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CLIFFORD LINE MANIFOLDS

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This article presents a new distribution of Clifford Klein manifolds. Special kinds of the distribution, under some assumption, are studies. The geometrical properties of the K-manifolds in the considered distribution are given. The relations between Gauss, mean, scalar normal and Lipschitz-Killing curvatures are obtained. The methods adapted here as in [1], [2] and [5].

1. Geometric preliminaries.

In this section, we will review the notations used in our previous paper [5]. Consider a Klein 5-dimensional Pseudo-Euclidean space H_5^3 of index three. The most convenient model of the space H_5^3 for the present work is the spherical one (Pseudo sphere of imaginary radius) which might be defined as follows

$$H_5^3 = \{(p^{ij}) \in \mathbb{R}^6 : \sum_{\alpha<\beta}(p^{\alpha\beta}) - \sum_{\alpha}(p^{\circ\alpha})^2 = -1,\, p^{ij} = -p^{ji},\, p^{\circ\alpha} > 0\}$$

The space $\mathbb{R}^6$ denotes the Euclidean space $(\mathbb{R}^6, < \ >_3)$ with the Pseudo-Riemannian metric

$$< p, p >_3 = \sum_{\alpha>\beta}(p^{\alpha\beta})^2 - \sum_{\alpha}(p^{\circ\alpha})^2$$

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Here and in the sequel, the Latin and Greek indices run over the rangers $\{0, 1, 2, 3\}$ and $\{1, 2, 3\}$ respectively, except the indices μ, ν taking on the values 1, 2.

Since the metric $<,>_3$ is not positive definite, the set of all vectors in H_5^3 can be decomposed into space-like, time-like and isotropic like vectors according to $<,>_3$ is positive, negative and zero respectively. The above metric when restricted to H_5^3 yields a Riemannian metric with constant sectional curvature $k_0 = -1$.

Using the Klein-bijective mapping (K-mapping) between the lines of a hyperbolic space H_3^1 (Minkowski space with the Pseudometric $<,>_1$ and the K-points of H_5^3. The K-image of the set of all lines in H_3^1 is a quadratic hypersurface (Grassmann manifold) immersed in H_5^3. The Grassmann manifold is denoted by $G_{1,4}^2$ (the absolutum of the space H_5^3) and is given by

$$\sum_{\alpha,\beta,\gamma} p^{0\alpha} p^{\beta\gamma} = 0, \quad (\beta < \gamma, \alpha \neq \beta) \tag{1}$$

where p^{ij} are the Plûcker coordinates of a point $p \in G_{1,4}^2$ (homogeneous coordinates in a 5-dimensional projective space) [1].

Attach to each K-point of $G_{1,4}^2$ a K-frame field $\{A_{ij}\}$ (orthonormalized frame) where $\{A_{ij}\}$ is the K-image (the Grassmann product of the two points A_i, A_j) of the edge (A_i, A_j) of the orthonormalized frame $\{A_i\}$ (polar tetrahedron with respect to the absolutum $\sum_\alpha (x^\alpha)^2 - (x^0)^2 = 0, x^0 > 0$) of the space H_3^1, in which x^i are the homogeneous coordinates of a 3-dimensional projective space. Thus, the infinitesimal displacements of the frames are given as:

$$dA_i = \omega_i^j A_j \quad \text{and} \quad dA_{ij} = \omega_i^h A_{hj} + \omega_j^h A_{ih} \tag{2}$$

with the conditions of normalization

$$A_i \equiv (\delta_i^j), \; < A_0, A_\alpha >_1 = -\delta_0^\alpha, \; < A_\alpha, A_\beta >_1 = \delta_\alpha^\beta,$$

$$< A_{0\alpha}, A_{i\alpha} >_3 = -\delta_0^i, \; < A_{\alpha\beta}, A_{\gamma\eta} >_3 = \delta_{\gamma\eta}^{\alpha\beta}$$

Here the vectors $A_0, A_{0\alpha}$ are time-like and $A_\alpha, A_{\beta\gamma}$ are space-like vectors.
The structure equations are given by:

$$D\omega_0^\alpha = \omega_0^\beta \Lambda \omega_\beta^\alpha, \; D\omega_\alpha^\beta = \omega_\alpha^\gamma \Lambda \omega_\gamma^\beta + \omega_0^\alpha \Lambda \omega_0^\beta \tag{3}$$

with the stationary conditions:

$$\omega_\alpha^\beta + \omega_\beta^\alpha = 0, \; \omega_i^i = 0, \; \omega_0^\alpha = \omega_\alpha^0$$

where ω_i^j are the Pfaff's forms.

