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REGULARITY RESULTS FOR NONLINEAR ANISOTROPIC PARABOLIC EQUATIONS WITH MEASURE DATA

A. SABIRY - G. ZINEDDAINE - A. KASSIDI - L. S. ZHADLI

This study focuses on a class of nonlinear anisotropic parabolic equations involving measurement data. We prove, under which condition on $s_i(y)$, some existence and regularity results for solutions in anisotropic Sobolev spaces using compactness arguments and convergence results.

1. Introduction

This paper discusses existence and regularity results for weak solutions of the following problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \displaystyle\sum_{i=1}^{N} \partial_i \left[a_i(t,y,v)(1+|v|)^{s_i(y)} |\partial_i v|^{p_i(y)-2} \partial_i v \right] = \mu & \\ & \text{in } Q := (0,T) \times \Omega, \\ v(t,y) = 0 & \text{on } (0,T) \times \partial\Omega, \\ v(0,y) = v_0(y) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$, $T > 0$) characterised by a smooth boundary $\partial\Omega$. The vector field $a_i(t,y,v)$ satisfies the conditions given below, μ is a bounded Radon measure on Q, and v_0 belongs to $L^1(\Omega)$.

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The classical model represented by the problem (1) simplifies to the well-known anisotropic evolving p-Laplacian equation. Such problems have received increasing attention in recent years due to their importance in modelling physical and mechanical processes in anisotropic continua. Although it is impossible to provide an exhaustive list, several studies have addressed these issues, as reported in [17] (see also the cited references for further investigation). It should be noted, however, that the parabolic operator of the problem (1) can degenerate once the solution is no longer bounded. In this case, the diffusion coefficient may tend towards zero as the solution v increases, indicating a slow diffusion effect. There is already evidence for the existence of such cases in stationary and evolving settings, assuming that the growth conditions are isotropic. For example, the isotropic elliptic case, where $p_i = p$ for $i = 1,2,\dots,N$ in the problem (1), was first studied in [6] and further investigated in [10, 15]. In the isotropic parabolic case, the existence and regularity results for problem (1) were established in [18] (and also in [7, 16, 25, 30, 31]), and the study of problem (1) with non-zero initial data is addressed in [24].

In a recent study [14], an attempt was made to investigate the regularity results associated with weak solutions within the anisotropic elliptic context of the problem (1). The work presented in [2] dealt with the existence and regularity of entropy solutions for the stationary problem (1) with lower order terms. However, it is noteworthy that, to the best of our knowledge, there is a significant gap in the literature concerning the existence and regularity of nonlinear anisotropic parabolic equations with measure data.

Therefore, the primary objective of this paper is to address highly complicated issues, specifically nonlinear $p(y)$ anisotropic parabolic problems. The idea used to establish our main results relies on a fusion of compactness estimates and convergence results within variable exponent Sobolev spaces, exploiting certain approximate problems. Consequently, a key aspect of the main results is the derivation of a priori estimates, initially using weak solutions. To obtain global estimates, it becomes necessary to introduce additional assumptions on $s_i(y)$ in order to obtain approximate estimates.

The remainder of this paper is organized as follows. In Section 2, we give some preliminary results and state the main results. Section 3 is devoted to give some technical results. In Section 4, we give the proof of our main result.

2. Preliminaries and Statement of the Result

To deal with problem (1), we will employ certain definitions and fundamental characteristics of anisotropic variable exponent Lebesgue-Sobolev spaces such as $L^{p_i(y)}(\Omega)$ and $W_0^{1,p_i(y)}(\Omega)$, along with the properties of parabolic capacities. It's worth noting that we will only revisit essential findings that will be subsequently applied, and for a more comprehensive exploration, we refer to [1, 9, 11–13].

2.1. Variable Exponent Spaces

Let Ω denote a bounded open subset of $\mathbb{R}^N$, where $N \geq 2$ and $\overline{\Omega}$ denote its closure. We define a real-valued continuous function p as log-Hölder continuous in Ω if it satisfies the condition

$$|p(z)-p(y)| \leq \frac{C}{|\log|z-y||} \text{ for all } z,\ y \in \overline{\Omega} \text{ such that } |z-y| < \frac{1}{2},$$

where C is a constant. We denote the set of such log-Hölder continuous functions as

$$C_+(\overline{\Omega}) = \Big\{ \text{ log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \\ \text{with } 1 < p^- \leq p(y) \leq p^+ < N \Big\},$$

where

$$p^- = \min\Big\{p(y) : y \in \overline{\Omega}\Big\} \text{ and } p^+ = \max\Big\{p(y) : y \in \overline{\Omega}\Big\}.$$

The Lebesgue space with a variable exponent is defined as

$$L^{p(y)}(\Omega) = \Big\{v : \Omega \to \mathbb{R} \text{ is measurable such that } \int_\Omega |v(y)|^{p(y)} dy < +\infty\Big\},$$

which is equipped with the Luxembourg norm

$$\|v\|_{p(y)} = \inf\Big\{\varsigma > 0; \int_\Omega |\frac{v(y)}{\varsigma}|^{p(y)} dy \leq 1\Big\}.$$

It's important to note that we will use the following inequality

$$\min\Big\{\|v\|_{p(y)}^{p^-}\ ;\ \|v\|_{p(y)}^{p^+}\Big\} \leq \int_\Omega |v(y)|^{p(y)} dy \leq \max\Big\{\|v\|_{p(y)}^{p^-}\ ;\ \|v\|_{p(y)}^{p^+}\Big\}.$$

Additionally, if $1 < p^- < \infty$, then $L^{p(y)}(\Omega)$ is reflexive, and its dual is denoted as $L^{p'(y)}(\Omega)$, where $\frac{1}{p(y)} + \frac{1}{p'(y)} = 1$. For any $v \in L^{p(y)}(\Omega)$ and $w \in L^{p'(y)}(\Omega)$, the Hölder-type inequality holds

$$\int_\Omega |vw| dy \leq \Big(\frac{1}{p(y)} + \frac{1}{p'(y)}\Big) \|v\|_{p(y)} \|w\|_{p'(y)}.$$

Next, when $p(y), p'(y) \in C_+(\overline{\Omega})$, we can apply Young's type inequality, defined as

$$ab \leq \frac{a^{p(y)}}{p(y)} + \frac{b^{p'(y)}}{p'(y)},$$

subject to the condition $\frac{1}{p(y)} + \frac{1}{p'(y)} = 1$, for any positive values of a and b. Extending a variable exponent $p : \overline{\Omega} \to [1, +\infty)$ to $\overline{Q} = \overline{\Omega} \times [0, T]$ by setting $p(y) := p(t, y)$ for every $(y,t) \in \overline{Q}$, we may also consider the generalized Lebesgue space

$$L^{p(y)}(Q) = \Big\{ v : Q \to \mathbb{R};\ \text{measurable such that } \int_Q \Big| v(y,t) \Big|^{p(y)} dydt < \infty \Big\},$$

endowed with the norm

$$\|v\|_{L^{p(y)}(Q)} = \inf \Big\{ \varsigma > 0; \int_Q \Big| \frac{v(y,t)}{\varsigma} \Big|^{p(y)} dydt < 1 \Big\}.$$

This space maintains the same properties as $L^{p(y)}(\Omega)$. Furthermore, the variable exponent Sobolev space is defined as

$$W^{1,p(y)}(\Omega) = \Big\{ v \in L^{p(y)}(\Omega)\ ;\ |\nabla v| \in L^{p(y)}(\Omega) \Big\},$$

endowed with the norm

$$\|v\|_{1,p(y)} = \|v\|_{p(y)} + \|\nabla v\|_{p(y)},$$

and satisfies

$$\|v\|_{1,p(y)} = \inf \Big\{ \varsigma > 0; \int_\Omega \Big(\Big| \frac{\nabla v(y)}{\varsigma} \Big|^{p(y)} + \Big| \frac{v(y)}{\varsigma} \Big|^{p(y)} \Big) dy \leq 1 \Big\}. \qquad (2)$$

