CLIFFORD LINE MANIFOLDS

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This article presents a new distribution of Clifford Klein manifolds. Special kinds of the distribution, under some assumption, are studied. The geometrical properties of the K-manifolds in the considered distribution are given. The relations between Gauss, mean, scalar normal and Lipschitz-Killing curvatures are obtained. The methods adapted here as in [1], [2] and [5].

1. Geometric preliminaries.

In this section, we will review the notations used in our previous paper [5]. Consider a Klein 5-dimensional Pseudo-Euclidean space $H_5^3$ of index three. The most convenient model of the space $H_5^3$ for the present work is the spherical one (Pseudo sphere of imaginary radius) which might be defined as follows

$$H_5^3 = \{(p^{ij}) \in \mathbb{R}^6 : \sum_{\alpha < \beta} (p^{\alpha \beta}) - \sum_{\alpha} (p^{\alpha \alpha})^2 = -1, \ p^{ij} = - p^{ji}, \ p^{\alpha \alpha} > 0\}$$

The space $\mathbb{R}^6$ denotes the Euclidean space $(\mathbb{R}^6, <, >_3)$ with the Pseudo-Riemannian metric

$$< p, p >_3 = \sum_{\alpha > \beta} (p^{\alpha \beta})^2 - \sum_{\alpha} (p^{\alpha \alpha})^2$$

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Here and in the sequel, the Latin and Greek indices run over the ranges \{0, 1, 2, 3\} and \{1, 2, 3\} respectively, except the indices \(\mu, \nu\) taking on the values 1, 2.

Since the metric \(<, >_3\) is not positive definite, the set of all vectors in \(H^3_5\) can be decomposed into space-like, time-like and isotropic like vectors according to \(<, >_3\) is positive, negative and zero respectively. The above metric when restricted to \(H^3_5\) yields a Riemannian metric with constant sectional curvature \(k_0 = -1\).

Using the Klein-bijective mapping (K-mapping) between the lines of a hyperbolic space \(H^1_3\) (Minkowski space with the Pseudometric \(<, >_1\) and the K-points of \(H^3_5\). The K-image of the set of all lines in \(H^1_3\) is a quadratic hypersurface (Grassmann manifold) immersed in \(H^3_5\). The Grassmann manifold is denoted by \(G^2_{1, 4}\) (the absolutum of the space \(H^3_5\)) and is given by

\[
\sum_{\alpha, \beta, \gamma} p^{\alpha \beta} p^{\delta \gamma} = 0, \quad (\beta < \gamma, \alpha \neq \beta)
\]

where \(p^{ij}\) are the Plücker coordinates of a point \(p \in G^2_{1, 4}\) (homogeneous coordinates in a 5-dimensional projective space) [1].

Attach to each K-point of \(G^2_{1, 4}\) a K-frame field \(\{A_{ij}\}\) (orthonormalized frame) where \(\{A_{ij}\}\) is the K-image (the Grassmann product of the two points \(A_1, A_2\)) of the edge \(\{A_1, A_2\}\) of the orthonormalized frame \(\{A_i\}\) (polar tetrahedron with respect to the absolutum \(\sum (x^0)^2 - (x^i)^2 = 0, x^0 > 0\) of the space \(H^1_3\), in which \(x^i\) are the homogeneous coordinates of a 3-dimensional projective space. Thus, the infinitesimal displacements of the frames are given as:

\[
dA_i = \omega^j_i A_j \quad \text{and} \quad dA_{ij} = \omega^h_i A_{hj} + \omega^h_j A_{ih}
\]

with the conditions of normalization

\[
A_i \equiv (\delta^i_j), \quad < A_0, A_{\alpha} >_1 = -\delta^a_0, \quad < A_{\alpha}, A_{\beta} >_1 = \delta^a_\beta,
\]

\[
< A_{0\alpha}, A_{\beta\gamma} >_3 = -\delta_1^0, \quad < A_{\alpha\beta}, A_{\gamma\eta} >_3 = \delta^\alpha_\beta \delta^\gamma_\eta
\]

