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ON THE QUALITATIVE STUDY OF AN ABSTRACT FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION VIA THE Ψ-HILFER DERIVATIVE

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In this article, we investigate the existence, uniqueness, Ulam-Hyers and generalized Ulam-Hyers stability of the fractional functional differential equation: ${}^{H}D_{0}^{\alpha,\beta;\Psi}(u(t)+g(t,u(t))) = Au(t)+f(t,u(t)), t \in [0,T]$, with initial condition $I_{0^+}^{1-\gamma;\Psi}u(0) = u_0$,where ${}^{H}D^{\alpha,\beta;\Psi}$ is the Ψ-Hilfer operator. We first establish a variation of constants formula, and then use the Banach fixed point principle and the Krasnoselskii's fixed point theorem to achieve our existence and uniqueness results. We also explore the stability of this equation under some appropriate conditions. Our results generalize some recent ones on the subject. We finally give an example to illustrate our main result.

1. Introduction

In 1695, a huge and unusual idea was introduced in Calculus when the concept of a derivative of fractional order was raised in by Leibniz. It was raised in a letter which he had sent to L'Hospital by discussing the possibility of defining $\frac{d^n x}{dx^n}$ for non-integer values of n, to which L'Hospital replied "what if $n = \frac{1}{2}$?". Leibniz replied prophetically by saying "it leads to a paradox, from which one

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day useful consequences will be drawn." Indeed, fractional calculus has been found to be extremely helpful for mathematicians and other scientists alike in fields such as physics, economics, aerodynamics, dynamical systems, and so on. For instance, due to their nonlocal nature, fractional operators have been found helpful in researching and studying memory systems and dynamical systems.

During the last two decades, several concepts of fractional derivatives were introduced in the literature. It was essential to introduce a fractional derivative of a function f with respect to another function, by using the fractional derivative in the sense of Riemann-Liouville given by Kilbas *et al.* [19],

$$\mathcal{D}^{\alpha;\psi} f(x) = (\frac{1}{\psi'(x)}\frac{d}{dx})^n I^{n-\alpha;\psi} f(x),$$

where $n-1 < \alpha < n, n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$. Such a definition is confined to the possible fractional derivatives that contains the differentiation operator acting on the integral operator.
Also, Almeida [2], by using the idea of the fractional derivative in the sense of Caputo, proposed a new concept of fractional derivative called ψ-Caputo derivative with respect to another function ψ which generalizes a class of fractional derivative as follows

$${}^{C}\mathcal{D}^{\alpha;\psi} f(x) = I^{n-\alpha;\psi}(\frac{1}{\psi'(x)}\frac{d}{dx})^n f(x),$$

where $n-1 < \alpha < n, n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$.

From above, there was the possibility of proposing a fractional differential operator that unifies these above operators and can overcome the wide number of definitions to propose a fractional differential equation and to prove the existence and uniqueness of solutions. So, the solution is to work with more general fractional operators, especially such as the ψ-Hilfer fractional operator (differentiation) and the Riemann-Liouville fractional operator with respect to another function (integration) [4–6]. Therefore, Vanterler and Capelas de Oliveira in their very interesting paper [34], presented a fractional differential operator of a function with respect to another ψ function, the so-called ψ-Hilfer fractional derivative as following:

$${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\beta(n-\alpha);\psi}(\frac{1}{\psi'(x)}\frac{d}{dx})^n I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x).$$

In his article [25], the author studied the Cauchy problem with nonlocal conditions in a finite dimensional space $D^q x(t) = f(t,x(t)),\ t \in [0,T],\ x(0)+g(x) =$

x_0, $0 < q < 1$, where D^q is the Caputo derivative operator, $0 < q < 1$. This same problem was studied by Ding *et al.* [12] in an infinite dimensional space. Both articles dealt with the existence of a mild solution to this fractional differential equation.

As indicated in Deng's pioneering paper [3], the nonlocal condition $x(0) + g(x) = x_0$ can be applied in physics with better effect than the classical Cauchy problem with initial condition $x(0) = x_0$. For instance the author used $g(x) = \sum_{k=1}^{m} c_k x(t_k)$.

The problem has been subsequently studied by several mathematicians ([8, 10, 16, 26]).

In the present article and motivated by the considerations above, we consider a more general fractional functional equation (FFE) with the Ψ-Hilfer fractional derivative operator, such as introduced by C. Sousa *et al.* in 2017 (see [4–7]). We study the existence and uniqueness of mild solutions to (FFE) the with non-local conditions and establish sufficient conditions for (FFE) to be Ulam-Hyers stable.

The work is organized as follows: in Section 2, we recall some notations and preliminary results to be used in this work. In Section 3, we establish a variation of constant formula. In Section 4, we presents our main results on the existence of mild solutions to our Cauchy problem. Finally, we study the stability in the sense of Ulam-Hyers and Ulam-Hyers-Rassias of the problem in Section 5. Our results generalize previous ones, including [25].

2. Preliminaries and Notations

In this paper, $(X, \|\cdot\|)$ will be a complex Banach space. Let $[a,b] (0 < a < b < \infty)$ be a finite interval on the half-axis $\mathbb{R}^+$, and $C^n[a,b]$ be the space of n-times continuously differentiable functions on $[a,b]$ with values in X .

