(x) & F(x) \\ \Psi_1(x) & H(x) \end{vmatrix},$$

$$\Delta(x)P_m(x) = \begin{vmatrix} E(x) & \Phi_2(x) \\ G(x) & \Psi_2(x) \end{vmatrix}.$$

This implies that any common factor of Φ_1 and Φ_2 is a factor of Δ; in particular, the highest common factor of Φ_1 and Φ_2, say Φ, is a factor of Δ. But from Lemma 3., thers exists a polynomial Ψ such that $D_\omega(\Phi u_0) + \Psi u_0 = 0$, where Ψ is given by $\Psi_i + \Phi D_\omega \check{\Phi}_i = \Psi(\tau_{-\omega}\check{\Phi}_i)$, $\Phi_i = \Phi\check{\Phi}_i$.
We have max $(\deg\Phi + \deg\check{\Phi}_i - 1,\ \deg\Psi_i) = \deg\Psi + \deg\check{\Phi}_i$, with $\deg\Phi_i = \deg\Phi + \deg\check{\Phi}_i$, max $(\deg\Phi_i - 1,\ \deg\Psi_i) = \deg\Psi + \deg\Phi_i - \deg\Phi$.
In the case $a)$, we have $\deg\Phi_i - 1 = \deg\Psi + \deg\Phi_i - \deg\Phi$. Therefore, $\deg\Psi = \deg\Phi - 1 \geq 1$, since u_0 is regular, thus $\deg\Phi \geq 2$. But $\deg\Phi \leq 2$. Consequently, $\deg\Phi = 2$ and $\deg\Psi = 1$. Then, the form u_0 is D_ω-classical.
In the case $b)$, we have $\deg\Phi_i = \deg\Psi + \deg\Phi_i - \deg\Phi$. Therefore, $\deg\Psi = \deg\Phi \geq 1$, but $\deg\Phi \leq 1$, thus, $\deg\Phi = 1$ and $\deg\Psi = 1$. Then, the form u_0 is D_ω-classical.
In the case $c)$, we have $\deg\Phi_i + 1 = \deg\Psi + \deg\Phi_i - \deg\Phi$. This implies $\deg\Psi = \deg\Phi + 1$, with $\deg\Phi = 0$. Thus $\deg\Phi = 0$ and $\deg\Psi = 1$. Then, the form u_0 is D_ω-classical. □

Theorem 2.4. *Let $\{P_n\}_{n\geq 0}$ be an orthogonal sequence; there exists an integer $k \geq 1$ such that $\{P_n^{[k]}\}_{n\geq 0}$ is also orthogonal. Then $\{P_n\}_{n\geq 0}$ is a $D_{-\omega}$-classical sequence.*

Proof. For the sake of simplicity, let us denote $Q_n(x) := P_n^{[k]}(x)$ and $\{v_n\}_{n\geq 0}$ the dual sequence of $\{Q_n\}_{n\geq 0}$ $(v_n = u_n^{[k]})$.
On account of assumptions, we can write the following recurrence relations

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0, \end{cases} \tag{23}$$

$$\begin{cases} Q_0(x) = 1,\ Q_1(x) = x - \zeta_0, \\ Q_{n+2}(x) = (x - \zeta_{n+1})Q_{n+1}(x) - \rho_{n+1}Q_n(x),\ n \geq 0. \end{cases} \tag{24}$$

Equivalently, we also have [9]

$$u_n = (< u_0, P_n^2 >)^{-1} P_n u_0, \; n \geq 0, \tag{25}$$

$$(x - \zeta_n) v_n = v_{n-1} + \rho_{n+1} v_{n+1}, \; n \geq 0, \; v_{-1} = 0. \tag{26}$$

By applying D_ω k times to two sides of (26) and with (5), we get

$$k D_\omega^{k-1} v_n = D_\omega^k v_{n-1} + \rho_{n+1} D_\omega^k v_{n+1} - (x - \zeta_n + k\omega) D_\omega^k v_n, \; n \geq 0.$$

But, since $v_n = u_n^{[k]}$, we easily see that

$$D_\omega^k v_n = (-1)^k \prod_{\mu=1}^{k} (n+\mu) u_{n+k}, \; n \geq 0. \tag{27}$$