We define $W_0^{1,p(y)}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{1,p(y)}(\Omega)$. Assuming $p^- > 1$, it follows that both $W_0^{1,p(y)}(\Omega)$ and $W^{1,p(y)}(\Omega)$ are separable and reflexive Banach spaces. Additionally, we adopt the standard notation for Bochner spaces. For $q \geq 1$ and $\mathcal{D}$ as a Banach space, $L^q(0,T;\mathcal{D})$ denotes the space of strongly measurable functions $v : (0,T) \to X$ such that $t \mapsto \|v(t)\|_{\mathcal{D}} \in L^q(0,T)$. Furthermore, $C([0,T];\mathcal{D})$ represents the space of continuous functions $v : [0,T] \to \mathcal{D}$ endowed with the norm $\|v\|_{C([0,T];\mathcal{D})} = \max_{t\in[0,T]} \|v(t)\|_{\mathcal{D}}$.
We also define

$$L^{p^-}(0,\ T; W_0^{1,p(y)}(\Omega)) = \Big\{ v : (0,T) \to W_0^{1,p(y)}(\Omega) \text{ measurable with } \Big(\int_0^T \|v(t)\|^{p^-}_{W_0^{1,p(y)}(\Omega)} \Big)^{\frac{1}{p^-}} dt < +\infty \Big\}.$$

Lemma 2.1. *If we denote $\rho(v) = \int_\Omega |v|^{p(y)}dy$, for all $v \in L^{p(y)}(\Omega)$. Then*

1. $|v|_{p(y)} < 1(= 1; > 1) \Leftrightarrow \rho(v) < 1(= 1; > 1)$;
2. $|v|_{p(y)} > 1 \Rightarrow |v|^{p^-}_{p(y)} \leq \rho(v) \leq |v|^{p^+}_{p(y)}$;
3. $|v|_{p(y)} < 1 \Rightarrow |v|^{p^-}_{p(y)} \geq \rho(v) \geq |v|^{p^+}_{p(y)}$;
4. $|v|_{p(y)} \to 0 \Leftrightarrow \rho(v) \to 0$, *because* $p^+ < \infty$.

Proof. See [11, Theorem 1.3]. □

2.2. The anisotropic variable exponent Sobolev space

We now introduce $p_i(y) : \Omega \to (1, \infty)$ as a continuous function for all $i = 1, \ldots, N$. The anisotropic variable exponent Sobolev spaces are then defined as

$$W^{1,p_i(y)}(\Omega) = \left\{ v \in L^{p_i(y)}(\Omega) \mid \partial_i v \in L^{p_i(y)}(\Omega) \right\},$$
$$W_0^{1,p_i(y)}(\Omega) = \left\{ v \in W_0^{1,1}(\Omega) \mid \partial_i v \in L^{p_i(y)}(\Omega) \right\}.$$

There are Banach spaces equipped with norms defined as

$$\|v\|_i = \|v\|_{L^{p_i(y)}(\Omega)} + \|\partial_i v\|_{L^{p_i(y)}(\Omega)}, \quad i = 1, \ldots, N.$$

We can now state the following lemma, often referred to as the Anisotropic Sobolev inequality, originally found in [28, 29].

Lemma 2.2. *Let Q be a cube in $\mathbb{R}^N$ with faces aligned with the coordinate planes. If $p_i \geq 1$ for $i = 1, \ldots, N$ and v belongs to $\bigcap_{i=1}^N W^{1,p_i}(Q)$ spaces, then the following inequality holds*

$$\|v\|_{L^s(Q)} \leq K \prod_{i=1}^N \left(\|v\|_{L^{p_i}(Q)} + |\partial_i v|_{L^{p_i}(Q)} \right)^{\frac{1}{N}},$$

where $s = \bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}$, if $\bar{p} < N$ such that $\bar{p}$ is defined by $\dfrac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}$. The constant K depends on N and the values of p_i. Moreover, if $\bar{p} \geq N$, the inequality holds for all $s \geq 1$, and K depends on s and the volume of Q.

Theorem 2.3. *Suppose a bounded domain $\Omega \subset \mathbb{R}^N$ and continuous functions $p_i(y) > 1$. Consider that $p_i(y) < \bar{p}^*(y)$, for $i = 0, \ldots, N$ where*

$$\bar{p}^*(y) = \begin{cases} \frac{N\bar{p}(y)}{N-\bar{p}(y)}, & \text{if } \bar{p}(y) < N \\ +\infty, & \text{if } \bar{p}(y) \geq N. \end{cases}$$

Then, the following Poincar-type inequality holds

$$\|v\|_{L^{p_i^+(y)}(\Omega)} \leq C \sum_{i=1}^{N} |\partial_i v|_{L^{p_i(y)}(\Omega)}, \quad \text{for all } v \in \bigcap_{i=1}^{N} W_0^{1,p_i(y)}(\Omega).$$

where C is a positive constant independent of v. Thus, $\sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i(y)}(\Omega)}$ is an equivalent norm on $\bigcap_{i=1}^{N} W_0^{1,p_i(y)}(\Omega)$.

Proof. See [21, Theorem 2.3] and [13]. □

We naturally introduce the functional space

$$W_0 = \{v \in L^{p_i^-}(0,T,W_0^{1,p_i(y)}(\Omega)),\ |\partial_i v| \in L^{p_i(y)}(Q)\},$$

which endowed with the norm

$$\|v\|_{w_0} := \|\partial_i v\|_{L^{p_i(y)}(Q)},$$

or, the equivalent norm

$$\|v\|_{w_0} := \|v\|_{L^{p_i^-}(0,T,W_0^{1,p_i(y)}(\Omega))} + \|v\|_{L^{p_i(y)}(Q)}$$

is a separable and Banach space.

Remark 2.4. . Let $\Omega \subseteq \mathbb{R}^N, Q = (0,T) \times \Omega$, and $p_i : \Omega \to (1,\infty)$ be a continuous function. We have the following continuous dense embeddings

$$L^{p_i^+}\left(0,T;L^{p_i(y)}(\Omega)\right) \hookrightarrow L^{p_i(y)}(Q) \hookrightarrow L^{p_i^-}\left(0,T;L^{p_i(y)}(\Omega)\right).$$

Proof. See [3, Remark 3.1]. □

2.3. Measures and parabolic capacity

Let $Q = (0,T) \times \Omega$ for each fixed $T > 0$. It is worth noting that $V = \bigcap_{i=0}^{N} W_0^{1,p_i(y)}(\Omega) \cap L^2(\Omega)$ equipped with its appropriate norm $\|\cdot\|_{W_0^{1,p_i(y)}} + \|.\|_{L^2(\Omega)}$, the space $W_{p_i(y)}(0,T)$ setting by

$$W_{p_i(y)}(0,T) = \Big\{v \in L^{p_i^-}(0,T,V);\ \partial_i v \in (L^{p_i(y)}(Q))^N \text{ and } v_t \in L^{(p_i^-)'}(0,T,V'),\ \text{for all } i = 1,\dots,N\Big\},$$

equipped with the following norm

$$\|v\|_{W_{p_i(y)}(0,T)} = \|v\|_{L^{p_i^-}(0,T,V)} + \|\partial_i v\|_{(L^{p_i(y)}(Q))^N} + \|v_t\|_{L^{(p_i^-)'}(0,T,V')}.$$

Note that $W_{p_i(y)}(0,T) \hookrightarrow C([0,T],L^2(\Omega))$ continuously. We introduce the (generalized) parabolic capacity of a set $\mathcal{D}$ in Q as

$$cap_{p_i(y)}(\mathcal{D}) = \inf\Big\{\|v\|_{W_{p_i(y)}(0,T)} : \mathcal{D} \in W_{p_i(y)}(0,T),\ s \geq \chi_{\mathcal{D}} \text{ a.e. in } Q\Big\}.$$

This capacity can be extended to Borel sets $\mathcal{B} \subseteq Q$ as

$$cap_{p_i(y)}(\mathcal{B}) = \inf\Big\{cap_{p_i(y)}(\mathcal{D}) : \mathcal{D} \text{ open subset of } Q, \mathcal{B} \subseteq \mathcal{D}\Big\}.$$

In the following , $\mathcal{M}_b(Q)$ represents the set of Radon measures characterized by a bounded variation on the set Q, while $\mathcal{M}_0(Q)$ is defined as follows

$$\mathcal{M}_0(Q) = \Big\{\mu \in \mathcal{M}_b(Q) : \mu(E) = 0 \text{ for every } E \subset Q \text{ such that } cap_{p_i(y)}(E) = 0\Big\}.$$

To better understand the nature of a measure in $\mathcal{M}_0(Q)$, we detail the structure of the dual space $(W_{p_i(y)}(0,T))'$.