Here the vectors \(A_0, A_{0\alpha}\) are time-like and \(A_{\alpha}, A_{\beta\gamma}\) are space-like vectors. The structure equations are given by:

\[
D\omega^a_0 = \omega_0^a \Lambda \omega^a_\beta, \quad D\omega^a_\alpha = \omega^a_\beta \Lambda \omega^\beta_\gamma + \omega^a_0 \Lambda \omega^0_\beta
\]

with the stationary conditions:

\[
\omega^a_\alpha + \omega^a_\beta = 0, \quad \omega^a_i = 0, \quad \omega^0_0 = \omega^0_a
\]
where \( \omega^I \) are the Pfaff’s forms.

In the following, we shall identify the \( \alpha \)-parametric families of straight lines in \( H^1 \) and their K-images under the K-mapping, that is, a ruled surface, a line congruence and a line complex immersed in \( H^1 \) is an \( \alpha \)-dimensional K-manifold immersed in \( G^2_{1,4} \subset H^3 \) respectively [2]. Consider a general K-point \( p \in G^2_{1,4} \), say without loss of generality \( p \equiv A_{03} \), and making use of the formulas (2) we find that the principal forms on \( G^2_{1,4} \) are \( \omega^\mu, \omega^\nu \). Thus, the immersion

\[
\omega^1 = b_\alpha \theta^\alpha, \quad (\theta^\alpha) \equiv (\omega^0, \omega^1, \omega^2)
\]

define a K-manifold of dimension three, denote it by \( M_3 \), immersed in the K-absolutum \( G^2_{1,4} \). The invariants \( b_\alpha \) are real valued functions defined on an open neighborhood of \( p \).

We may specialize the frames such that the K-inverse of \( M_3 \) is a line complex in the canonical form [7]. Thus, we have

\[
\omega^1 = b_1 \theta^1
\]

Exterior differentiation according to (3) and using Cartan’s lemma yields

\[
(4) \quad \omega^2 = b_1 \theta^1, \quad \theta^\alpha = a_{\alpha\beta} \theta^\beta
\]

where \( (\theta^\alpha) \equiv (db_1, -\omega^3_0 - b_1 \omega^2_1, b_1 \omega^3_0 - \omega^2_2) \) with the condition \( \Delta_0 = 1 + \varepsilon b_1^2 \neq 0, (\varepsilon = \pm 1) \). The matrix \( (a_{\alpha\beta}) \) is a nonsingular symmetric matrix of the invariants \( a_{\alpha\beta} \) defined in the 2nd order contact elements of \( p \in M_3 \).

From (2) and (4) we have the formulas

\[
(5) \quad dp = \psi^\alpha E^\alpha, \quad d^2 p \equiv I^\nu N_\nu (\text{mod} p, dp)
\]

where

\[
(E^\alpha) \equiv (A_{13}, (A_{23} + b_1 A_{01}) \sqrt{\Delta_{-1}}, A_{02}),
\]

\[
(\psi^\alpha) \equiv (\theta^2, \theta^1 \sqrt{\Delta_{-1}}, \theta^3),
\]

\[
(N_\nu) \equiv (A_{12}, A_{01}) \text{ and } \Delta_\varepsilon = 1/\Delta_\varepsilon.
\]

The tangent K-space \( T_p(M_3) \) consists of the K-points \( p, E^\alpha \) and the normal bundle \( T^\nu_p(M_3) \) consists of the K-points \( N_\nu \). The verification of the formulas (5) is routine and is left to the reader. Here, and in later formulas we will agree on the following. The forms \( \psi^\alpha, \tilde{\psi}^\alpha, \tilde{\phi}^\alpha, \phi^\alpha, \hat{\phi}^\alpha \) and \( \hat{\phi}^\nu \) are the dual coframes to the local orthonormal tangent frame fields \( E^\alpha, \tilde{E}^\alpha, \tilde{e}_\mu, \hat{e}_\mu \) and \( \hat{e}_\mu \) on the K-submanifolds \( M_3, M^1_2, M^2_2, M_2, M_2 \) and \( M_2 \) of \( G^2_{1,4} \) respectively. The forms \( I^\nu \) are the 2nd fundamental forms in the normal directions \( N_\nu \) respectively.
2. General Clifford distribution.