In the following, we shall identify the α-parametric families of straight lines in H_3^1 and their K-images under the K-mapping, that is, a ruled surface, a line congruence and a line complex immersed in H_3^1 is an α-dimensional K-manifold immersed in $G_{1,4}^2 \subset H_5^3$ respectively [2]. Consider a general K-point $p \in G_{1,4}^2$, say without loss of generality $p \equiv A_{03}$, and making use of the formulas (2) we find that the principal forms on $G_{1,4}^2$ are $\omega_0^\mu, \omega_3^\mu$. Thus, the immersion

$$\omega_3^1 = b_\alpha\, \theta^\alpha\ ,\ (\theta^\alpha) \equiv (\omega_0^2, \omega_0^1, \omega_3^2)$$

define a K-manifold of dimension three, denote it by M_3, immersed in the K-absolutum $G_{1,4}^2$. The invariants b_α are real valued functions defined on an open neighborhood of p.

We may specialize the frames such that the K-inverse of M_3 is a line complex in the canonical form [7]. Thus, we have

$$\omega_3^1 = b_1\, \theta^1$$

Exterior differentiation according to (3) and using Cartan's lemma yields

$$\omega_3^1 = b_1\theta^1, \theta_\alpha = a_{\alpha\beta}\, \theta^\beta \tag{4}$$

where $(\theta_\alpha) \equiv (db_1, -\omega_0^3 - b_1\omega_1^2, b_1\omega_0^3 - \omega_1^2)$ with the condition $\Delta_\omega = 1 + \varepsilon b_1^2 \neq 0, (\varepsilon = \pm 1)$. The matrix $(a_{\alpha\beta})$ is a non singular symmetric matrix of the invariants $a_{\alpha\beta}$ defined in the $2nd$ order contact elements of $p \in M_3$.

From (2) and (4) we have the formulas

$$dp = \psi^\alpha\, E_\alpha\ ,\ d^2p \equiv II^\nu\ N_\nu\ (\mathrm{mod}\, p, dp) \tag{5}$$

where

$$(E_\alpha) \equiv (A_{13}, (A_{23} + b_1 A_{01})\sqrt{\hat{\Delta}_{-1}}, A_{02}),$$
$$(\psi^\alpha) \equiv (\theta^2, \theta^1\sqrt{\Delta_{-1}},\ \theta^3),$$
$$(N_\nu) \equiv (A_{12}, A_{01}) \text{ and } \hat{\Delta}_\varepsilon = 1/\Delta_\varepsilon$$

The tangent K-space $T_p(M_3)$ consists of the K-points p, E_α and the normal bundle $T_p^l(M_3)$ consists of the K-points N_ν. The verification of the formulas (5) is routine and is left to the reader. Here, and in later formulas we will agree on the following. The forms $\psi^\alpha, \tilde{\psi}^\alpha, \tilde{\phi}^\mu, \phi^\mu, \hat{\phi}^\mu$ and $\bar{\phi}^\mu$ are the dual coframes to the local orthonormal tangent frame fields $E_\alpha, \tilde{E}_\alpha, \tilde{e}_\mu, \hat{e}_\mu$ and $\bar{e}_\mu$ on the K-submanifolds $M_3, M_3^c, M_2^c, M_2, \hat{M}_2$ and $\bar{M}_2$ of $G_{1,4}^2$ respectively. The forms II^ν are the $2nd$ fundamental forms in the normal directions N_ν respectively.