We can define the Marcinkiewicz space $\mathcal{M}_0^{q_i(y)}(Q)$ for every $0 < q_i(y) < \infty$ and for $i = 0,\dots,N$ as the space of measurable functions g satisfying the following condition

$$\exists C > 0 \ \text{ with } \ meas\big\{(t,y) \in Q \quad |g(t,y)| \geq h\big\} \leq \frac{C}{h^{q_i^-}}.$$

This space is equipped with the semi-norm

$$\|g\|_{\mathcal{M}_0^{q_i(y)}(Q)} = \inf\Big\{C > 0 : meas\big\{(t,y) : |g(t,y)| \geq h\big\} \leq \Big(\frac{C}{k}\Big)^{q_i(y)}\Big\}.$$

It's important to note that if $q_i(y) \geq q_i^- > 1$, we have the following continuous embedding

$$L^{q_i(y)}(Q) \hookrightarrow \mathcal{M}_0^{q_i(y)}(Q) \hookrightarrow L^{q_i(y)-\varepsilon}(Q), \text{ for all } \varepsilon \in (0, q_i(y)-1].$$

The standard p-capacity of a Borel set $E \subset Q$ is then defined by

$$cap_p(y)(E,Q) = \inf\{\|v\|_{W_0} \text{ with } v \in W_0 \text{ and } v \geq 1 \text{ a.e. in a neighbourhood of } E\}.$$

A function v is said to be $cap_p(y)$-quasi continuous if for every $\varepsilon > 0$ there exists an open set $E \subset Q$ such that $\text{cap}_{p(y)}(E) < \varepsilon$ and $v_{|Q\setminus E}$ is continuous in $Q\setminus E$. Moreover, for every $v \in W_0$ there exist a $cap_p(y)$-quasi continuous representative $\tilde{v}$ yielding $v = \tilde{v}$ a.e. in Q.
We will also use the truncation function and its auxiliary function defined by

$$T_k(r) = \max\{-k, min(k,r)\}, \qquad \Theta_k(r) = T_1(r - T_k(r)).$$

and define $\Lambda_K(r) = \int_0^r T_k(s)ds$ as the primitive function of the truncation function.

Proposition 2.5. *Any weak solution of $(\mathcal{P})$ with the initial datum $v_0 \in L^1(\Omega)$ satisfies the following estimates*

$$\|v\|_{\mathcal{M}_0^{p_i(y)-1+\frac{p_i(y)}{N}}(Q)} \leq C_1, \quad \|\partial_i v\|_{\mathcal{M}_0^{p_i(y)-\frac{N}{N+1}}(Q)} \leq C_2, \tag{3}$$

where $C_j,\ j = 1,\ 2,$ are positive constants that depend solely on $v_0,\ \mu,\ N,\ T,$ and the range of $p_i(y)$ such that $p_i^- < p_i(y) < p_i^+$, for $i = 0,\dots,N$.

Proof. Let $v^\varepsilon(y,t)$ as the entropy solution of the initial boundary value problem

$$\begin{cases} (v_\varepsilon)_t - \sum_{i=1}^{N} \partial_i[a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon)] = \mu_\varepsilon \quad in\ Q, \\ \\ v_\varepsilon(t,y) = 0 \quad on\ (0,T)\times\partial\Omega, \\ v_\varepsilon(0,y) = v_0^\varepsilon(y)\ in\ \Omega. \end{cases}$$

Let us now seek some a priori estimates for the sequence v_i^ε for all $i = 1,\dots,N$. Throughout, the symbol C will represent a generic positive constant, which may vary from one step to another.
To proceed, we fix ε and set $\varphi = 0$ in the entropy formulation for v^ε, yielding the following

$$\begin{aligned} &\int_\Omega \Lambda_k(v^\varepsilon)(1) + \alpha\sum_{i=0}^{N}\int_Q |\partial_i T_k(v^\varepsilon)|^{p_i(y)}\,dydt \\ &\leq k\left(|\mu|_{\mathcal{M}_0(Q)} + \|v_i(y,\varepsilon)\|_{L^1(\Omega)}\right) \\ &\leq k\left(|\mu|_{\mathcal{M}_0(Q)} + \|v_i\|_{L^1(\Omega)}\right) = Ck \end{aligned} \tag{4}$$

Thus, for any fixed $k > 0$, starting from the first term on the left-hand side of (4), and recalling that $v_i^\varepsilon(t,y)$ is nondecreasing in t, we can deduce, following a similar reasoning to [23, Theorem 1.7], that v_i^ε is uniformly bounded in

$L^\infty\left(0,1;L^1(\Omega)\ \right)$. Moreover, from the second term, we conclude $T_k\left(v_i^\varepsilon\right)$ is uniformly bounded in $L^{p_i(y)}\left(0,1;W_0^{1,p_i(y)}(\Omega)\right)$ for all $i=0,\ldots,N$, and for any fixed $k>0$.
We can obtain a better estimate by using a Gagliardo-Nirenberg type inequality which allows us to conclude that,
if $u\in L^{q_i(y)}\left(0,T;W_0^{1,q_i(y)}(\Omega)\right)\cap L^\infty\left(0,T;L^{\rho_i(y)}(\Omega)\right)$, with $q_i(y)\geq 1,\rho_i(y)\geq 1$. Then $u\in L^{\sigma_i(y)}(Q)$ with $\sigma_i(y)=q_i(y)\frac{N+\rho_i(y)}{N}$ and

$$\int_Q |u_i|^{\sigma_i(y)}dydt\leq C\|u_i\|_{L^\infty(0,T;L^{p_i(y)}(\Omega))}^{\frac{\rho q_i(y)}{N}}\int_Q |\partial_i u|^{q_i(y)}dydt$$

In fact, we obtain

$$\int_Q |T_k\left(v^\varepsilon\right)|^{p_i(y)+\frac{p_i(y)}{N}}\,dydt\leq Ck \tag{5}$$

and therefore we have that

$$\begin{aligned}k^{p_i(y)+\frac{p_i(y)}{N}}\,\mathrm{meas}\left\{|v_i^\varepsilon|\geq k\right\}&\leq \int_{\left\{|v_i^\varepsilon|\geq k\right\}}|T_k\left(v_i^\varepsilon\right)|^{p_i(y)+\frac{p_i(y)}{N}}\,dydt\\&\leq \int_Q |T_k\left(v_i^\varepsilon\right)|^{p_i(y)+\frac{p_i(y)}{N}}\,dydt\leq Ck\end{aligned}$$

thus,

$$\mathrm{meas}\left\{|v_i^\varepsilon|\geq k\right\}\leq \frac{C}{k^{p_i(y)-1+\frac{p_i(y)}{N}}} \tag{6}$$