In general, the system

\[ \theta_2 = -\omega_3^2 - b_1\omega_1^2, \quad \theta_3 = b_1\omega_3^2 - \omega_1^2 \]

is non-singular \((\Delta_1 \neq 0)\). If the system (6) is singular \((\Delta_1 = 0)\), we have a K-manifold \(M_3\) such that its K-inverse is a Clifford line complex [8]. Here the K-manifold \(M_3\) is called a K-Clifford manifold, we denote it by \(M_3^2\), immersed in \(G_{1,4}^2\). Henceforth, the K-manifolds \(M_3^2\) is characterized by the system of equations

\[ \omega_3^1 = i\theta^1, \quad \omega_0^3 + i\omega_1^2 = -a_{22}(\theta^2 - i\theta^3), \quad (i = \sqrt{-1}) \]

Thus the formulas (5) take the form

\[ dp = \tilde{\psi}^\alpha \tilde{E}_\alpha, \quad d^2 p \equiv \tilde{I} \tilde{I}^* N_\psi \quad (\text{mod } p, dp) \]

where

\[ \begin{align*}
\tilde{\psi}^\alpha & \equiv (\theta^2, \sqrt{2}\theta^1, \theta^3), \\
\tilde{E}_\alpha & \equiv (E_1, (A_{23} + iA_{01})/\sqrt{2}, E_3), \\
\tilde{I} & \equiv (2\tilde{\psi}^1 \tilde{\psi}^3 - i(\tilde{\psi}^2)^2), \\
\tilde{I}^2 & \equiv a_{22}((\tilde{\psi}^1)^2 + (\tilde{\psi}^2)^2 + 2\tilde{\psi}^1 \tilde{\psi}^3),
\end{align*} \]

From (9), one can prove that the Gauss curvature and the normal mean curvature vector are given by

\[ G = -1 + i \quad \text{and} \quad H = (2a_{22}N_2 - iN_1)/3 \]

The \(\alpha\)th mean curvatures \(G_\alpha^\mu\) in the normal direction \(N_\mu\) satisfy

\[ G_1^1G_1^1 + G_2^1 = 0, \quad G_3^3 = G_2^2 = 0, \quad G_1^7 = (2a_{22})/3 \]

Then, it follows easily from (9) that we have

**Theorem 1.** The lines of curvatures on the manifold \(M_3^2\) in the normal direction \(N_1\) consist of the families of curves

\[ \tilde{\psi}^1 \pm \tilde{\psi}^3 = 0, \quad \tilde{\psi}^2 = 0, \quad \text{and} \quad \tilde{\psi}^1 = \tilde{\psi}^3 = 0. \]

corresponding to the principal curvature \(\pm 1\) and \(-i\).
From (3) and (7) its easy to see that
\[ D(\tilde{\psi}^1 - i\tilde{\psi}^3) \equiv 0 \pmod{\tilde{\psi}^1 - i\tilde{\psi}^3} \]
on the manifold \( M_2^c \). Thus, with the Pfaffian equation \( \tilde{\psi}^1 - i\tilde{\psi}^3 = 0 \) there is associated a field of planes, that is, a function that assigns to each point \( p \in M_2^c \) a plane of the tangent K-space \( T_p(M_2^c) \). Thus, the Pfaffian system
\[
(12) \quad \sqrt{2} \omega^1_0 - i\tilde{\psi}^3 = 0, \omega^3_0 + i\omega^2_0 = 0, \tilde{\psi}^1 - i\tilde{\psi}^3 = 0
\]
is completely integrable and through each point of \( M_2^c \) there passes one and only one integral (holonomic) submanifold of dimension two. Hence, the system (12) determines a distribution (stratification) of one parametric family of 2-dimensional K-surfaces. The inverse K-representation of the family of K-surfaces is a family of Clifford line congruences [8]. Therefore, we introduce the definition.