Therefore
$(-1)^k k D_\omega^{k-1} v_n =$

$$\prod_{\mu=1}^{k} (n+\mu) \left\{ \frac{n}{n+k} u_{n-1+k} + \frac{n+k+1}{n+1} \rho_{n+1} u_{n+1+k} - (x - \zeta_n + k\omega) u_{n+k} \right\}.$$

Taking account of (25) et (23), we obtain

$$D_\omega^{k-1} v_n = N_n^k \phi_{n+k+1} u_0, \; n \geq 0, \tag{28}$$

where ϕ_{n+k+1} is monic and

$$N_n^k \phi_{n+k+1}(x) = L_n^k \Bigg\{ \left(\tfrac{n+k+1}{n+1} \rho_{n+1} \gamma_{n+k+1}^{-1} - \tfrac{n}{n+k} \right) P_{n+k+1}(x) - - \left(\tfrac{k}{n+k} x + \beta_{n+k} - \zeta_n + k\omega \right) P_{n+k}(x) \Bigg\},$$

$$L_n^k = (-1)^k k^{-1} \prod_{\mu=1}^{k} (n+\mu) \left(< u_0, P_{n+k}^2 > \right)^{-1}.$$

From (28) and (27), we get

$$N_n^k D_\omega(\phi_{n+k+1} u_0) = D_\omega^k(v_n) = (-1)^k \prod_{\mu=1}^{k} (n+\mu) \left(< u_0, P_{n+k}^2 > \right)^{-1} P_{n+k} u_0.$$

Hence

$$D_\omega(\phi_{n+k+1} u_0) + \lambda_n^k P_{n+k} u_0 = 0, \; n \geq 0, \tag{29}$$

with

$$\lambda_n^k = (-1)^{k-1} \prod_{\mu=1}^{k} (n+\mu) \Big(< u_0, P_{n+k}^2 > \Big)^{-1} (N_n^k)^{-1}.$$

Without going into details, we can read

$$\phi_{n+k+1}(x) = A_n^k P_{n+k+1}(x) - (B_n^k x + C_n^k) P_{n+k}(x),\ n \geq 0. \tag{30}$$

In particular, for $n = 0$ and $n = 1$

$$\phi_{k+1}(x) = A_0^k P_{k+1}(x) - (B_0^k x + C_0^k) P_k(x),\ n \geq 0. \tag{31}$$

$$\phi_{k+2}(x) = A_1^k P_{k+2}(x) - (B_1^k x + C_1^k) P_{k+1}(x),\ n \geq 0. \tag{32}$$

Taking into account (23), (32) becomes

$$\phi_{k+2}(x) = \{(A_1^k - B_1^k)x - (A_1^k \beta_{k+1} + C_1^k)\} P_{k+1}(x) - A_1^k \gamma_{k+1} P_k(x),\ n \geq 0. \tag{33}$$

Let us introduce the determinant Δ of (31), (33) (see (22)). Since $\deg \Delta \leq 2$, the form u_0 is D_ω-classical by virtue of Lemma 4. □

3. An extension of discrete Hahn's theorem

First a lemma.

Lemma 3.1. *[10] Let $\{Q_n\}_{n\geq 0}$ be any sequence with its dual sequence $\{v_n\}_{n\geq 0}$. Then, for any integer $m \geq 1$, the dual sequence $\{v_n^{(m)}\}_{n\geq 0}$ of the associated sequence $\{Q_n^{(m)}\}_{n\geq 0}$ fulfils*

$$v_n^{(m)} v_{m-1} = x v_{n+m},\ n \geq 0. \tag{34}$$

When $\{Q_n^{(m)}\}_{n\geq 0}$ is orthogonal, the sequence $\{v_n\}_{n\geq 0}$ fulfils

$$S_n^{(m)} v_{n+m} = Q_n^{(m)} v_m - Q_{n-1}^{(m+1)} v_{m-1},\ n \geq 0, \tag{35}$$

where

$$S_n^{(m)} = < v_0^{(m)}, (Q_n^{(m)})^2 >,\ n \geq 0,\ m \geq 1. \tag{36}$$

Now, our aim is to determine all orthogonal sequence $\{P_n\}_{n\geq 0}$ for which there exist two integer $k,\ m \geq 1$ such that, putting $P_n^{[k]} = Q_n,\ n \geq 0$, the associated sequence $\{Q_n^{(m)}\}_{n\geq 0}$ is also orthogonal. When $m = 0$, it is discrete Hahn's problem. When $m \geq 1$, the answer is giving by the following theorem.