As a result, the sequence v_i^ε is uniformly bounded in the Marcinkiewicz space $\mathcal{M}_0^{p_i(y)-1+\frac{p_i(y)}{N}}(Q)$; This, in turn, implies that, since $p_i(y)>\frac{2N}{N+1}$ he sequence v_i^ε is uniformly bounded $L^{m_i(y)}(Q)$ for all $i=1,\ldots,N$ and for every $1\leq m_i(y)<p_i(y)-1+\frac{p_i(y)}{N}$.
Next, we focus on a similar estimate for the gradients of the functions v_i^ε. It is important to note that these estimates apply to any function satisfying (4), and therefore we will omit the index ε for simplicity. First, observe that

$$\mathrm{meas}\{|\partial_i v|\geq\lambda\}\leq\mathrm{meas}\{|\partial_i v|\geq\lambda;|v_i|\leq k\}+\mathrm{meas}\{|\partial_i v|\geq\lambda;|v_i|>k\} \tag{7}$$

for all $i=1,\ldots,N$. Now, for the first term on the right-hand side of (8), we get

$$\begin{aligned}&\mathrm{meas}\{|\partial_i v|\geq\lambda;|v_i|\leq k\}\leq\frac{1}{\lambda^{p_i()}}\int_{\{|\partial_i v|\geq\lambda;|v_i|\leq k\}}|\partial_i v_i|^{p_i(y)}dy \qquad (8)\\&=\frac{1}{\lambda^{p_i(y)}}\int_{\{|v_i|\leq k\}}|\partial_i v|^{p_i(y)}dy=\frac{1}{\lambda^{p_i(y)}}\int_Q|\partial_i T_k(v)|^{p_i(y)}\,dy\leq\frac{Ck}{\lambda^{p_i(y)}}\end{aligned}$$

For the second term in (8), by applying (6), we obtain

$$\text{meas}\{|\partial_i v| \geq \lambda; |v_i| > k\} \leq \text{meas}\{|v_i| \geq k\} \leq \frac{\bar{C}}{k^{\sigma_i(y)}},$$

where $\sigma_i(y) = p_i(y) - 1 + \frac{p_i(y)}{N}$. Thus, combining both terms, we infer

$$\text{meas}\{|\partial_i v| \geq \lambda\} \leq \frac{\bar{C}}{k^{\sigma_i(y)}} + \frac{Ck}{\lambda^{p_i(y)}}.$$

A better estimate can be achieved by minimizing the right-hand side with respect to k. The optimal value of k is given by

$$k_0 = \left(\frac{\sigma_i(y)C}{\bar{C}}\right)^{\frac{1}{\sigma_i(y)+1}} \lambda^{\frac{\sigma_i(y)}{\sigma_i(y)+1}},$$

which leads to the desired estimate

$$\text{meas}\{|\partial_i v| \geq \lambda\} \leq C\lambda^{-\gamma_i(y)},$$

where$\gamma_i(y) = p_i(y)\left(\frac{\sigma_i(y)}{\sigma_i(y)+1}\right) = \frac{Np_i(y)+p_i(y)-N}{N+1} = p_i(y) - \frac{N}{N+1}$.
Returning to our context, we conclude that for all $\varepsilon \geq 0, |\partial_i v^\varepsilon|$ is equi-bounded bounded in $\mathcal{M}_0^{\gamma_i(y)}(Q)$, where $\gamma_i(y) = p_i(y) - \frac{N}{N+1}$. Moreover, since $p_i(y) > \frac{2N+1}{N+1}$, we have that $|\partial_i v^\varepsilon|$ is uniformly bounded in $L^{s_i(y)}(Q)$ for all $1 \leq s_i(y) < p_i(y) - \frac{N}{N+1}$. □

3. Assumptions and technical results

3.1. Assumption and lemmas

Our work is based on the following assumptions. We are concerned with the problem represented by

$$\begin{cases} v_t - \sum_{i=1}^{N} \partial_i\big[a_i(t,y,v,\partial_i v)\big] = \mu \quad in\ Q = (0,T)\times\Omega, \\ v(t,y) = 0 \quad on\ (0,T)\times\partial\Omega,\ v(0,y) = v_0(y) \quad in\ \Omega, \end{cases} \tag{9}$$

where μ is a bounded Radon measure on Q, $v_0 \in L^1(\Omega)$ and $a_i : \Omega \times [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $i = 0,\ldots,N$ is a Carathéodory function satisfying the following condition, there exist $\theta \in L^{p_i(y)}(Q)$ and $\alpha,\ \beta > 0$ such that, for each $(t,y) \in Q$ and all $(v,\xi_i) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$a_i(t,y,v,\xi_i)\cdot\xi_i \geq L(|v|)|\xi_i|^{p_i(y)}, \tag{10}$$

$$|a_i(t,y,v,\xi_i)| \leq \beta[\theta(t,y) + L(|v|)|\xi_i|^{p_i(y)-1}], \tag{11}$$

$$[a_i(y,t,v,\xi_i) - a_i(y,t,v,\eta)](\xi_i - \eta_i) > 0 \quad \text{for all } \xi_i \neq \eta_i. \tag{12}$$

Furthermore, the function L satisfies

$$L(|v|) \geq \alpha > 0, \; for\ all\ v \in \mathbb{R}^N. \tag{13}$$

In this section, we will present the most important technical results that will be needed for the rest of the article. In our study we are mainly interested in measurable functions having truncations in the energy space $L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))$. To this aim, let us define $\mathcal{T}_0^{1,\vec{p}(y)}(\Omega)$ as the set of measurable functions $v : Q \to \mathbb{R}$ such that $T_k(v)$ belongs to $L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))$ for every $k > 0$ and $i = 1,\ldots,N$.

Lemma 3.1. *Let $v \in \mathcal{T}_0^{1,\vec{p}(y)}(Q)$, where $i = 1,\ldots,N$. Then, there exists a unique measurable function $u : Q \mapsto \mathbb{R}^N$ such that $\partial_i Tk(v) = u\chi_{|v|\leq k}$ a.e. in Q for each $k > 0$, where χ_E is the characteristic function of the measurable set E. Furthermore, if*

$$\sum_{i=1}^{N} \int_Q |\partial_i T_k(v)|^{p_i(y)} dydt \leq C(k+1),$$

then u coincides with the classical gradient of v and is denoted as $\partial_i u = v$. where v is $cap_{p_i(y)}$ almost everywhere finite, i.e. $cap_{p_i(y)}\{(t,y) \in Q \,:\, |v(t,y)| = +\infty\} = 0$, and there exists a $cap_{p_i(y)}-$quasi-continuous representative. of v, namely a function $\tilde{v}$ such that $\tilde{v} = v$ almost everywhere in Q and $\tilde{v}$ is $cap_{p_i(y)}$-quasi continuous.

Proof. A similar argument to that presented in [4, Lemma 2.1] can be applied. Based on the proof of this lemma, we have established that the following formula holds

$$\partial_i T_k(v) = u_i \chi_{|v_i|\leq k} \text{ a.e. in } Q, \tag{14}$$

for each $k > 0$ and for all $i = 1,\ldots,N$, where $v_i \in W_{\text{loc}}^{1,P_i(y)}(\Omega)$ and $u_i = \partial_i v$. Additionally, for any $k,\varepsilon > 0$, we have $T_k(T_{k+\varepsilon}(v_i)) = T_k(v_i)$. Hence, $\Omega_k = |v_i| < k$, we obtain for almost everywhere $\partial_i T_{k+\varepsilon}(v) = \partial_i T_k(v_i)$. Since $\bigcup_{k>0} \Omega_k = \Omega$, the assertion (14) follows.