2. General Clifford distribution.

In general, the system

$$\theta_2 = -\omega_0^3 - b_1\omega_1^2\ ,\ \theta_3 = b_1\omega_0^3 - \omega_1^2 \tag{6}$$

is non-singular ($\Delta_1 \neq 0$). If the system (6) is singular ($\Delta_1 = 0$), we have a K-manifold M_3 such that its K-inverse is a Clifford line complex [8]. Here the K-manifold M_3 is called a K-Clifford manifold, we denote it by M_3^c, immersed in $G_{1,4}^2$. Henceforth, the K-manifolds M_3^c is characterized by the system of equations

$$\omega_3^1 = i\theta^1, \omega_0^3 + i\omega_1^2 = -a_{22}(\theta^2 - i\theta^3),\ (i = \sqrt{-1}) \tag{7}$$

Thus the formulas (5) take the form

$$dp = \tilde{\psi}^\alpha \tilde{E}_\alpha, d^2p \equiv \tilde{I}I^\nu N_\nu\ (\mathrm{mod}\, p, dp) \tag{8}$$

where

$$\begin{cases} (\tilde{\psi}^\alpha) \equiv (\theta^2, \sqrt{2}\theta^1, \theta^3), \\ (\tilde{E}_\alpha) = (E_1, (A_{23} + iA_{01})/\sqrt{2}, E_3), \\ \tilde{I}I^1 \quad \equiv (2\tilde{\psi}^1\ \tilde{\psi}^3 - i(\tilde{\psi}^2)^2), \\ \tilde{I}I^2 \quad \equiv a_{22}((\tilde{\psi}^1)^2 + (\tilde{\psi}^3)^2 + 2\tilde{\psi}^1\ \tilde{\psi}^3), \end{cases} \tag{9}$$

From (9), one can prove that the Gauss curvature and the normal mean curvature vector are given by

$$G = -1 + i \ \text{ and } \ H = (2a_{22}N_2 - iN_1)/3 \tag{10}$$

The αth mean curvatures G_α^μ in the normal direction N_μ satisfy

$$G_3^1G_1^1 + G_2^1 = 0,\ G_3^2 = G_2^2 = 0,\ G_1^2 = (2a_{22})/3$$

Then, it follows easily from (9) that we have

Theorem 1. *The lines of curvatures on the manifold M_3^c in the normal direction N_1 consist of the families of curves*

$$\tilde{\psi}^1 \pm \tilde{\psi}^3 = 0, \tilde{\psi}^2 = 0,\ \textit{and}\ \tilde{\psi}^1 = \tilde{\psi}^3 = 0. \tag{11}$$

corresponding to the principal curvature $\pm\ 1$ and $-i$.

From (3) and (7) its easy to see that

$$D(\tilde{\psi}^1 - i\tilde{\psi}^3) \equiv 0 \pmod{\tilde{\psi}^1 - i\tilde{\psi}^3}$$

on the manifold M_3^c. Thus, with the Pfaffian equation $\tilde{\psi}^1 - i\tilde{\psi}^3 = 0$ there is associated a field of planes, that is, a function that assigns to each point $p \in M_3^c$ a plane of the tangent K-space $T_p(M_3^c)$. Thus, the Pfaffian system

$$\sqrt{2}\,\omega_3^1 - i\tilde{\psi}^2 = 0,\ \omega_0^3 + i\omega_1^2 = 0,\ \tilde{\psi}^1 - i\tilde{\psi}^3 = 0 \tag{12}$$

is completely integrable and through each point of M_3^c there passes one and only one integral (holonomic) submanifold of dimension two. Hence, the system (12) determines a distribution (stratification) of one parametric family of 2-dimensional K-surfaces. The inverse K-representation of the family of K-surfaces is a family of Clifford line congruences [8]. Therefore, we introduce the definition.

Definition 1. A K-surface of the family (12) is called a Clifford K-surface immersed in M_3^c and we denote it by M_2^c.