The space of the continuous function f on [a,b] with the norm is defined by [3] as follows

$$\|f\|_{c[a,b]} = \max_{t \in [a,b]} |f(t)|.$$

On the other hand, we have n-times absolutely continuous given by

$$C^n[a,b] = \{f : [a,b] \to \mathbb{R}; f^{(n-1)} \in C[a,b]\}.$$

The weighted space $C_{\gamma,\Psi}[a,b]$ of a function f on (a,b] is defined by

$$C_{\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{R}; (\Psi(t) - \Psi(a))^{\gamma} f(t) \in C[a,b]\}, \;\; 0 \leq \gamma < 1,$$

and

$$C_{1-\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{R}; (\Psi(t) - \Psi(a))^{1-\gamma} f(t) \in C[a,b]\}, \;\; 0 \leq \gamma < 1,$$

with the norm

$$\|f\|_{c_{\gamma,\Psi[a,b]}} = \|(\Psi(t)-\Psi(a))^{\gamma} f(t)\|_{c[a,b]} = \max_{t\in[a,b]} |(\Psi(t)-\Psi(a))^{\gamma} f(t)|.$$

The weighted space $C^{n}_{\sigma;\Psi}[a,b]$ of function f on $(a,b]$ is defined by :

$$C^{n}_{\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{R}; f(t) \in C^{n-1}[a,b] : f^{(n)}(t) \in C_{\gamma;\psi}[a,b]\},\ \ 0 \le \gamma < 1,$$

with the norm

$$\|f\|_{c^{n}_{\gamma;\psi}[a,b]} = \sum_{k=0}^{n-1} \|f^{(k)}\|_{c[a,b]} + \|f^{(n)}\|_{c_{\gamma;\psi[a,b]}}.$$

For $n = 0$, we have $C^{0}_{\gamma}[a,b] = C_{\gamma}[a,b]$.
The weighted space $C^{\alpha,\beta}_{\gamma,\psi}[a,b]$ is defined by:

$$C^{\alpha,\beta}_{\gamma,\psi}[a,b] = \{f \in C_{\gamma;\psi}[a,b] :^{H} \mathbb{D}^{\alpha,\beta;\Psi}_{a^+} f \in C_{\gamma;\Psi}[a,b]\},\quad \gamma = \alpha + \beta(1-\alpha).$$

Definition 2.1. [4]
Let $f : [a,b] \to X$ be a function. The fractional integral of order $\alpha > 0$ for a function f is defined as

$$_{a}I^{\alpha}_{t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,$$

provided the right-hand side is point-wise defined on $[0,\infty)$.

Definition 2.2. [4]
Let $I = (a,b), f(x) \in AC^{n}(a,b)$, and $m-1 < \alpha < m$ where $m \in N$. The Riemann-Liouville derivatives (left-sided and right-sided) of f of order α are given by:

$$^{RL}D^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dx})^{n} \int_a^x (x-t)^{n-\alpha-1} f(t)dt, \tag{1}$$

and

$$^{RL}D^{\alpha}_{b^-} f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} (\frac{d}{dx})^{n} \int_x^b (t-x)^{n-\alpha-1} f(t)dt,$$

respectively.

Definition 2.3. [4]
Let [a,b]$(-\infty < a < b < \infty)$ be a finite interval on the real-axis $\mathbb{R}$. The Riemann-Liouville fractional integrals (left-sided and right-sided) of a function f of order α, with $\alpha > 0$, are defined by

$$^{RL}I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > a, \tag{2}$$

and

$$^{RL}I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad x < b,$$

respectively.

Definition 2.4. [25]

The fractional derivative of order α in the sense of Caputo is

$$^{c}D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds,$$

where $f^{(n)}$ is the derivative of order n of the function $f \in C[a,b]$, with $\alpha \in (n-1,n)$ and $n \in \mathbb{N}$ such as $n = [\alpha]+1$. If $[\alpha] = 0$, then $n = 1$ and any derivative of a function of order α in the sense of Caputo will be written in the following manner (left-sided and right-sided respectively):

$$^{c}D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \tag{3}$$

and

$$^{c}D_{b^-}^{\alpha} f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b \frac{f'(s)}{(s-t)^{\alpha}} ds.$$

Definition 2.5. [4]

Let $(a,b)(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also, let $\Psi(t)$ be an increasing and positive monotone function on $(a,b]$ having a continuous derivative $\Psi'(t)$ on (a,b). The left and right-sided fractional integrals of a function f with respect to another function $\Psi(t)$ on $[a,b]$ are defined by

$$I_{a^+}^{\alpha;\Psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1} f(s) ds, \tag{4}$$

and

$$I_{b^-}^{\alpha;\Psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \Psi'(s)(\Psi(s)-\Psi(t))^{\alpha-1} f(s) ds.$$

Definition 2.6. [4]
Let $n-1 < \alpha < n$ with $n \in N, I = [a,b]$ is the interval such that $-\infty \leq a < b \leq \infty$

and $f,\Psi \in C^n([a,b],R)$ two functions such the Ψ is increasing and $\Psi'(x) \neq 0$ for all $x \in I$. The Ψ-Hilfer fractional derivative (left-sided and right-sided) ${}^H D_{a^+}^{\alpha,\beta;\Psi}(.)$ and ${}^H D_{b^-}^{\alpha,\beta;\Psi}(.)$ of a function of order α and type $0 \leq \beta \leq 1$, are defined by

$$ {}^H D_{a^+}^{\alpha,\beta;\Psi} f(x) = I_{a^+}^{\beta(n-\alpha);\Psi}\left(\frac{1}{\Psi'(x)}.\frac{d}{dx}\right)^n\left(I_{a^+}^{(1-\beta)(n-\alpha);\Psi}\right) f(x) \quad (5) $$

and

$$ {}^H D_{b^-}^{\alpha,\beta;\Psi} f(x) = I_{b^-}^{\beta(n-\alpha);\Psi}\left(\frac{-1}{\Psi'(x)}.\frac{d}{dx}\right)^n\left(I_{b^-}^{(1-\beta)(n-\alpha);\Psi}\right) f(x). $$