**Definition 1.** A K-surface of the family (12) is called a Clifford K-surface immersed in \( M_2^c \) and we denote it by \( M_2^c \).

Thus, we have proved the following:

**Theorem 2.** The K-manifold (7) admits an arbitrary distribution of one-parametric family of K-surfaces (12).

The infinitesimal displacements on \( M_2^c \) are given by
\[
(13) \quad dp = \hat{\phi}^\mu \hat{e} \quad \text{and} \quad d^2p \equiv II N_1 \pmod{p, dp}
\]
where
\[
(14) \quad \begin{cases} 
(\hat{\phi}^\mu) \equiv \sqrt{2} (\theta^2, \theta^4), \\
(\hat{e}_\mu) \equiv ((\hat{E}_1 - i\hat{E}_3)/\sqrt{2}, \hat{E}_2), \\
II \equiv -i((\hat{\phi}^1)^2 + (\hat{\phi}^3)^2).
\end{cases}
\]

From (13) and (14), one can see that the K-surface \( M_2^c \) consist of umbilical points. The Gauss and mean curvatures are given by \( \tilde{G} = -2 \) and \( \tilde{H} = -i \) respectively \( (\tilde{G} = \hat{H}^2 - 1) \). Thus, we have proved the following [9].

**Theorem 3.** The K-surface \( M_2^c \subset M_2^c \) is a Pseudo K-sphere.

**Definition 2.** The distribution (12) is called a Clifford distribution.
3. General normal distribution.

It is well-known that K-manifold $M_3 \subset G^3_{14} \subset H^3_5$ admits a normal distribution (brevity $N$-distribution) of one-parametric family of K-surfaces $M_2$ of type normal [6] (normal line congruences in $H^3_5$). The rays of the K-inverse of $M_2$ cut orthogonally holonomic surface described by the proper point $A_0 + tA_3 \in (A_0, A_3)$. The differential equation of this surface is given by

$$\omega_0^3 + D(\text{arctanh} \, t) = 0$$

In [10], it has been proved that the Gauss and mean curvatures of the surface (15) are given by

$$K = \xi (l - t^2) \Delta_1 \quad \text{and} \quad H = \xi (2t \Delta_1 + c_1 (l + t^2))$$

respectively, where $\xi = 1/(1 + c_1 t - b_1^2 t^2)$. Thus, using (15) and (4), the $N$-distribution is given by the involutive equation

$$\theta^3 = c_3 \theta^\nu, (c_\nu) \equiv (c_1, b_1)$$

with the system (4), where $c_1$ is an invariant given as a differentiable function of the invariants $b_1, a_{\alpha \beta}$. It is convenient to rewrite the system of equations which characterizes the $N$-distribution as the following

$$\omega_\alpha^1 = b_1 \theta^1, \quad \theta_\alpha = b_{a\nu} \theta^\nu, \quad \theta^3 = c_3 \theta^\nu,$$

$$dc_1 = B \theta^1 + (b_{11} + c_1 b_{31}) \theta^2 - 2b_1 \omega_1^2 + \Delta \omega_0^3,$$

where $b_{a\nu} = a_{a\nu} + c_\nu a_{\nu 3}$, $\Delta = \Delta_{-1} - c_1^2$ and $B$ is an invariant of the second order.

The immersion (18) characterizes the K-surface $M_2$ of the $N$-distribution. Making use of (18) and (2) we get

$$dp = \phi^\nu \epsilon_\nu \quad \text{and} \quad d^2 p \equiv \overline{T^\alpha}_a \overline{N}_a \mod p, dp$$

where

$$(\phi^\mu) \equiv (\theta^2 \sqrt{\Delta_{-1}}, \theta^1 \sqrt{\Delta}), \quad (\epsilon_\mu) \equiv (\sqrt{\hat{\Delta}_{-1} E_1 + E_3}, \sqrt{\hat{\Delta} (\sqrt{\Delta_{-1}} E_2 + c_1 E_3)}).$$