Thus, we have proved the following:

Theorem 2. *The K-manifold* (7) *admits an arbitrary distribution of one-parametric family of K-surfaces (12).*

The infinitesimal displacements on M_2^c are given by

$$dp = \tilde{\phi}^\mu\, \tilde{e} \ \text{ and } \ d^2p \equiv II\, N_1 \pmod{p, dp} \tag{13}$$

where

$$\begin{cases} (\tilde{\phi}^\mu) \equiv \sqrt{2}\,(\theta^2, \theta^1), \\ (\tilde{e}_\mu) \equiv ((\tilde{E}_1 - i\tilde{E}_3)/\sqrt{2}, \tilde{E}_2), \\ II \quad \equiv -i((\tilde{\phi}^1)^2 + (\tilde{\phi}^2)^2). \end{cases} \tag{14}$$

From (13) and (14), one can see that the K-surface M_2^c consist of umbilical points. The Gauss and mean curvatures are given by $\tilde{G} = -2$ and $\tilde{H} = -i$ respectively ($\tilde{G} = \tilde{H}^2 - 1$). Thus, we have proved the following [9].

Theorem 3. *The K-surface $M_2^c \subset M_3^c$ is a Pseudo K-sphere.*

Definition 2. The distribution (12) is called a Clifford distribution.

3. General normal distribution.

It is well-known that K-manifold $M_3 \subset G^3_{1,4} \subset H^3_5$ admits a normal distribution (brevity N-distribution) of one-parametric family of K-surfaces M_2 of type normal [6] (normal line congruences in H^1_3). The rays of the K-inverse of M_2 cut orthogonally holonomic surface described by the proper point $A_0 + tA_3 \in (A_0, A_3)$. The differential equation of this surface is given by

$$\omega^3_0 + D(\operatorname{arctanh} t) = 0 \tag{15}$$

In [10], it has been proved that the Gauss and mean curvatures of the surface (15) are given by

$$K = \xi(l - t^2)\Delta_1 \quad \text{and} \quad H = \xi(2t\Delta_1 + c_1(l + t^2)) \tag{16}$$

respectively, where $\xi = 1/(1 + c_1 t - b^2_1 t^2)$.
Thus, using (15) and (4), the N-distribution is given by the involutive equation

$$\theta^3 = c_\nu \theta^\nu, (c_\nu) \equiv (c_1, b_1) \tag{17}$$

with the system (4), where c_1 is an invariant given as a differentiable function of the invariants $b_1, a_{\alpha\beta}$. It is convenient to rewrite the system of equations which characterizes the N-distribution as the following

$$\begin{cases} \omega^1_3 = b_1\theta^1, \ \theta_\alpha = b_{\alpha\nu}\theta^\nu, \ \theta^3 = c_\nu\theta^\nu, \\ dc_1 = B\theta^1 + (b_{11} + c_1 b_{31})\theta^2 - 2b_1 \ \omega^2_1 + \Delta\omega^3_0, \end{cases} \tag{18}$$

where $b_{\alpha\nu} = a_{\alpha\nu} + c_\nu a_{\alpha 3}$, $\Delta = \Delta_{-1} - c^2_1$ and B is an invariant of the second order.
The immersion (18) characterizes the K-surface M_2 of the N-distribution. Making use of (18) and (2) we get

$$dp = \phi^\nu e_\nu \quad \text{and} \quad d^2 p \equiv \overline{II}^\alpha \ \overline{N}_\alpha \ (\operatorname{mod} p, dp) \tag{19}$$

where

$$(\phi^\mu) \equiv (\theta^2\sqrt{\Delta_{-1}}, \theta^1\sqrt{\Delta}) \ , \ (e_\mu) \equiv (\sqrt{\hat{\Delta}_{-1}}(E_1 + E_3), \sqrt{\hat{\Delta}}(\sqrt{\Delta_{-1}}E_2 + c_1 E_3)),$$

$$\overline{N}_3 \equiv A_{02}, \overline{N}_\mu = N_\mu \ , \ \hat{\Delta} = l/\Delta.$$