The Ψ- Hilfer fractional derivative as above defined, can be written in the following form

$$ {}^H D_{a^+}^{\alpha,\beta;\Psi} f(x) = I_{a^+}^{\gamma-\alpha;\Psi} D_{a^+}^{\gamma;\Psi} f(x) $$

and

$$ {}^H D_{b^-}^{\alpha,\beta;\Psi} f(x) = I_{b^-}^{\gamma-\alpha;\Psi}(-1)^n D_{b^-}^{\gamma;\Psi} f(x), $$

with $\gamma = \alpha - \beta(n-\alpha)$ and $I_{a^+}^{\gamma-\alpha;\Psi}(.)$, $D_{a^+}^{\gamma;\Psi}(\cdot)$, $I_{b^-}^{\gamma-\alpha;\Psi}(\cdot)$, $D_{b^-}^{\gamma;\Psi}(\cdot)$ as defined above.

Definition 2.7. Banach Fixed Point Theorem
Let (X,d) be a non-empty complete metric space and $T : C_{1-\gamma;\Psi} \to C_{1-\gamma;\Psi}$ be a contraction mapping where $C_{1-\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{X}; (\Psi(t) - \Psi(a))^{1-\gamma} f(t) \in C[a,b]\}$, $0 \leq \gamma < 1$ is the weighted space.Then T admits a unique fixed point $x^\star \in [0,T]$.

Theorem 2.8. *[4, 5]*
Let $f \in C^1[0,T]$, $\alpha > 0$, and $0 \leq \beta \leq 1$. We have

(i) ${}^H D_0^{\alpha,\beta;\Psi} I_0^{\alpha,\Psi} f(x) = f(x)$,
(ii) $I_0^{\alpha,\Psi H} D_0^{\alpha,\beta;\Psi} f(x) = f(x) - \sum_{k=1}^n \frac{(\Psi(x)-\Psi(0))^{\gamma-k}}{\Gamma(\gamma+k-1)} f_\Psi^{(n-k)} I_0^{(1-\beta)(n-\alpha);\Psi} u(0)$.

If $n = 1$,the sum will have just one element and we will have:

$$ I_0^{\alpha,\Psi H} D_0^{\alpha,\beta;\Psi} f(x) = f(x) - \frac{(\Psi(x)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)} f_\Psi I_0^{(1-\beta)(1-\alpha);\Psi} u(0). $$

3. A variation of constants formula

In this section, we like to establish a variation of constants formula for the following Cauchy Problem in $C_{1-\gamma;\Psi}$[0,T], where A is a closed linear operator.

Consider the abstract fractional functional differential equation

$$ {}^{H}D_{0}^{\alpha,\beta;\Psi}(u(t)+g(t,u(t))) = Au(t)+f(t,u(t)) \;\; where \;\; 0<\alpha\leq 1, \;\; t\in[0,T], \tag{6} $$

$$ I_{0^{+}}^{1-\gamma;\Psi}u(0) = u_0, \tag{7} $$

where ${}^{H}D_{0}^{\alpha,\beta;\Psi}$ denotes Ψ-Hilfer's fractional derivative, $C_{1-\gamma;\psi}[0,T] = \{f : (0,T] \to \mathbb{X}; (\Psi(t)-\Psi(0))^{1-\gamma}f(t) \in C[0,T]\}$, $\alpha>0$ and $0\leq\beta\leq 1$, $t\in[0,T]$, and $A : X \to X$ is a closed linear operator.

Theorem 3.1. *The system Eq.(6) is equivalent to the Volterra integral equation*

$$ u(t) = \tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_{\Psi}\, I_{0}^{(1-\beta)(1-\alpha);\Psi}u(0) + AI_{0}^{\alpha,\Psi}u(t) + I_{0}^{\alpha,\Psi}(f(t,u(t))-g(t,u(t))). \tag{8} $$

Proof. Assume that Eq.(6) is true. Let's show that Eq.(8) is also true. To this end, let's apply the fractional integral operator $I_{0}^{\alpha,\Psi}(\cdot)$ to both sides of Eq.(6). So we get

$$ I_{0}^{\alpha,\Psi}[{}^{H}D_{0}^{\alpha,\beta;\Psi}(u(t)+g(t,u(t)))] = I_{0}^{\alpha,\Psi}[Au(t)+f(t,u(t))] $$

or

$$ I_{0}^{\alpha,\Psi}[{}^{H}D_{0}^{\alpha,\beta;\Psi}(u(t)+g(t,u(t)))] = AI_{0}^{\alpha,\Psi}u(t) + I_{0}^{\alpha,\Psi}f(t,u(t)). $$

By using Theorem 2.8 (i), the left hand side becomes

$$ (u(t)+g(t,u(t))) - \tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_{\Psi}\, I_{0}^{(1-\beta)(1-\alpha);\Psi}u(0). $$

The right hand side will be

$$ I_{0}^{\alpha,\beta}[Au(t)+f(t,u(t))] = AI_{0}^{\alpha,\Psi}u(t) + I_{0}^{\alpha,\Psi}f(t,u(t)). $$

Finally, we obtain

$(u(t)+g(t,u(t)))-\frac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}u(0)=AI_0^{\alpha,\Psi}$ $u(t)+I_0^{\alpha,\Psi}f(t,u(t))$.

Therefore we get

$u(t)=\frac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}u(0)+AI_0^{\alpha,\Psi}u(t)+$ $I_0^{\alpha,\Psi}(f(t,u(t))-g(t,u(t)))$.

Conversely, suppose Eq.(8) is true. Then we show that Eq.(6) is also true.