We now need to prove that $v_i \in W_{\text{loc}}^{1,p_i(y)}(\Omega)$ if $u_i \in L_{\text{loc}}^{P_i(y)}(\Omega)$. Indeed, under this condition, $\partial_i T_k(v) \to u_i$ in $L_{\text{loc}}^{p_i(y)}(\Omega)$. Furthermore, we must show that $v_i \in$

$L^{p_i(y)}_{\text{loc}}(\Omega)$. If this were not the case, then there would exist a closed ball $B \subset \Omega$ such that

$$S_k = \|T_k(u)\|_{L^{p_i(y)}(B)} \to \infty$$

as $k \to \infty$. By normalizing, let $u_i^k = T_k(v_i)/S_k$. Then, $u_i^k \to 0$ almost everywhere, $\|u_i^k\|_{L^{p_i(y)}(B)} = 1$, and $\|\partial_i u_i^k\|_{L^{p_i(y)}(B)} \to 0$. This contradicts the compactness of the embedding $W^{1,p_i(y)}(B) \subset L^{p_i(y)}(B)$. □

Definition 3.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset where $N \geq 2$ and $\mu \in \mathcal{M}_0(Q)$ (the space of Radon measures on Q with total bounded variation). A measurable function $v \in C([0,T];L^1(\Omega))$ is a weak solution of the problem (1), if $a_i(t,y,v,\partial_i v) \in L^1(Q)^N$, $T_k(v) \in L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))$ for $i = 1,\dots,N$, and

$$\int_0^T \langle v_t, \varphi\rangle dt + \sum_{i=1}^N \int_Q a_i(t,y,v)\Big(1+|v_\varepsilon|\Big)^{s_i(y)} |\partial_i v_\varepsilon|^{p_i(y)-2}\partial_i v_\varepsilon \cdot \partial_i \varphi dydt$$
$$= \int_Q f_\varepsilon \varphi d\mu, \text{ for all } \varphi \in C_c^\infty(Q). \quad (15)$$

Lemma 3.3. *[8, Lemma 7.1] Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1-function such that $g(0) = 0$ and g' is zero away from a compact set of $\mathbb{R}$.*
Let us denote $G(r) = \int_0^r g(\delta)d\delta$. If $v \in L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))$ is such that $v_t \in L^{(p_i^-)'}(0,T;W^{-1,p_i'(y)}(\Omega)) + L^1(Q)$ and if $\psi \in C^\infty(\bar{Q})$, for $i = 0,\dots,N$, then we have

$$\int_0^T \langle v_t, g(v)\psi\rangle\, dt = \int_\Omega G(v(T))\psi(T)dy - \int_\Omega G(v(0))\psi(0)dy - \int_Q \psi_t G(v)dydt.$$

3.2. Technical results

It is crucial to introduce a key element in our arguments, namely a generalised existence result that extends the results established in [27] to cases involving measure data. To achieve this, we need to introduce an approximation problem for each natural number ε, along with its associated properties, which will play a crucial role in our subsequent analysis. It's worth noting that both μ and v can be effectively approximated by sequences of smooth functions (μ_ε) and (v_0^ε) generated by convolution. To understand the convolution functions used in our problem, let's start by defining the notion of convolution, and then specify the convolution functions applied to μ and v_0.
The convolution of a function f with a function η (often called the convolution kernel) is defined by :

$$(f * \eta)(y) = \int_{\mathbb{R}^n} f(y)\eta(y-z)dz.$$

The function μ is approximated by a sequence of smooth functions μ_ε obtained through convolution with a regularizing kernel η_ε. Generally, the kernel η_ε is used, such that:

$$\eta_\varepsilon(y) = \frac{1}{\varepsilon^n}\eta\left(\frac{y}{\varepsilon}\right),$$

where η is a smooth function with compact support, and it satisfies $\eta(y) \geq 0$ for all $\in \mathbb{R}^n$, and $\int_{\mathbb{R}^n}\eta(y)\,dy = 1$. Thus, μ_ε is defined by the convolution $\mu_\varepsilon = \mu * \eta_\varepsilon$. Similarly, the initial function v_0 is approximated by a sequence of smooth functions v_0^ε obtained through convolution with a regularizing kernel η_ε. This means that $v_0^\varepsilon = v_0 * \eta_\varepsilon$.

We will now examine the behaviour of the sequence (v_ε) which consists of solutions to the following problems

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} (v_\varepsilon)_t - \sum_{i=1}^{N} \partial_i[a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon)] = \mu_\varepsilon \quad in\ Q, \\ \\ v_\varepsilon(t,y) = 0 \quad on\ (0,T)\times\partial\Omega, \quad v_\varepsilon(0,y) = v_0^\varepsilon(y)\ in\ \Omega. \end{cases}$$

Proposition 3.4. *Let* $1 < p_i(y) < N$, $\mu \in \mathcal{M}_0(Q)$ *and assume that* $a_i(t,y,s,\xi)$ *satisfies (10)-(13). Then, there exists a function* $v \in \mathcal{T}_0^{1,\vec{p}(y)}(\Omega)$ *with* $a_i(t,y,v,\partial_i v)$ *belongs to* $L^{q_i(y)}(Q)$ *for all* $q_i(y) < p_i(y) - \dfrac{N}{N+1}$, $i = 0,\ldots,N$ *and* v *verifies (15) in the sense of Definition 3.2.*

Proof. Let μ_ε be a sequence of $C_c^\infty(Q)$-functions such that $\mu_\varepsilon \to \mu$ *tightly in* $\mathcal{M}_0(Q)$, with $\|\mu_\varepsilon\|_{L^1(Q)} \leq \|\mu\|_{\mathcal{M}_0(Q)}$, and (v_0^ε) a sequence of $C_c^\infty(\Omega)$-functions such that $v_0^\varepsilon \to v_0$ *in* $L^1(\Omega)$, with $\|v_0^\varepsilon\|_{L^1(\Omega)} \leq \|v_0\|_{L^1(\Omega)}$. In addition, let v_ε be a weak solution of the problem $(\mathcal{P}_\varepsilon)$. Observe that, according to the results of [20], there is one weak solution for $(\mathcal{P}_\varepsilon)$, i.e., a function $v_\varepsilon \in L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))$ such that $(v_\varepsilon)_t \in L^{p_i^-}(0,T;W^{-1,p_i(y)}(\Omega)) \cap L^\infty(Q)$ an the following identity holds true

$$\int_0^t \langle (v_\varepsilon)_t, \varphi\rangle dt + \sum_{i=1}^{N}\int_0^t\int_\Omega a_i(s,y,v_\varepsilon,\partial_i v_\varepsilon)\cdot\partial_i\varphi\, dy\, ds = \int_0^t\int_\Omega \varphi\, d\mu_\varepsilon, \tag{16}$$

by taking $\varphi = T_k(v_\varepsilon)$, for each $\varphi \in \bigcap_{i=1}^{N} L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega)) \cap L^\infty(Q)$ in (16) and integrating in $]0,T[$ we have

$$\int_\Omega \Theta_k(v_\varepsilon)(t)\,dy + \sum_{i=1}^{N}\int_0^t\int_\Omega a_i(s,y,v_\varepsilon,\partial_i v_\varepsilon)\cdot\partial_i T_k(v)\,dy\,ds$$
$$= \int_0^t\int_\Omega T_k(v_\varepsilon)\,d\mu_\varepsilon + \int_\Omega \Theta_k(v_0^\varepsilon)\,dy,$$

which implies, from (10) and as $\|v_0^\varepsilon\|_{L^1(\Omega)}$ and $\|\mu_\varepsilon\|_{L^1(Q)}$ are bounded, that

$$\int_\Omega \Theta_k(v_\varepsilon)(t)dy + \alpha \sum_{i=1}^N \int_0^t \int_\Omega |\partial_i T_k(v_\varepsilon)|^{p_i(y)} dyds \\ \leq k(\|\mu\|_{\mathcal{M}_0(Q)} + \|v_0^\varepsilon\|_{L^1(\Omega)}) = Ck.$$

As $\Theta_k(\ell) \geq 0$ and $|\Theta_1(\ell)| \geq |\ell| - 1$, we obtain

$$\int_\Omega |v_\varepsilon|(t)dy + \alpha \sum_{i=1}^N \int_0^t \int_\Omega |\partial_i T_k(v_\varepsilon)|^{p_i(y)} dydt \\ \leq C(k+1), \text{ for all } k > 0, \ \forall t \in [0,T],$$

choosing the supremum on $(0,T)$, we have

$$\int_\Omega |v_\varepsilon|(t)dy \leq C, \ for \ all \ t \in [0,T], \tag{17}$$

which gives an estimate of v_ε in $L^\infty(0,T;L^1(\Omega))$ and also

$$\sum_{i=1}^N \int_Q |\partial_i T_k(v_\varepsilon)|^{p_i(y)} dydt \leq C(k+1), \ for \ i = 0, \ldots, N. \tag{18}$$