Theorem 3.2. *Let $\{P_n\}_{n\geq 0}$ be an orthogonal sequence; for any integer $k \geq 1$ fixed, let us put $P_n^{[k]} := Q_n$. Suppose that there exists an integer $m \geq 1$ such that the associated sequence $\{Q_n^{(m)}\}_{n\geq 0}$ is orthogonal. Then, $\{P_n\}_{n\geq 0}$ is a $D_{-\omega}$-classical sequence.*

Proof. For simplifying, we put $Q_n^{(m)} = R_n$ et $Q_n^{(m+1)} = S_n$. By applying D_ω k times both sides of (35) where $n \to n+1$ and taking into account (10), we have

$$\sum_{\nu=1}^{k} \binom{k}{\nu} \Big(D_{-\omega}^{\nu} \circ \tau_{-k\omega} R_{n+1} \Big) \Big(D_{\omega}^{k-\nu} v_m \Big) - \sum_{\nu=1}^{k} \binom{k}{\nu} \Big(D_{-\omega}^{\nu} \circ \tau_{-k\omega} S_n \Big) \Big(D_{\omega}^{k-\nu} v_{m-1} \Big)$$
$$= S_{n+1}^{(m)} D_\omega^k v_{n+1+m} - \Big(\tau_{-k\omega} R_{n+1} \Big) \Big(D_\omega^k v_m \Big) + \Big(\tau_{-k\omega} S_n \Big) \Big(D_\omega^k v_{m-1} \Big).$$

With (27), we obtain

$$\sum_{\nu=1}^{k} \binom{k}{\nu} \Big(D_{-\omega}^{\nu} \circ \tau_{-k\omega} R_{n+1} \Big) \Big(D_{\omega}^{k-\nu} v_m \Big) - \sum_{\nu=1}^{k} \binom{k}{\nu} \Big(D_{-\omega}^{\nu} \circ \tau_{-k\omega} S_n \Big) \Big(D_{\omega}^{k-\nu} v_{m-1} \Big)$$
$$= A_{n+1+m+k} u_0, \ n \geq 0, \tag{37}$$

where

$$A_{n+1+m+k} = (-1)^k \Big(< u_0, P_{m-1+k}^2 > \Big)^{-1} \times$$
$$\Big\{ L_n^{(m)}(k) P_{n+1+m+k} - \frac{(m+k)!}{m!} \gamma_{m+k}^{-1} \Big(\tau_{-k\omega} R_{n+1} \Big) P_{m+k} + (m)_k \Big(\tau_{-k\omega} S_n \Big) P_{m-1+k} \Big\}, n \geq 0, \tag{38}$$

$$L_n^{(m)}(k) = \prod_{\mu=1}^{k} (n+1+m+\mu) \frac{< u_0, P_{m-1+k}^2 >}{< u_0, P_{n+1+m+k}^2 >} < v_0^{(m)}, R_{n+1}^2 >, \ n \geq 0, \tag{39}$$

For $n = 0$ in (37)

$$k D_\omega^{k-1} v_m = A_{m+1+k} u_0. \tag{40}$$

By virtue of (40), the equality (37) becomes

$$\sum_{\nu=2}^{k} \binom{k}{\nu} \Big(D_{-\omega}^{\nu} \circ \tau_{-k\omega} R_{n+1} \Big) \Big(D_{\omega}^{k-\nu} v_m \Big)$$

$$-\sum_{\nu=1}^{k}\binom{k}{\nu}\left(D_{-\omega}^{\nu}\circ\tau_{-k\omega}S_n\right)\left(D_{\omega}^{k-\nu}v_{m-1}\right)$$

$$=\left\{A_{n+1+m+k}-\left(D_{-\omega}\circ\tau_{-k\omega}R_{n+1}\right)A_{m+1+k}\right\}u_0. \tag{41}$$