We apply ${}^{H}D_a^{\alpha,\beta;\Psi}(\cdot)$ to both sides of Eq.(7) to get

$$\begin{aligned}{}^{H}D_0^{\alpha,\beta;\Psi}u(t) ={}^{H}D_0^{\alpha,\beta;\Psi}[\tfrac{(\Psi(t)-\Psi(0))^{(\gamma-1)}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_{\Psi}I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\\ +AI_0^{\alpha,\Psi}u(t)+I_0^{\alpha,\Psi}(f(t,u(t))-g(t,u(t)))].\end{aligned}$$

Then we have

$$\begin{aligned}{}^{H}D_0^{\alpha,\beta;\Psi}u(t) ={}^{H}D_0^{\alpha,\beta;\Psi}(\tfrac{(\Psi(t)-\Psi(0))^{(\gamma-1)}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_{\Psi}I_0^{(1-\beta)(1-\alpha);\Psi}u(0))\\ +{}^{H}D_0^{\alpha,\beta;\Psi}(AI_0^{\alpha,\Psi}u(t)+I_0^{\alpha,\Psi}f(t,u(t))-g(t,u(t)))\end{aligned}$$

$${}^{H}D_0^{\alpha,\beta;\Psi}u(t)=A{}^{H}D_0^{\alpha,\beta;\Psi}I_0^{\alpha,\Psi}u(t)+{}^{H}D_0^{\alpha,\beta;\Psi}I_0^{\alpha,\Psi}f(t,u(t))-{}^{H}D_0^{\alpha,\beta;\Psi}g(t,u(t)).$$

By using Theorem 2.8 (ii), we obtain

$${}^{H}D_0^{\alpha,\beta;\Psi}(u(t)+g(t,u(t)))=Au(t)+f(t,u(t)),$$

which is exactly Eq.(6). The proof is complete.

□

4. Existence results

In this section we assume X is a finite dimensional space. Let's consider the following assumptions:

(**G1**) $f : C_{1-\gamma;\psi}[0,T] \times X \to X$ is a Caratheodory function, for any $u \in X, f(t,u)$ is measurable with respect to first variable and for every $t \in I, f(t,u)$ is continuous with respect to the second variable.

(**G2**) There exists $L > 0$ such that $\|f(t,u) - f(t,v)\| \leq L\|u-v\| \quad \forall u,v \in X$.

(**G3**) There exists $b > 0$ such that $g : [0,T] \times X \to X$ is continuous and

$$\|g(t,u) - g(t,v)\| \leq b\|u-v\| \qquad \forall u,v \in X.$$

Definition 4.1. A continuous function $u : [a,b] \to X$ is said to be a mild solution of equation (6) if it satisfies the equation

$$u(t) = \frac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_\Psi\, I_0^{(1-\beta)(1-\alpha);\Psi}u(0) + AI_0^{\alpha,\Psi}u(t) + I_0^{\alpha,\Psi}(f(t,u(t)) - g(t,u(t))).$$

Theorem 4.2. *Assume that assumptions (G1-G3) hold, and* $0 < \mathcal{D} < 1$ *where* $\mathcal{D} = \frac{(\|A\|+L+b)(\Psi(T)^\alpha)}{\Gamma(\alpha+1)}$. *Then there is a unique mild solution to the system (6).*

Proof. Consider the operator $H : C([0,T],X) \to C([0,T],X)$ such that

$$\begin{aligned}(Hu)(t) &= \frac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_\Psi\, I_0^{(1-\beta)(1-\alpha);\Psi}u(0) + AI_0^{\alpha,\Psi}u(t) \\ &\quad + I_0^{\alpha,\Psi}(f(t,u(t)) - g(t,u(t))),\end{aligned}$$

such that $u,v \in C([0,T],X)$.

It's obvious that this operator H is well defined. Now let $u,v \in ([0,T],X)$. Then

$$\begin{aligned}
\|(Hu)(t)-(Hv)(t)\| &= \|\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\\
&\quad +AI_0^{\alpha,\Psi}u(t)+I_0^{\alpha,\Psi}(f(t,u(t))-g(t,u(t)))\\
&\quad -[\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(v(t)+g(t,v(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}v(0)\\
&\quad +AI_0^{\alpha,\Psi}v(t)+I_0^{\alpha,\Psi}((f(t,v(t))-g(t,v(t))))]\|\\
&= \|\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\\
&\quad +AI_0^{\alpha,\Psi}u(t)+I_0^{\alpha,\Psi}(f(t,u(t))-g(t,u(t)))\\
&\quad -\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(v(t)+g(t,v(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}v(0)\\
&\quad -AI_0^{\alpha,\Psi}v(t)+I_0^{\alpha,\Psi}(f(t,v(t))-g(t,v(t)))]\|
\end{aligned}$$

$$\begin{aligned}
\leq\ & \|\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\\
& -\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(v(t)+g(t,v(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}v(0)\|+\|AI_0^{\alpha,\Psi}u(t)-AI_0^{\alpha,\Psi}v(t)\|\\
& +\|I_0^{\alpha,\Psi}f(t,u(t))+I_0^{\alpha,\Psi}f(t,v(t))\|+\|I_0^{\alpha,\Psi}(g(t,u(t))-g(t,v(t)))\|
\end{aligned}$$

$$\begin{aligned}
\leq\ & |\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}|\|((u(t)-v(t))\|\|I_0^{(1-\beta)(1-\alpha);\Psi}(u(0)-v(0))_{\psi}\|+|\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}|\\
& \|(g(t,u(t))-(g(t,v(t))))\|\|I_0^{(1-\beta)(1-\alpha);\Psi}(u(0)-v(0))_{\psi}\|+\|A\|\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t\Psi'(s)(\Psi(t)-\\
& \|(u(s)-v(s)\|ds+\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}\|f(s,u(s))-f(s,v(s))\|ds\\
& +\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t\Psi'(s)(\Psi(t)-\Psi(s)^{\alpha-1})\|g(s,u(s))-g(s,v(s))\|ds.
\end{aligned}$$