This means that for each $k > 0$, $T_k(v)$ is bounded in $\bigcap_{i=1}^N L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))$. As a result, there exists a function $v \in \mathcal{T}_0^{1,\vec{p}(y)}(\Omega)$ such that, up to subsequences,

$$\begin{cases} v_\varepsilon \to v \ a.\ e. \text{ in } Q, \\ T_k(v_\varepsilon) \rightharpoonup T_k(v) \text{ weakly in } L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega)), \\ \ \text{strongly in } L^{p_i(y)}(Q) \text{ and a.e. in } Q. \end{cases} \tag{19}$$

Choosing $\varphi = T_k(B(v_\varepsilon))$ with $B(\ell) = \int_0^\ell b(|\sigma|)^{\frac{1}{p_i(y)-1}} d\sigma$ as test function in the weak formulation of (16) for each $p_i(y) > 1$ (which is an eligible choice because $T_k(B(v_\varepsilon)) \in L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))$). According to the definition of $T_k(\ell)$, we obtain

$$\int_0^T \langle (v_\varepsilon)_t, T_k(B(v_\varepsilon))\rangle dt + \sum_{i=1}^N \int_Q a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon)\partial_i v_\varepsilon b(|v_\varepsilon|)^{\frac{1}{p_i(y)-1}} dydt \\ \leq \int_Q T_k(B(v_\varepsilon)) d\mu_\varepsilon,$$

applying (10), we get (by fixing $\Lambda(\ell)=\displaystyle\int_0^\ell T_k(B(\sigma))d\sigma$) that

$$\int_\Omega \Lambda(v_\varepsilon)(t)dy+\sum_{i=1}^N\int_{\{|B(u_\varepsilon)|\le k\}}|b(|v_\varepsilon|)|^{p_i'(y)}|\partial_i v_\varepsilon|^{p_i(y)}dydt \le k\Big[\|\mu_\varepsilon\|_{\mathcal{M}_0(Q)}+\int_\Omega \Lambda(v_0^\varepsilon)dy\Big],$$

and since μ_ε is bounded in $L^1(Q)$ and v_0^ε is bounded in $L^1(\Omega)$ we obtain

$$\int_\Omega \Lambda(v_\varepsilon(t))dy\le C,\ for\ all\ t\in[0,T],$$

which means that $|\partial_i B(v_\varepsilon)|$ is bounded in the Marcinkiewicz space

$$\mathcal{M}_0^{p_i(y)-1+\frac{p_i(y)}{N}}(Q).$$

Hence $b(|v_\varepsilon|)|\partial_i v_\varepsilon|^{p_i(y)-1}$ is bounded in the Marcinkiewicz space

$$\mathcal{M}_0^{1+\frac{p_i(y)N-N+p_i(y)}{N(p_i(y)-1)}}(Q),$$

and as

$$\Big\{(t,y):|a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon)|>k\Big\}\subset\Big\{(t,y):\beta(\theta(t,y)+b(|v_\varepsilon|)|\partial_i v_\varepsilon|^{p_i(y)-1})>k\Big\},$$

then, $a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon)$ is bounded in $L^{q_i(y)}(Q)$, but we can not yet prove that its weak limit is $a_i(t,y,v,\partial_i v)$; this will be accomplished by demonstrating that $\partial_i v_\varepsilon$ converges to $\partial_i v$ almost everywhere. For this, we will employ the technique used in [26], with minor alterations related to the hypothesis (11). For $m,\ k>0$, we choose $T_m(v_\varepsilon-T_k(v))$ as the test function in the weak formulation of $(\mathcal{P}_\varepsilon)$. Since, $\|\mu_\varepsilon\|_{L^1(Q)}\le C_0$ (we will designate from now on by C_i positive constants independent of ε and m), we obtain

$$\sum_{i=1}^N\int_Q a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon)\cdot\partial_i T_m(v_\varepsilon-T_k(v))dydt \tag{20}$$
$$\ge\sum_{i=1}^N\int_Q a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v_\varepsilon))\cdot\partial_i T_m(T_k(v_\varepsilon)-T_k(v))dydt$$
$$-\sum_{i=1}^N\int_{\{|v_\varepsilon|>k\}}|a_i(t,y,T_{k+m}(v_\varepsilon),\ \partial_i T_{k+m}(v_\varepsilon))||\partial_i T_k(v)|dydt.$$

Using (19), $|\partial_i T_k(v)|\chi_{|v_\varepsilon|>k}$ converges strongly to zero in $L^{p_i(y)}(Q)$, by tending ε to infinity, the last term goes to zero for each $m>0$ fixed. This means, by (20), that

$$\sum_{i=1}^{N}\int_Q a_i(t,y,T_k(v_\varepsilon)),\ \partial_i T_k(v_\varepsilon))\cdot\partial_i T_m(T_k(v_\varepsilon)-T_k(v_\varepsilon))dydt \leq\ m\,C_0+\varpi_m(\varepsilon). \quad (21)$$

On the other hand, let's $0<\varsigma_i(y)<1$ and $E_k^m=\{(t,y)\in Q\ :\ |T_k(v_\varepsilon)-T_k(v)|>m\}$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_Q\Big[[a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v_\varepsilon))\\
&\qquad - a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v)]\partial_i(T_k(v_\varepsilon)-T_k(v))\Big]^{\varsigma_i(y)}dydt\\
&=\sum_{i=1}^{N}\int_Q\Big[[a_i(t,y,T_k(v_\varepsilon),\ \partial_i T_k(v_\varepsilon))-a_i(t,y,T_k(v_\varepsilon),\ \partial_i T_k(v))]\\
&\qquad\times\partial_i T_m(T_k(v_\varepsilon)-T_k(v))\Big]^{\varsigma_i(y)}dydt\\
&+\sum_{i=1}^{N}\int_{E_k^m}\Big[[a_i(t,y,T_k(v_\varepsilon),\ \partial_i T_k(v_\varepsilon))-a_i(t,y,T_k(v_\varepsilon),\ \partial_i T_k(v))]\\
&\qquad\times\partial_i(T_k(v_\varepsilon)-T_k(v))\Big]^{\varsigma_i(y)}dydt,
\end{aligned}$$

Hence, combined (21) with Hölder's inequality of exponent $\frac{1}{\varsigma_i(y)}$, we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_Q\Big[[a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v_\varepsilon))-a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v))]\\
&\times\partial_i(T_k(v_\varepsilon)-T_k(v))\Big]^{\varsigma_i(x)}dydt\leq\ \text{meas }(Q)^{1-\varsigma^+}\Big[mc_0+\varpi_m(\varepsilon)\\
&-\sum_{i=1}^{N}\int_Q a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v))\partial_i T_m(T_k(v_\varepsilon)-T_k(v))\Big]^{\varsigma_i(y)}dydt\\
&+\sum_{i=1}^{N}\int_{E_k^m}\Big[\big(a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v_\varepsilon))-a_i(t,y,T_k(v_\varepsilon),\ \partial_i T_k(v))\big)\\
&\qquad\times\partial_i(T_k(v_\varepsilon)-T_k(v))\Big]^{\varsigma_i(y)}dydt,
\end{aligned}\quad (22)$$