The quadratic differential forms $\overline{T^\alpha}_a \equiv c_{a\mu}^\alpha \phi^\mu \phi^\nu$ are the $2nd$ fundamental forms of the K-surface $M_2$, where the quantities $c_{a\mu}^\alpha$ are the components of quadratic
symmetric covariant tensors associated with the forms $\overline{T}^\alpha$. The components $c^\alpha_{\mu\nu}$ are given from the relations

\begin{equation}
(20) \quad c^1_{11} = -\Delta \hat{\Delta}_{-1} c^1_{21} = 2b_1 \hat{\Delta}_{-1}, \quad c^1_{12} = c_1 \sqrt{\hat{\Delta}_{-1}},
\end{equation}

\begin{equation*}
c^2_{11} = \hat{\Delta}_{-1}(b^1_{22} + b_1 b_{32}), \quad c^2_{22} = \sqrt{\hat{\Delta}_{-1}} c^3_{12} = \hat{\Delta}(b_{11} + c_1 b_{31}),
\end{equation*}

\begin{equation*}
c^3_{12} = \sqrt{\hat{\Delta}_{-1}} c^3_{11} = \sqrt{\hat{\Delta}_{-1}} (b_{21} + b_1 b_{31}) \quad \text{and} \quad c^3_{22} = B \hat{\Delta}.
\end{equation*}

The Lipschitz-Killing curvatures $\overline{G}^\alpha_3$ corresponding to the normal directions $\overline{N}_\alpha$ are given by $\overline{G}^\alpha_3 = \text{Det}(C^\alpha)$, where $C^\alpha \equiv (c^\alpha_{\mu\nu})$ are the symmetric matrices attached to the forms $\overline{T}^\alpha$. The Gauss curvature on $M_2$ is given by the formula [11]

\begin{equation*}
G_{M_2} = -1 + \sum \overline{G}^\alpha_3.
\end{equation*}

Explicitly, we have

\begin{equation}
(21) \quad G_{M_2} = -1 + \hat{\Delta}_{-1}(4b^3_1 + c^3_1 + (b_{21} + b_1 B_{31})^2
\end{equation}

\begin{equation*}
+ (b_{11} + c_1 b_{31})^2 - B(b_{21} + b_1 b_{31})
\end{equation*}

\begin{equation*}
- (b_{11} + c_1 b_{31}) (b_{22} + b_1 b_{32}))
\end{equation*}

The normal mean curvature vector $\overline{H}_{M_2}$ [3] is given by

\begin{equation*}
\overline{H}_{M_2} = (\text{tr}(C^\alpha)/2)\overline{N}_\alpha,
\end{equation*}

or equivalently

\begin{equation}
(22) \quad \overline{H}_{M_2} = -b_1 c^2_1 \hat{\Delta}_{-1} \hat{\Delta}_{-1} \overline{N}_1 + (\hat{\Delta}_{-1}(b_{22} + b_1 b_{32})/2
\end{equation}

\begin{equation*}
+ \hat{\Delta}_{-1}(b_{11} + c_1 b_{31}/2) \overline{N}_2 + ((\hat{\Delta}_{-1}(b_{21} + b_1 b_{31}))/2) + B/2)\overline{N}_3
\end{equation*}

The scalar normal curvature of $M_2$ is given by

\begin{equation*}
K_{M_2} = \sum_{\alpha, \beta} n(C^\alpha C^\beta - C^\beta C^\alpha), \quad \text{where} \quad n(C^\alpha) = \sum_{\mu, \nu} (c^\alpha_{\mu\nu})^2
\end{equation*}

Thus, by virtue of (20), we have

\begin{equation}
(23) \quad K_{M_2} = 4\hat{\Delta} \hat{\Delta}_{-1}\{(\zeta_1 Y_2 - \zeta_2 Y_1)^2 + (2b_1 \zeta_1 Y_3 - c_1 Y_1)^2 + (2b_1 \zeta_2 Y_3 - c_1 Y_2)^2\}
where \( Y_1 = -B \Delta + \sqrt{\Delta} \Delta_1 c_{12}^2 \),
\[
Y_2 = \hat{\Delta} \Delta_{-1} c_{11}^2 - \Delta^2 c_{22}^2, \quad Y_3 = \Delta + \hat{\Delta},
\]
\[
\zeta_1 = b_{11} + c_1 b_{31} \quad \text{and} \quad \zeta_2 = b_{21} + b_1 b_{31}.
\]
The asymptotic K-manifold at the K-point \( p \in M_2 \subset M_3 \subset G_{1,4}^2 \) is given by
\[
\det(d^2 p, p, N_2, A_{13}, A_{23}) = 0 \quad \text{or} \quad \text{equivalently} \quad \hat{\Theta}^T = 0.
\]
This equation characterizes the family of developables on the inverse K-image. Thus, we have proved the following:

**Theorem 4.** The asymptotic K-manifold on \( M_2 \) is in one-to-one correspondence with the developables of its inverse K-image.

4. **Clifford N-distribution.**

Here, consider the \( N \)-distribution (18) for which the holonomic surface (15) is a Clifford surface (ruled surface with zero Gauss and constant mean curvatures) [11]. In this distribution, the K-surface \( M_2 \) is a Clifford surface and the K-manifold \( M_3 \) is a Clifford manifold. Thus, the distribution is called \( N \)-distribution of Clifford K-surfaces (brevity \( CN \)-distribution). Using (16) and (18) one can be easily verified that the \( CN \)-distribution is decomposed into two subclasses, denoted by \( S_\mu \), characterized by the following:

\[
(S_1) \quad \left\{ \begin{array}{l}
\omega_1^2 - i \theta^1 = 0, \quad \theta^3 - 2 \theta^1 - i \theta^2 = 0, \\
\omega_0^3 + i \omega_1^3 + 2a_{22}(\theta^2 - i \theta^1) = 0
\end{array} \right.
\]

\[
(S_2) \quad \left\{ \begin{array}{l}
\omega_1^3 - i \theta^1 = 0, \quad \theta^3 - c_1 \theta^1 - i \theta^2 = 0, \\
dc_1 = (4 - c_1^2) \omega_0^3 + B \theta^1
\end{array} \right.
\]

respectively.

For a K-surface of \( S_1 \), we denote it by \( \hat{M}_2 \), we have
\[
(24) \quad dp = \hat{\psi}^\mu \hat{e}_\mu \quad \text{and} \quad d^2 p \equiv (\hat{\psi}^1 - \hat{\psi}^2)^2 N \pmod{dp, dp}
\]
where
\[
(\hat{\psi}^\mu) \equiv \sqrt{2}(\theta^2, \theta^1),
\]
\[
(\hat{e}_\mu) \equiv (A_{13} + i A_{02}, A_{01} - i A_{23} - 2i A_{02})/\sqrt{2}
\]
and \( N \equiv i A_{12} + 2a_{22}(i A_{23} - A_{13}) \) is a time like normal vector. From (24), it follows that the \( 2nd \) osculating space \( T^2_p(M_2) \) has dimension three. Thus, the inverse K-image is a linear parabolic line congruence in \( H^3_4 \) [4]. Thus, we have proved.
Lemma 1. The tangent K-plane $T_p(\tilde{M}_2)$ to the K-surface $\tilde{M}_2$ contain a fixed isotropic vector $\tilde{Q} = \sqrt{2}(\hat{e}_1 + \hat{e}_2)$.

Consequently, we have:

Corollary 1. The inverse K-image of $\tilde{Q}$ is the directrix of the inverse K-image of the K-surface $\tilde{M}_2$.

From the foregoing results, we have the geometrical characterization of the K-surface $\tilde{M}_2$ as the following.

Theorem 5. The K-surface $\tilde{M}_2$ consists of all K-points which are polar conjugate to the isotropic vector $\tilde{Q}$ with respect to the K-absolutum $G_{1,4}^2 \subset H^3_2$.

From (24) one can prove the following [10], [11].

Lemma 2. The principal direction on $\tilde{M}_2$ corresponding to the principal curvatures $0, 2\iota$ are $\hat{\phi}^1 + \hat{\phi}^2 = 0$ respectively.