Since we have a Cauchy Problem and $I_0^{\alpha;\psi}u(0)$ is the initial condition with respect to Riemann Liouville definition, $I_0^{\alpha;\psi}u(0)=I_0^{\alpha;\psi}v(0)$. Thus, the inequality becomes
$\|(Hu)(t)-(Hv)(t)\|$

$$\leq \|A\|\frac{1}{\Gamma(\alpha)}\int_0^t|\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}|\|(u(s)-v(s)\|ds+ \frac{1}{\Gamma(\alpha)}\int_0^t|\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}|\|(f(s,u(s))-f(s,v(s))\|ds+ \frac{1}{\Gamma(\alpha)}\int_0^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}|\|(g(s,u(s))-g(s,v(s)))\|ds$$

$$\leq \|A\|\frac{1}{\Gamma(\alpha)}\|u-v\|_\infty\frac{(\Psi(t)-\Psi(0))^\alpha}{\alpha}+\frac{1}{\Gamma(\alpha)}L\|u-v\|_\infty\frac{(\Psi(t)-\Psi(0))^\alpha}{\alpha}+ \frac{1}{\Gamma(\alpha)}b\|u-v\|_\infty\frac{(\Psi(t)-\Psi(0))^\alpha}{\alpha}$$

$$\leq \frac{\|A\|\ (\Psi(T)^\alpha)}{\Gamma(\alpha+1)}\|u-v\|_\infty+\frac{L(\Psi(T)^\alpha)}{\Gamma(\alpha+1)}\|u-v\|_\infty+\frac{b(\Psi(T)^\alpha)}{\Gamma(\alpha+1)}\|u-v\|_\infty.$$

So we get

$$\|(Hu)(t)-(Hv)(t)\|\leq\|u-v\|_\infty\left[\frac{(\|A\|+L+b)(\Psi(T)^\alpha)}{\Gamma(\alpha+1)}\right].$$

Therefore $\|(Hu)-(Hv)\|_\infty\leq\mathcal{D}\|u-v\|_\infty$, where $\mathcal{D}=\frac{(\|A\|+L+b)(\Psi(T)^\alpha)}{\Gamma(\alpha+1)}$. Since $0<\mathcal{D}<1$ and $\|(Hu)-(Hv)\|_\infty\leq\mathcal{D}\|u-v\|_\infty$, we obtain that H has a unique fixed point by the Banach's fixed point principle.The proof is complete. □

Let's recall the following:

Theorem 4.3. (Krasnoselskii[25].
Let D be a closed convex and nonempty subset of a Banach space X and P,Q be two mappings such that
i) $Pu+Qv\in D$ for every u, v $\in D$,
ii) Q is compact and continuous,
iii) P is a contraction.
Then there exists $u\in D$ such that $u=Pu+Qu$.
We make the following assumption
(**G4**) $\rho h<1$, where $\rho=\sup\|I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\|$ $\quad, and\ h=|\frac{(\Psi(T)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}|$.

Theorem 4.4. *Suppose assumptions (G1)-(G4) hold and $\rho h(1+b)<1$. Then there exists at least one solution to Eq(6).*

Proof. Consider $B_r=\{u\in C[0,T]:\ \|u\|\leq r\}$, where $r\geq\frac{K_2}{1-K_1}$ with $K_1=\rho h+\rho hb+\|A\|\frac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}+L+b$, and $K_2=[\rho h+\|A\|\frac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}]c_g+\frac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}c_f$.

Define on B_r the operators P and Q as follows:

$$Qu(t)=AI_0^{\alpha,\Psi}u(t)+I_0^{\alpha,\Psi}(f(t,u(t))-g(t,u(t))),$$

$$Pu(t) = \frac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_\Psi\, I_0^{(1-\beta)(1-\alpha);\Psi}u(0).$$

Let's prove that if $u, v \in B_r$, then $Pu + Qv \in B_r$.

$$\begin{aligned}
\|Qv(t)+Pu(t)\| &= \|\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_\Psi\, I_0^{(1-\beta)(1-\alpha);\Psi}u(0) \\
&\quad +AI_0^{\alpha,\Psi}u(t)+I_0^{\alpha,\Psi}(f(t,u(t))-g(t,u(t)))\| \\
&\le \|\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u(t)+g(t,u(t)))_\Psi I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\|+\|AI_0^{\alpha,\Psi}u(t)\|+ \\
&\quad \|I_0^{\alpha,\Psi}(f(t,u(t))-g(t,u(t)))\| \\
&\le \|\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}|\|(u(t)+g(t,u(t)))_\Psi I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\|+\|A\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t\Psi'(s) \\
&\quad (\Psi(t)-\Psi(s))^{\alpha-1}u(s)ds\|+\|\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1} \\
&\quad (f(s,u(s))-g(s,u(s)))ds\| \\
&\le |\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}|\|(u(t)+g(t,u(t)))_\Psi I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\|+\|A\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t\Psi'(s) \\
&\quad (\Psi(t)-\Psi(s))^{\alpha-1}u(s)ds\|+\|\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1} \\
&\quad (f(s,u(s))-g(s,u(s)))ds\| \\
&\le |\tfrac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}|\|(u(t)+g(t,u(t)))_\Psi I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\|+\|A\|\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t|\Psi'(s) \\
&\quad (\Psi(t)-\Psi(s))^{\alpha-1}\|u(s)\|ds+\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t|\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1} \\
&\quad |\|(f(s,u(s))-g(s,u(s)))\|ds \\
&\le \rho h\|u(t)\|+\|g(t,u(t))_\Psi\|+\|A\|\tfrac{1}{\Gamma(\alpha)}\tfrac{\Psi(T)^\alpha}{\alpha}\|u(s)\|+\tfrac{1}{\Gamma(\alpha)}\tfrac{\Psi(T)^\alpha}{\alpha}\|f(s,u(s))-g(s,u(s))\| \\
&\le \rho h(r+\|g(t,u(t))-g(t,0)+g(t,0)\|)+\|A\|\tfrac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}(r)+\tfrac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}(\|f(s,u(s))-f(s,0)+ \\
&\quad f(s,0)\|+\|g(s,u(s))-g(s,0)+g(s,0)\|) \\
&\le \rho h(r+\|g(t,u(t))-g(t,0)\|+\|g(t,0)\|)+\|A\|\tfrac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}(r)+\tfrac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}(\|f(s,u(s))-f(s,0)\| \\
&\quad +\|f(s,0)\|+\|g(s,u(s))-g(s,0)\|+\|g(s,0)\|).
\end{aligned}$$