As $T_k(v_\varepsilon)$ converges to $T_k(v)$ in $L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))$ By applying the hypothesis $(\mathcal{P}_\varepsilon)$, it is simple to show that the first term on the right hand side of (22) becomes zero when ε approaches infinity. As a result, the sequence $\left(\left|a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v_\varepsilon))-a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v))\right|\right)_\varepsilon$ is bounded in $L^{p_i(y)}(Q)$ for each $m>0$, and for each $i=1,\ldots,N$, then by Hölder's inequality, we obtain

$$\sum_{i=1}^{N}\int_Q\Big[\big(a_i(t,\varepsilon,y,T_k(v_\varepsilon),\partial_i T_k(v_\varepsilon))-a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v))\big)\Big]$$
$$\times\,\partial_i\Big(T_k(v_\varepsilon)-T_k(v)\Big)^{\varsigma_i(y)}dydt\leq \text{ meas }(Q)^{1-\varsigma_i^+}[mc_0+2\varpi_m(\varepsilon)]^{\varsigma_i^-}$$
$$+c_1[\text{ meas }\{(t,y)\in Q\;:\;|T_k(v_\varepsilon)-T_k(v)|>m\}]^{1-\varsigma_i^+}.$$

By tending ε to infinity and then m to zero, and since $T_k(v_\varepsilon)$ converges in measure to $T_k(v)$, we state that

$$\lim_{\varepsilon\to\infty}\sum_{i=1}^{N}\int_Q\Big[\big((a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v_\varepsilon))-a_i(t,y,T_k(v_\varepsilon),\partial_i T_k(v))\big)$$
$$\times\,\partial_i(T_k(v_\varepsilon)-T_k(v))\Big]^{\varsigma_i(y)}dydt=0,$$

we conclude, by reasoning as in [26], for $i=1,\ldots,N$ that $\partial_i T_k(v_\varepsilon)$a.e. converges to $\partial_i T_k(v)$ for all $k>0$, in fact, $\partial_i v_\varepsilon$ converges to $\partial_i v$ a.e. in Q, which proves that

$$a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon)\to a_i(t,y,v,\partial_i v)\text{ strongly in }L^{q_i(y)}(Q),$$

for all $q_i(y)<p_i(y)-\frac{N}{N+1}$. Finally, by passing to the limit, tending v to infinity, in the weak formulation of (16) for each $\varphi\in C_c^\infty([0,T]\times\Omega)$ to conclude that v satisfies (15) in the distributional sense (this means that v is a weak solution of (9) and this concludes the proof of Proposition 3.4. □

4. Main result and proof

In this section we define the notion of weak solution to problem (1) and we give the existence result for such a solution

Theorem 4.1. *Assume that a_i satisfies (11)-(13), $\mu\in\mathcal{M}_0(Q)$ and $v_0\in L^1(\Omega)$. Let $q_i(y)>1$, $s_i(y)\geq 0$, $2-\frac{1}{N+1}<p_i(y)<N$ and suppose that there are positive constants ζ^- and m_0, where*

$$b(|m|)\geq\zeta^-|m|^{s_i(y)},\quad \textit{for all } m\in\mathbb{R}:\;|m|>m_0. \tag{23}$$

Then, (1) *has a weak solution v such that*

(i) if $s_i(y)>1$, then $v\in\bigcap_{i=1}^{N}W_0\cap L^{\zeta_i(y)}(Q)$ for each $\zeta_i^-<\frac{(p_i^-N+p_i^--N)(s_i^-+1)}{N+1}$,

(ii) *if* $0 \le s_i^- \le s_i(y) \le s_i^+ \le 1$ *and* $p_i^- > 2 - \dfrac{1+s_i^-(N-1)}{N}$, *then we have* v *belongs to* $\bigcap_{i=1}^N L^{q_i^-}(0,T;W_0^{1,q_i(y)}(\Omega))$ *for every* $q_i^- < \frac{N(p_i^- - 1 + s_i^-)}{N-(1-s_i^-)}$.

In addition, if μ *is a function in* $\bigcap_{i=1}^N L^{\sigma_i'(y)}(Q)$ *where* $1 < \sigma_i(y) < (p_i^\star(y))'$, *then* (1) *has a weak solution* v *such that*

(iii) *if* $s_i(y) > 1 - \dfrac{p_i^\star(y)}{\sigma_i'(y)}$, *then* $v \in \bigcap_{i=1}^N W_0 \cap L^{\frac{(N(p_i^-+1)+p_i^-)\sigma_i^-(s_i^-+1)}{N+p_i^- - \sigma_i^- p_i^-}}(Q)$.

(iv) *if* $0 \le s_i(y) < 1 - \dfrac{p_i^\star(y)}{\sigma_i'(y)}$ *and* $p_i^- > max\left\{1, 2 - \frac{\sigma_i^-(1+s_i^-(N-1))+N(\sigma_i^- -1)}{N\sigma_i^-}\right\}$, *then* v *belongs to* $\bigcap_{i=1}^N L^{q_i^-}(0,T;W_0^{1,q_i(y)}(\Omega))$ *such that* $q_i^- = \frac{N\sigma_i^-(p_i^- - 1 + s_i^-)}{N - \sigma_i^-(1-s_i^-)}$.

Proof. To prove Theorem 4.1, let $\mu \in \mathcal{M}_0(Q)$ and $v_0 \in L^1(\Omega)$. Consider the two sequences (v_0^ε) of $L^\infty(\Omega)$-functions and (f_ε) of $L^{p'(y)}(Q)$-functions satisfying

$$\begin{cases} f_\varepsilon \to \mu \ \text{ in the weak* topology of measures } \ \textit{and} \quad \|f_\varepsilon\|_{L^1(Q)} \le C, \\ v_0^\varepsilon \to v_0 \quad \textit{in } L^1(\Omega) \quad \textit{and} \quad \|v_0^\varepsilon\|_{L^1(\Omega)} \le C. \end{cases} \tag{24}$$

Consider that v_ε is the weak solution of $(\mathcal{P}_\varepsilon)$

$$(\mathcal{P}_\varepsilon) \begin{cases} (v_\varepsilon)_t - \displaystyle\sum_{i=1}^N [a_i(t,y,v_\varepsilon)(1+|v_\varepsilon|)^{s_i(y)}|\partial_i v_\varepsilon|^{p_i(y)-2}\partial_i v_\varepsilon] = f_\varepsilon \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad Q := (0,\, T) \times \Omega, \\ v_\varepsilon(0,y) = v_0^\varepsilon(y) \quad \text{in } \Omega,\ v_\varepsilon(t,y) = 0 \qquad \text{on} \quad (0,T) \times \partial\Omega, \end{cases}$$

where μ_ε and v_0^ε are specified as before.
Such a solution is established by well-known results (see [1, 19]) and belongs to $\bigcap_{i=1}^N L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega)) \cap C(0,T;L^2(\Omega))$. As $\{f_\varepsilon\}$ is bounded in $L^1(Q)$, and according to Proposition 3.4, v_ε is bounded in $\mathcal{T}_0^{1,\vec{p}(y)}(Q)$ such that

$$a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon) \in \bigcap_{i=1}^N L^{q_i(y)}(Q),$$

for each $q_i(y) < p_i(y) - \dfrac{N}{N+1}$, and v_ε solves $(\mathcal{P}_\varepsilon)$ in the sense of distributions.