The quadratic differential forms $\overline{II}^\alpha \equiv c^\alpha_{\mu\nu}\phi^\mu\phi^\nu$ are the $2nd$ fundamental forms of the K-surface M_2, where the quantities $c^\alpha_{\mu\nu}$ are the components of quadratic symmetric covariant tensors associated with the forms $\overline{II}^{\alpha}$. The components $c^{\alpha}_{\mu\nu}$ are given from the relations

(20) $$c^1_{11} = -\Delta\hat{\Delta}_{-1}\, c^1_{21} = 2b_1\hat{\Delta}_{-1}\ ,\ c^1_{12} = c_1\sqrt{\hat{\Delta}\hat{\Delta}_{-1}},$$

$$c^2_{11} = \hat{\Delta}_{-1}(b^1_{22} + b_1 b_{32})\ ,\ c^2_{22} = \sqrt{\hat{\Delta}\Delta_{-1}}\, c^3_{12} = \hat{\Delta}(b_{11} + c_1 b_{31}),$$

$$c^2_{12} = \sqrt{\hat{\Delta}\Delta_{-1}}\, c^3_{11} = \sqrt{\hat{\Delta}\hat{\Delta}_{-1}}\,(b_{21} + b_1 b_{31}) \ \text{ and } \ c^3_{22} = B\hat{\Delta}.$$

The Lipschitz-Killing curvatures $\overline{G}^{\alpha}_3$ corresponding to the normal directions $\overline{N}_{\alpha}$ are given by $\overline{G}^{\alpha}_3 = Det(C^{\alpha})$, where $C^{\alpha} \equiv (c^{\alpha}_{\mu\nu})$ are the symmetric matrices attached to the forms $\overline{II}^{\alpha}$. The Gauss curvature on M_2 is given by the formula [11]

$$G_{M_2} = -1 + \sum_{\alpha} \ \overline{G}^{\alpha}_3$$

Explicitly, we have

(21) $$\begin{aligned} G_{M_2} = &-1 + \hat{\Delta}\Delta_{-1}\{4b_1^2 + c_1^2 + (b_{21} + b_1 B_{31})^2 \\ &+ (b_{11} + c_1 b_{31})^2 - B(b_{21} + b_1 b_{31} \\ &- (b_{11} + c_1 b_{31})\ (b_{22} + b_1 b_{32})\} \end{aligned}$$

The normal mean curvature vector $\vec{H}_{M_2}$ [3] is given by

$$\vec{H}_{M_2} = (\mathrm{tr}(C^{\alpha})/2)\overline{N}_{\alpha},$$

or equivalently

(22) $$\vec{H}_{M_2} = -b_1 c_1^2 \hat{\Delta}_{-1}\hat{\Delta}\overline{N}_1 + (\hat{\Delta}_{-1}(\hat{\Delta}_{-1}(b_{22} + b_1 b_{32})/2$$

$$+ \hat{\Delta}(b_{11} + c_1 b_{31})/2)\overline{N}_2 + ((\hat{\Delta}_{-1}(b_{21} + b_1 b_{31})/2) + \hat{\Delta}B/2)\overline{N}_3$$

The scalar normal curvature of M_2 is given by

$$K_{M_2} = \sum_{\alpha,\beta} n(C^{\alpha}C^{\beta} - C^{\beta}C^{\alpha}),\ \text{ where }\ n(C^{\alpha}) = \sum_{\mu,\nu}(c^{\alpha}_{\mu\nu})^2$$

Thus, by virtu of (20), we have

(23) $$K_{M_2} = 4\hat{\Delta}\hat{\Delta}_{-1}\{(\zeta_1 Y_2 - \zeta_2 Y_1)^2 + (2b_1\zeta_1 Y_3 - c_1 Y_1)^2 + (2b_1\zeta_2 Y_3 - c_1 Y_2)^2\}$$

where $Y_1 = -B\Delta + \sqrt{\hat{\Delta}\Delta_1}\, c_{12}^2$,

$$Y_2 = \hat{\Delta}\Delta_{-1}c_{11}^2 - \Delta^2 c_{22}^2,\ Y_3 = \Delta + \hat{\Delta},$$
$$\zeta_1 = b_{11} + c_1 b_{31} \text{ and } \zeta_2 = b_{21} + b_1 b_{31}.$$