If we let $c_f = Max_{t\in[0,T]}\|f(t,0)\| > 0$ and $c_g = Max_{t\in[0,T]}\|g(t,0)\| > 0$, the right hand side of the last inequality will satisfy

$$\le \rho h(r+br+c_g)+\|A\|\tfrac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}r+\|A\|\tfrac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}[L\|u(t)\|+c_f+b\|u(t)\|+c_g]$$

$$\le \rho h(r+br+c_g)+\|A\|\tfrac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}r+\|A\|\tfrac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}[Lr+c_f+br+c_g]$$

$\leq [\rho h+\rho hb+\|A\|\frac{\Psi(T)^{\alpha}}{\Gamma(\alpha+1)}+L+b]r+[\rho h+\|A\|\frac{\Psi(T)^{\alpha}}{\Gamma(\alpha+1)}]c_g+\frac{\Psi(T)^{\alpha}}{\Gamma(\alpha+1)}c_f.$

The last term on the right hand side of the inequality is K_2+K_1r which satisfies $K_2+K_1r\leq r$. Therefore $(P+Q)(B_r)\subset B_r$

Let's prove now that P is a contraction. Take $u_1,u_2\in B_r$. Then

$\|Pu_1(t)-Pu_2(t)\|$

$=\|\frac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u_1(t)+g(t,u_1(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}u_1(0)-\frac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(u_2(t)+$
$g(t,u_2(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}u_2(0)\|$

$=\frac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}[\|(u_1(t)+g(t,u_1(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}u_1(0)-(u_2(t)+g(t,u_2(t)))_{\Psi}$
$I_0^{(1-\beta)(1-\alpha);\Psi}u_2(0)\|]$

$\leq h[\|(u_1(t)+g(t,u_1(t)))_{\Psi}\, I_0^{(1-\beta)(1-\alpha);\Psi}u_1(0)\|-\|(u_2(t)+g(t,u_2(t)))_{\Psi}$
$I_0^{(1-\beta)(1-\alpha);\Psi}u_2(0)\|]$

$\leq h[\|(u_1(t)+g(t,u_1(t)))_{\Psi}\|\|I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\|-\|(u_2(t)+g(t,u_2(t)))_{\Psi}\|$
$\|I_0^{(1-\beta)(1-\alpha);\Psi}u(0)\|]$

$\leq \rho h[\|u_1(t)-u_2(t)\|+\|g(t,u_1(t))-g(t,u_2(t))\|]$

$\leq \rho h\|u_1(t)-u_2(t)\|+\rho hb\|u_1(t)-u_2(t)\|$

$\leq \rho h(1+b)\|u_1-u_2\|_{\infty}.$

So we get $\|Pu_1-Pu_2\|_{\infty}\leq \mu\|u_1-u_2\|_{\infty}$, with $\mu=\rho h(1+b)<1$.

Thus P is a contraction.

Let's notice that since $u(t)$ is continuous on $[0,T]$, then $(Qu(t))$ is also continuous on $[0,T]$ in view of (G1) and (G3), and A being a bounded linear operator.
Let's prove that $(Qu(t))$ is uniformly bounded on B_r.

Let $K=\frac{\Psi(T)^{\alpha}}{\Gamma(\alpha+1)}[r\|A\|+Lr+c_f+br+c_g]$.
Taking $u\in B_r$, we get

$\|Qu(t)\| = AI_0^{\alpha,\Psi}u(t) + I_0^{\alpha,\Psi}(f(t,u(t)) - g(t,u(t)))$

$\leq \|AI_0^{\alpha,\Psi}u(t)\| + \|I_0^{\alpha,\Psi}(f(t,u(t)) - g(t,u(t)))\|$

$\leq \|A\frac{1}{\Gamma(\alpha)}\int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}u(s)ds\| + \|\frac{1}{\Gamma(\alpha)}\int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}$
$(f(s,u(s)) - g(s,u(s)))ds\|$

$\leq \|A\|\frac{1}{\Gamma(\alpha)}\int_0^t |\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}|\|u(s)\|ds + \frac{1}{\Gamma(\alpha)}\int_0^t |\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}|$
$\|(f(s,u(s)) - g(s,u(s)))\|ds$

$\leq \|A\|\frac{1}{\Gamma(\alpha)}\frac{\Psi(T)^\alpha}{\alpha}r + \frac{1}{\Gamma(\alpha)}\frac{\Psi(T)^\alpha}{\alpha}\|f(s,u(s)) - g(s,u(s))\|$

$\leq \|A\|\frac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}(r) + \frac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}(\|f(s,u(s)) - f(s,0) + f(s,0)\| + \|g(s,u(s))$
$- g(s,0) + g(s,0)\|)$

$\leq \frac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}(\|A\|r + (\|f(s,u(s)) - f(s,0) + f(s,0)\| + \|g(s,u(s)) - g(s,0) +$
$g(s,0)\|).$

By (G2), (G3) and the assumptions above, the right hand side of this inequality will satisfy

$\leq \frac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}[r\|A\| + L\|u(t)\| + c_f + b\|u(t)\| + c_g]$

$\leq \frac{\Psi(T)^\alpha}{\Gamma(\alpha+1)}[r\|A\| + Lr + c_f + br + c_g].$
So we obtain that $\|Qu(t)\| \leq K$.