Lemma 3. The asymptotic K-manifold at $p \in \tilde{M}_2$ is degenerate into one family of principal directions given by $\hat{\phi}^1 - \hat{\phi}^2 = 0$.

From (7) and (S), one can see that the system (S) is involutive on the K-manifold given by (7). Hence this characterizes a distribution on $M^*_3$. Thus, we have

Lemma 4. The K-surface $\tilde{M}_2$ given as an immersion in the Clifford manifold $M^*_3$.

The proof is omitted.

The distribution (S) is defined on a Clifford K-manifold, denote it by $\tilde{M}^*_3$, belonging to a subclass of the class (7) under the condition $a_{22} = 0$. It is easy to see that the 2nd osculating space $T^3_p(M^*_3)$ has the dimension four. Consequently the inverse K-image is a linear Clifford line complex in $H^1_3$ [4], [10]. Thus, we have proved the following:

Theorem 6. The K-manifold $\tilde{M}^*_3$ is a Wiengarten-manifold immersed in $G^2_{1,4}$.

The distribution (S) consists of K-surfaces, without loss of generality we take one of them, say $\tilde{M}_2 \subset M^*_3$. The fundamental equations on the K-surface $\tilde{M}_2$ are given as

\begin{equation}
 dp = \tilde{\phi}^\mu \tilde{e}_\mu \quad \text{and} \quad d^2 p \equiv \hat{I} I^\mu \hat{N}_\mu \quad \text{mod} \, p, dp
\end{equation}

where $(\tilde{\phi}^\mu) \equiv (\sqrt{2} \theta^2, \sqrt{2 - c^2_1} \theta^1)$,

$(\tilde{e}_\mu) \equiv ((A_{12} + i A_{02})/\sqrt{2}, (A_{23} + i A_{01} + c_1 A_{02})\Delta)$
and \((N_\mu) \equiv (\overline{N}_1, \overline{N}_3)\)

The forms \(I I^\mu\) are given by

\[
\begin{align*}
I I^1 &\equiv i((\overline{\phi}^1)^2 - 2\overline{\Delta}(\overline{\phi}^2)^2) - ic_1\sqrt{2\overline{\Delta}}\overline{\phi}^4 \\
I I^2 &\equiv B\overline{\Delta}(\overline{\phi}^3)^2,
\end{align*}
\]

where \(\overline{\Delta} = 1/(2 - c_1^2)\).

From (25) and (26), it follows that the Lipschitz-Killing curvatures in the normal directions \(\hat{N}_\mu\) are \(\hat{G}_1^1 = (4 - c_1^2)\overline{\Delta}/2\) and \(\hat{G}_3^3 = 0\), respectively.

As a similar way to [1], one can prove the following:

**Lemma 5.** The lines of curvatures on the normal sections \(\hat{N}_\mu\) are the family of curves

\[
ic\sqrt{2} ((\overline{\phi}^1)^2 - (\overline{\phi}^2)^2) + (4 - c_1^2)\sqrt{\overline{\Delta} \overline{\phi}^4} = 0
\]

and the family of parametric curves \([\overline{\phi}_\mu]\).

**Lemma 6.** The Gauss curvature and the normal mean curvature vector are \(\hat{G}_{\overline{M}_2} = c_1^4 \overline{\Delta}/2\), \(\hat{m}_{\overline{M}_2} = -(i\overline{\Delta}/2) (c_1^4 \hat{N}_1 + B \hat{N}_2)\) respectively.

**Corollary 2.** The Gauss and mean curvatures on \(\overline{M}_2\) are related by

\[
c_1^4 \hat{H}_{\overline{M}_2}^2 = (B^2 - c_1^4)\hat{G}_{\overline{M}_2}^2.
\]

So we have the following [9]:

**Lemma 7.** The K-surface \(\overline{M}_2\) is a Wiengarten-surface.

Equipped with the forms (26) and the associated quadratic tensors, we have the proof of the following [3]:

**Lemma 8.** The scalar normal curvature of the K-surface \(\overline{M}_2\) is

\[
K_{\overline{M}_2} = 2B^2 c_1^4 \overline{\Delta}^3.
\]
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