The asymptotic K-manifold at the K-point $p \in M_2 \subset M_3 \subset G_{1,4}^2$ is given by

$$\det(d^2p, p, N_2, \overline{N}_3, A_{13}, A_{23}) = 0 \text{ or equivalently } \overline{II}^1 = 0.$$

This equation characterizes the family of developables on the inverse K-image. Thus, we have proved the following:

Theorem 4. *The asymptotic K-manifold on M_2 is in one-to-one correspondence with the developables of its inverse K-image.*

4. Clifford N-distribution.

Here, consider the N-distribution (18) for which the holonomic surface (15) is a Clifford surface (ruled surface with zero Gauss and constant mean curvatures) [11]. In this distribution, the K-surface M_2 is a Clifford surface and the K-manifold M_3 is a Clifford manifold. Thus, the distribution is called N-distribution of Clifford K-surfaces (brevity CN-distribution). Using (16) and (18) one can be easily verified that the CN-distribution is decomposed into two subclasses, denoted by S_μ, characterized by the following:

$$(S_1) \qquad \begin{cases} \omega_3^1 - i\theta^1 = 0,\ \theta^3 - 2\theta^1 - i\theta^2 = 0, \\ \omega_0^3 + i\omega_1^2 + 2a_{22}(\theta^2 - i\theta^1) = 0 \end{cases}$$

$$(S_2) \qquad \begin{cases} \omega_3^1 - i\theta^1 = 0,\ \theta^3 - c^1\theta^1 - i\theta^2 = 0, \\ dc_1 = (4 - c_1^2)\,\omega_0^3 + B\theta^1 \end{cases}$$

respectively.

For a K-surface of S_1, we denote it by $\hat{M}_2$, we have

$$(24) \qquad dp = \hat{\psi}^\mu \hat{e}_\mu \text{ and } d^2p \equiv (\hat{\psi}^1 - \hat{\psi}^2)^2\, N\ (\operatorname{mod} p, dp)$$

where

$$(\hat{\psi}^\mu) \equiv \sqrt{2}(\theta^2, \theta^1),$$
$$(\hat{e}_\mu) \equiv (A_{13} + iA_{02}, A_{01} - iA_{23} - 2iA_{02})/\sqrt{2}$$

and $N \equiv iA_{12} + 2a_{22}(iA_{23} - A_{13})$ is a time like normal vector. From (24), it follows that the $2nd$ osculating space $T_p^2(\hat{M}_2)$ has dimension three. Thus, the inverse K-image is a linear parabolic line congruence in H_3^1 [4]. Thus, we have proved.

Lemma 1. *The tangent K-plane $T_p(\hat{M}_2)$ to the K-surface $\hat{M}_2$ contain a fixed isotropic vector $\vec{Q} = \sqrt{2}\,(\hat{e}_1 + \hat{e}_2)$.*

Consequently, we have:

Corollary 1. *The inverse K-image of $\vec{Q}$ is the directrix of the inverse K-image of the K-surface $\hat{M}_2$.*

From the foregoing results, we have the geometrical characterization of the K-surface $\hat{M}_2$ as the following.

Theorem 5. *The K-surface $\hat{M}_2$ consists of all K-points which are polar conjugate to the isotropic vector $\vec{Q}$ with respect to the K-absolutum $G^2_{1,4} \subset H^3_5$.*

From (24) one can prove the following [10], [11].

Lemma 2. *The principal direction on $\hat{M}_2$ corresponding to the principal curvatures $0, 2i$ are $\hat{\phi}^1 \pm \hat{\phi}^2 = 0$ respectively.*

Lemma 3. *The asymptotic K-manifold at $p \in \hat{M}_2$ is degenerate into one family of principal directions given by $\hat{\phi}^1 - \hat{\phi}^2 = 0$.*

From (7) and (S_1), one can see that the system (S_1) is involutive on the K-manifold given by (7). Hence this characterizes a distribution on M^c_3. Thus, we have

Lemma 4. *The K-surface $\hat{M}_2$ given as an immersion in the Clifford manifold M^c_3.*

The proof is omitted.