Therefore (Qu) is uniformly bounded on B_r.

In what follows, $\mu = \sup_{(t,u)\in I\times B_r}\|f(t,u)\|$, $\lambda = \sup_{(t,u)\in I\times B_r}\|g(t,u)\|$ and $\Lambda = \frac{\|A\|}{\Gamma(\alpha+1)}(\mu+\lambda+r)$.

Let $t_1,t_2 \in [0,T]$, with $t_1 < t_2$ and $u \in B_r$. we have

$\|Qu(t_1) - Qu(t_2)\|$

$$= \|\frac{A}{\Gamma(\alpha)}[[\int_0^{t_1}\Psi'(s)(\Psi(t_1)-\Psi(s))^{\alpha-1}u(s)ds + \Psi'(s)(\Psi(t_1)-\Psi(s))^{\alpha-1}(f(s,u(s)) - g(s,u(s))ds)] - [\int_0^{t_2}\Psi'(s)(\Psi(t_2)-\Psi(s))^{\alpha-1}u(s)ds - \Psi'(s)(\Psi(t_2)-\Psi(s))^{\alpha-1} (f(s,u(s)) - g(s,u(s))ds)\|]]$$

$$= \frac{\|A\|}{\Gamma(\alpha)}[\| \int_0^{t_1} \Psi'(s)(\Psi(t_1)-\Psi(s))^{\alpha-1}u(s)ds - \int_0^{t_2} \Psi'(s)(\Psi(t_2)-\Psi(s))^{\alpha-1}u(s)ds + \int_0^{t_1} \Psi'(s)(\Psi(t_1)-\Psi(s))^{\alpha-1}(f(s,u(s))-g(s,u(s))ds) - \int_0^{t_2} \Psi'(s)(\Psi(t_2)-\Psi(s))^{\alpha-1}(f(s,u(s))-g(s,u(s)))ds\|]$$

$$= \frac{\|A\|}{\Gamma(\alpha)}[\| \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_1)-\Psi(s))^{\alpha-1}u(s)ds - \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_1)-\Psi(s))^{\alpha-1} (f(s,u(s))-g(s,u(s))ds)\|]$$

$$\leq \frac{\|A\|}{\Gamma(\alpha)}\| \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_1)-\Psi(s))^{\alpha-1}u(s)ds\| + \frac{\|A\|}{\Gamma(\alpha)}\| \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_1)-\Psi(s))^{\alpha-1} (f(s,u(s))-g(s,u(s))ds)\|$$

$$\leq \frac{\|A\|}{\Gamma(\alpha)}|\frac{(\Psi(t_1)-\Psi(t_2))^{\alpha}}{\alpha}|r + \frac{\|A\|}{\Gamma(\alpha)}|\frac{(\Psi(t_1)-\Psi(t_2))^{\alpha}}{\alpha}|(\mu+\lambda)$$

$$\leq \frac{\|A\|}{\Gamma(\alpha+1)}[|r(t_1-t_2)^{\alpha} + (t_1-t_2)^{\alpha}(\lambda+\mu)|]$$

$$\leq \frac{\|A\|}{\Gamma(\alpha+1)}(\mu+\lambda+r)(t_1-t_2)^{\alpha}$$

$$\leq \Lambda|(t_1-t_2)|^{\alpha}.$$

So we have that $\|Qu(t_1)-Qu(t_2)\| \leq \Lambda(t_2-t_1)^{\alpha}$.
Therefore $\|Qu(t_1)-Qu(t_2)\| \to 0$ when $t_2 \to t_1$, which means that $Qu(t)$ is equicontinuous.
Thus,Q is a compact operator on B_r.
We conclude the results using Krasnoselskii's theorem.

□

5. Stability results

Now we are ready to discuss the stability of (6) with regard to Ulam-Hyers stability. We define the mapping $\Phi : C([0,T],X) \to C([0,T],X])$ as follows:

$$\Phi v(t) =^H D_0^{\alpha,\beta;\Psi}(v(t)+g(t,v(t))) - Av(t) - f(t,v(t)).$$

Assume $v(t) \in C([0,T],X), \varepsilon > 0$, and $v \in C([0,T],X)$ satisfies the equation:

$$v(t) = \frac{(\Psi(t)-\Psi(0))^{\gamma-1}}{\Gamma(\gamma)}(v(t)+g(t,v(t)))_\Psi I_0^{(1-\beta)(1-\alpha);\Psi} v(0) + AI_0^{\alpha,\Psi}v(t) + I_0^{\alpha,\Psi}(f(t,v(t)) - g(t,v(t))).$$

Following the works of Vanterler and Oliveira in [5, 6], we will give the following definitions of stability:

Definition 5.1. Eq.(6) is said to be Ulam-Hyers stable if there exists $\omega > 0$ such that for each $\varepsilon^* > 0$, and for each solution $u \in C^1_{1-\gamma,\psi}(J,X)$ of $\|\Phi v\| \leq \varepsilon$ there exists a solution $v \in C^1_{1-\gamma,\psi}(J,X)$, of Eq.(6) such that

$$\|u(t) - v(t)\| \leq \omega\varepsilon^*,$$

where $\varepsilon^* > 0$ is dependent on ε.