Taking $n=1$ in (41), we get

$$k(k-1)D_{\omega}^{k-2}v_m-kD_{\omega}^{k-1}v_{m-1}=\left\{A_{m+2+k}-\left(D_{-\omega}\circ\tau_{-k\omega}R_2\right)A_{m+1+k}\right\}u_0. \tag{42}$$

Applying the operator D_ω to (40) and taking into account (25) and (27), we get

$$D_\omega(\phi_1u_0)+\lambda_1P_{m+k}u_0=0, \tag{43}$$

where

$$N_1\phi_1=A_{m+1+k},$$

$$\lambda_1=(-1)^{k+1}k\prod_{\mu=0}^{k}(m+\mu)(<u_0,P_{m+k}^2>)^{-1}N_1^{-1}.$$

Now, after applying D_ω both sides of (42), we have

$$k(k-1)D_{\omega}^{k-1}v_m-kD_{\omega}^{k}v_{m-1}=D_\omega\left(\left\{A_{m+2+k}-\left(D_{-\omega}\circ\tau_{-k\omega}R_2\right)A_{m+1+k}\right\}u_0\right).$$

Putting $N_2\phi_2=A_{m+2+k}-\left(D_{-\omega}\circ\tau_{-k\omega}R_2\right)A_{m+1+k}$ and on account of (40) and (27), we get

$$D_\omega(\phi_2u_0)+\left\{\lambda_2P_{m-1+k}-(k-1)N_2^{-1}A_{m+1+k}\right\}u_0=0, \tag{44}$$

where

$$\lambda_2=(-1)^k k\frac{(m-1+k)!}{(m-1)!}(<u_0,P_{m-1+k}^2>)^{-1}N_2^{-1}.$$

Finally, with (23), we can express ϕ_1, ϕ_2 as

$$\phi_1(x)=E(x)P_{m+k}(x)+F(x)P_{m-1+k}(x),\ \phi_2(x)=G(x)P_{m+k}(x)+H(x)P_{m-1+k}(x), \tag{45}$$

where

$$E(x)=(-1)^k(<u_0,P_{m-1+k}^2>)^{-1}N_1^{-1}\times$$
$$\left\{(x-\beta_{m+k})L_0^{(m)}(k)-\frac{(m+k)!}{m!}\gamma_{m+k}^{-1}(\tau_{-k\omega}R_1)(x)\right\},$$

$$F(x)=(-1)^k(<u_0,P_{m-1+k}^2>)^{-1}N_1^{-1}\times$$

$$\left\{\frac{(m-1+k)!}{(m-1)!}-\gamma_{m+k}L_0^{(m)}(k)\right\},$$

$$G(x)=(-1)^k(<u_0,P^2_{m-1+k}>)^{-1}N_2^{-1}\times$$

$$\left\{(x-\beta_{m+k})\left((x-\beta_{m+k+1})L_1^{(m)}(k)-(D_{-\omega}\circ\tau_{-k\omega}R_2)(x)L_0^{(m)}(k)\right)-\gamma_{m+k+1}L_1^{(m)}(k)\right.$$

$$\left.-\frac{(m+k)!}{m!}\gamma^{-1}_{m+k}\left((D_{-\omega}\circ\tau_{-k\omega}R_2)(x)(\tau_{-k\omega}R_1)(x)-(\tau_{-k\omega}R_1)(x)\right)\right\},$$

$$H(x)=(-1)^k(<u_0,P^2_{m-1+k}>)^{-1}N_2^{-1}\times$$

$$\left\{\frac{(m-1+k)!}{(m-1)!}\left((\tau_{-k\omega}S_1)(x)-(D_{-\omega}\circ\tau_{-k\omega}R_2)(x)\right)-\gamma_{m+k}\left((x-\beta_{m+k+1})L_1^{(m)}(k)\right.\right.$$

$$\left.\left.-(D_{-\omega}\circ\tau_{-k\omega}R_2)(x)L_0^{(m)}(k)\right)\right\}.$$

Since $\deg\Delta\leq 2$ with Δ given by (22), the form u is D_ω-classical. □

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