The distribution (S_2) is defined on a Clifford K-manifold, denote it by $\tilde{M}^c_3$, belonging to a subclass of the class (7) under the condition $a_{22} = 0$. It is easy to see that the $2nd$ osculating space $T^2_p(M^c_3)$ has the dimension four. Consequently the inverse K-image is a linear Clifford line complex in H^1_3 [4], [10]. Thus, we have proved the following:

Theorem 6. *The K-manifold $\tilde{M}^c_3$ is a Wiengarten- manifold immersed in $G^2_{1,4}$.*

The distribution (S_2) consists of K-surfaces, without loss of generality we take one of them, say $\overline{M}_2 \subset \tilde{M}^c_3$. The fundamental equations on the K-surface $\overline{M}_2$ are given as

$$dp = \overline{\phi}^{\mu}\overline{e}_{\mu} \ \text{ and } \ d^2p \equiv \hat{I}I^{\mu}\hat{N}_{\mu} \ (\operatorname{mod} p, dp) \tag{25}$$

where $(\overline{\phi}^{\mu}) \equiv (\sqrt{2}\,\theta^2, \sqrt{2 - c_1^2}\,\theta^1)$,

$$(\overline{e}_{\mu}) \equiv ((A_{13} + iA_{02})/\sqrt{2}, (A_{23} + iA_{01} + c_1 A_{02})\overline{\Delta})$$

and $(\hat{N}_\mu) \equiv (\overline{N}_1, \overline{N}_3)$
The forms $\hat{I}I^\mu$ are given by

$$
(26) \qquad \begin{cases} \hat{I}I^1 \equiv i((\overline{\phi}^1) - 2\overline{\Delta}(\overline{\phi}^2)^2 - ic_1\sqrt{2\overline{\Delta}}\overline{\phi}^1\overline{\phi}^2) \\ \hat{I}I^2 \equiv B\overline{\Delta}(\overline{\phi}^2)^2, \end{cases}
$$

where $\overline{\Delta} = 1/(2 - c_1^2)$.

From (25) and (26), it follows that the Lipschitz-Killing curvatures in the normal directions $\hat{N}_\mu$ are $\hat{G}_3^1 = (4 - c_1^2)\overline{\Delta}/2$ and $\hat{G}_3^2 = 0$, respectively.

As a similar way to [1], one can prove the following:

Lemma 5. *The lines of curvatures on the normal sections $\hat{N}_\mu$ are the family of curves*

$$
ic\sqrt{2}\,((\overline{\phi}^1)^2 - (\overline{\phi}^2)^2) + (4 - c_1^2)\sqrt{\overline{\Delta}}\ \overline{\phi}^1\overline{\phi}^2 = 0
$$

and the family of parametric curves $\{\overline{e}_\mu\}$.

Lemma 6. *The Gauss curvature and the normal mean curvature vector are $G_{\overline{M}_2} = c_1^2\,\overline{\Delta}/2$, $\vec{H}_{\overline{M}_2} = -(i\overline{\Delta}/2)\ (c_1^2\hat{N}_1 + B\hat{N}_2)$ respectively.*

Corollary 2. *The Gauss and mean curvatures on $\overline{M}_2$ are related by*

$$
c_1^4 H^2_{\overline{M}_2} = (B^2 - c_1^4)G^2_{\overline{M}_2}.
$$

So we have the following [9]:

Lemma 7. *The K-surface $\overline{M}_2$ is a Wiengarten-surface.*

Equipped with the forms (26) and the associated quadratic tensors, we have the proof of the following [3]:

Lemma 8. *The scalar normal curvature of the K-surface $\overline{M}_2$ is*

$$
K_{\overline{M}_2} = 2B^2\, c_1^2\, \overline{\Delta}^3.
$$

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