Definition 5.2. Let $\varphi \in C(R^+,R^+)$ so that for every mild solution v of (6), there exists a mild solution $u \in C([0,T],X)$ of problem (6) such that

$$\|u(t) - v(t)\| \leq \varphi\varepsilon^*, t \in [0,T].$$

Then Eq.(6) is generalized Ulam-Hyers stable.

Definition 5.3. Problem (6) is called Ulam-Hyers-Rassias stable with respect to $\Xi \in C([0,T],R^+)$ if for $\varepsilon > 0$,

$$\|\Phi v(t)\| \leq \varepsilon\Xi(t),\ t \in [0,T]$$

and there exists $\omega > 0$ and $v \in C([0,T],X)$ such that

$$\|u(t) - v(t)\| \leq \omega\varepsilon\Xi(t), t \in [0,T],$$

and $\varepsilon > 0$ depends on ε.

Theorem 5.4. *Assume $\|f(t,u(t))\| + \|Au(t)\| \leq p(t)q(\|u\|)$ where $p \in C([0,T] \times [0,T],R^+)$ and $q : R^+ \to R^+$, and $0 < \mathcal{D} < 1$. Then problem (6) is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $u \in C([0,T],X)$ be a solution of (6),and let v be any solution satisfying Definition 5.1. We determine that the operators Φ and $H - Id$ (identity operator) are equivalent for all solutions $v \in C([0,T],X)$ of (6) satisfying $0 < \mathcal{D} < 1$. Thus , using the fixed point property of operator H, we conclude that

$$\begin{aligned}
\|v(t)-u(t)\| &= \|v(t)-Hv(t)+Hv(t)-u(t)\| \\
&= \|v(t)-Hv(t)+Hv(t)-Hu(t)\| \\
&\leq \|Hv(t)-Hu(t)\| + \|Hv(t)-v(t)\| \\
&\leq \mathcal{D}\|u-v\| + \varepsilon.
\end{aligned}$$

Because $0 < \mathcal{D} < 1$ and and $\varepsilon > 0$,

$$\|u-v\| \leq \frac{\varepsilon}{1-\mathcal{D}}.$$

Fixing $\varepsilon = \dfrac{\varepsilon}{1-\mathcal{D}}$, and $\omega = 1$ we get the Ulam-Hyers stability.
Using $\varphi\varepsilon = \dfrac{\varepsilon}{1-\mathcal{D}}$, we get the generalized Ulam-Hyers stability. □

Corollary 5.5. *For $\zeta \in C([0,T],R^+)$ and*

$$L < \frac{\Gamma(\alpha+1)}{\Psi(T)^\alpha} - [\|A\|+b],$$

if Definition 5.3 is satisfied, then problem (6) is Ulam-Hyers-Rassias stable in reference to ζ.

Proof. We refer to the proof for theorem 5.4, where $\|u(t)-v(t)\| \leq \varepsilon\zeta(t), t \in [0,T]$, and

$$\varepsilon = \frac{\varepsilon}{1-\mathcal{D}}.$$

The proof is complete. □

6. Example

Consider the following equation:

$$^{H}D_0^{\frac{1}{2},\frac{1}{11};(\frac{\pi}{3})^{\frac{t}{2}}}\left(u(t)+\frac{\arctan(t)}{\sinh(t)+10}u(t)\right) = Au(t)+\frac{\ln(t+1)}{\ln(t+1)+20}u(t)$$

$$I_0^{\frac{5}{11};(\frac{\pi}{3})^{\frac{t}{2}}}u(0) = 0,$$

where $\alpha = \frac{1}{2}, \beta = \frac{1}{11}, t \in [0,2], \Psi(t) = (\frac{\pi}{3})^{\frac{t}{2}}, \gamma = \alpha+\beta-\alpha\beta = \frac{6}{11}$.

$$Let\ f(t,u(t)) = \frac{\ln(t+1)}{\ln(t+1)+20}u(t)\ \ ;\ g(t,u(t)) = \frac{\arctan(t)}{\sinh(t)+10}u(t).$$

For $t \in [0,2], u, v \in X = R^{+}$, we can see that

$$\begin{aligned} |f(t,u(t)) - f(t,v(t))| &\leq \left| \frac{\ln(t+1)}{\ln(t+1)+20} u(t) - \frac{\ln(t+1)}{\ln(t+1)+20} v(t) \right| \\ &\leq \frac{\ln(t+1)}{\ln(t+1)+20} |u-v| \\ &\leq \frac{2}{25} |u-v|, \end{aligned}$$

and

$$\begin{aligned} |g(t,u) - g(t,v)| &\leq \left| \frac{\arctan(t)}{\sinh(t)+10} u(t) - \frac{\arctan(t)}{\sinh(t)+10} v(t) \right| \\ &\leq \left| \frac{\arctan(t)}{\sinh(t)+10} \right| |u-v| \\ &\leq \frac{1}{10} |u-v|. \end{aligned}$$

Thus, assumptions G2 and G3 are satisfied for $L = \frac{2}{25}$ and $b = \frac{1}{10}$. Moreover, for $\|A\| \leq \frac{25\sqrt{3}-9}{50}, t \in [0,2]$, it is easy to see that $\mathcal{D} = \frac{(\|A\|+L+b)(\Psi(T)^{\alpha})}{\Gamma(\alpha+1)} < 1$, as required. Consequently, since the conditions of theorem 4.2 are met, problem (6)-(7) has a unique solution on $t \in [0,2]$.

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