Consequently, there is v and a subsequence (still denoted by v_ε) such that

$$\begin{cases} v_\varepsilon \rightharpoonup v \text{ a. e. in } Q, \\ T_k(v_\varepsilon) \rightharpoonup T_k(v) \text{ weakly in } \bigcap_{i=1}^{N} L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega)), \\ \text{ strongly in } \bigcap_{i=1}^{N} L^{p_i(y)}(Q) \text{ and a.e.in } Q. \end{cases} \tag{25}$$

On the other side, we take $\Psi_k(v_\varepsilon) = T_1(v_\varepsilon - T_k(v_\varepsilon))$, with $k \geq k_0$ where $k_0 \in \mathbb{N}$, as test function in the weak formulation of $(\mathcal{P}_\varepsilon)$ and given that $\partial_i \Psi_k(v_\varepsilon) = \partial_i v_\varepsilon \chi_{\{k \leq |v_\varepsilon| < k+1\}}$ and $\Psi_k(v_\varepsilon) = 0$ if $|v_\varepsilon| \leq k$, we can obtain easily

$$\int_0^T \langle (v_\varepsilon)_t,\ T_1(v_\varepsilon - T_k(v_\varepsilon))\rangle dt + \sum_{i=1}^{N} \int_{\{k \leq |v_\varepsilon| < k+1\}} a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon) \cdot \partial_i v_\varepsilon dydt \leq \int_{\{|v_\varepsilon| \geq k\}} |f_\varepsilon| dydt, \tag{26}$$

for every $k \geq k_0$.
Using the integration by parts formula of Lemma 3.1 and from (23), it follows that the some subsequence $\{u_\varepsilon\}$ verifies

$$\int_\Omega \Theta_k(v_\varepsilon(\tau))dy + \zeta_i^- k^{s_i^-} \sum_{i=1}^{N} \int_{\{k \leq |v_\varepsilon| < k+1\}} |\partial_i v_\varepsilon|^{p_i(y)} dydt \leq \int_{\{|v_\varepsilon| \leq k\}} |f_\varepsilon| dydt + \int_\Omega \Theta_k(v_\varepsilon(0))dy, \tag{27}$$

where $\Theta_k(\ell)$ defined as follows $\Theta_k(\ell) = \displaystyle\int_0^\ell \Lambda(\tau)d\tau$. Let us notice that we require to distinguish two cases:
1^{st} **case:** If $s_i(y) > 1$. For a.e. $t \in (0,T)$, by means (24), (27) and the fact that f_ε is bounded in $L^1(Q)$ and $|\theta_k(v_0^\varepsilon)| \leq |v_0^\varepsilon|$ a.e. in Ω, we can write that

$$\int_\Omega \Theta_k(v_\varepsilon)(t)dy \leq C, \text{ for all } t \text{ belongs to } [0,T]. \tag{28}$$

From now on, we will denote by C any constant that is dependent on particular variables and whose value can vary from one line to the next, implying the estimation of v in $L^\infty(0,T;L^1(\Omega))$ and

$$\sum_{i=1}^{N}\int_{\{k\le|v_\varepsilon|<k+1\}}|\partial_i v_\varepsilon|^{p_i(y)}dydt \tag{29}$$

$$\le \sum_{i=1}^{N}\int_Q|\partial_i T_{k_0}(v_\varepsilon)|^{p_i(y)}dydt+\sum_{i=1}^{N}\frac{1}{\zeta_i^-}\int_{\{|v_\varepsilon|\ge k\}}\frac{|f_\varepsilon|}{k^{s_i(y)}}dydt$$

$$\le \sum_{i=1}^{N}\int_Q|\partial_i T_{k_0}(v_\varepsilon)|^{p_i(y)}dydt+\frac{[\|v_\varepsilon\|_{L^1(\Omega)}+\|\mu\|_{\mathcal{M}_0(Q)}]}{\zeta_i^-}\sum_{k=1}^{\infty}\frac{1}{k^{s_i^-}}$$

$$\le C\Big[\|\mu\|_{L^1(Q)}+\|v_0^\varepsilon\|_{L^1(\Omega)}\Big]. \tag{30}$$

These previous results produce a bound for v_ε in $\bigcap_{i=1}^{N}L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))\cap L^\infty(0,T;L^1(\Omega))$; as a result, from (25), we obtain a bound for its weak limit v in

$$\bigcap_{i=1}^{N}L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))\cap L^\infty(0,T;L^1(\Omega)).$$

In the same way that the proof of Proposition 3.4, we get a bound of v in

$$\bigcap_{i=1}^{N}L^{q_i(y)}(Q)\text{ for each } q_i^-<\frac{(p_i^-N+p_i^--N)(s_i^-+1)}{(N+1)},$$

which is verified, as $|v_\varepsilon|^{s_i(y)+1}\le C(1+M(v))$ *and* $M(v)\in\bigcap_{i=1}^{N}L^{q_i(y)}(Q)$ for each

$$q_i^-<p_i^--\frac{N}{N+1}$$

where $M(\ell)$ is defined as $M(\ell)=\int_0^\ell b(|y|)^{\frac{1}{p_i(y)-1}}dy.$

2nd case: If $0\le s_i(y)\le 1$.

Let $\Gamma_i(y)\in\mathbb{R}$ where $\Gamma_i(y)>1-s_i(y)$ and $1<q_i(y)<2$, then, by applying the inequality of Hölder and (27) for a.e. $t\in(0,T)$, for $i=1,\dots,N$, we have

$$\sum_{i=1}^{N}\int_\Omega|\partial_i v_\varepsilon(t,y)|^{q_i(y)}dy=\sum_{i=1}^{N}\int_\Omega\frac{|\partial_i v(t,y)|^{q_i(y)}}{(|v(t,y)|+1)^{\frac{\Gamma_i(y)q_i(y)}{2}}}\big(|v_\varepsilon(t,y)|+1\big)^{p_i(y)}dydt \tag{31}$$

$$\le\sum_{i=1}^{N}\Big[\Big(\int_\Omega\frac{|\partial_i v_\varepsilon(t,y)|^{q_i(y)}}{(|v_\varepsilon(t,y)|+1)^{\Gamma(y)}}dydt\Big)^{\frac{q_i^-}{p_i^-}}\Big(\int_\Omega\big(|v_\varepsilon(t,y)|+1\big)^{\frac{\Gamma_i^-q_i^-}{p_i^--q_i^+}}dydt\Big)^{\frac{p_i^--q_i^-}{p_i^-}}\Big].$$

Similar to (29) and integrate over t, if $\Gamma_i(y) > 1 - s_i(y)$ and $\Gamma_i^- = \dfrac{N(p_i^-) - q_i^-)}{N - q_i^-}$ which leads to $q_i^- < \dfrac{N(s_i^- + 1))}{N - (1 - s_i^-)}$, we have by means the Sobolev's embedding theorem that

$$\Big(\int_Q |v_\varepsilon|^{q_i^\star(y)} dydt\Big)^{\frac{q_i^-}{(q_i^\star)^-}} \le \int_Q |\partial_i v_\varepsilon|^{q_i(y)} dydt \le \Big(C + \sum_{k=1}^{\infty} \frac{c}{k^{\Gamma_i(y)+s_i(y)}}\Big)^{\frac{q_i^-}{p_i^-}}$$
$$\times \Big(\int_Q \big(|v_\varepsilon(t,y)|+1\big)^{\frac{\Gamma_i^- q_i^-}{p_i^- - q_i^+}} dydt\Big)^{\frac{p_i^- - q_i^-}{p_i^-}} \tag{32}$$
$$\le \Big[\int_Q (|v_\varepsilon(t,y)|+1)^{\frac{\Gamma_i^- q_i^-}{p_i^- - q_i^+}} dydt\Big]^{\frac{p_i^- - q_i^-}{p_i(y)}}$$
$$\le \Big[\int_Q (|v_\varepsilon(t,y)|+1)^{\Upsilon^-} dydt\Big]^{\frac{p_i^- - q_i^-}{p_i^-}}$$

where $\Upsilon^- < \frac{p_i^- q_i^-}{p_i^- - q_i^+}$.
As a consequence, one may readily get a priori estimates on

$$v_\varepsilon \; in \; \bigcap_{i=1}^{N} L^{q_i(y)}(0,T;W_0^{1,q_i(y)}(\Omega)) \text{ for each } q_i^- < \frac{Ns_i^- + N}{N - 1 + s_i^-}.$$

We now suppose that the assumptions (11)-(13) and (23) are satisfied, and that the datum $\mu = f$ such that $f \in L^{\sigma_i(y)}(Q)$ with $s_i^- \ge 1 - \dfrac{(p_i^\star)^+}{(\sigma_i^\star)^+}$, then it is possible to use the results of the above calculations to find the solution's summability and its gradient with respect to time and space. Let us consider v_ε the solution of problem (11) with (f_ε) a sequence of regular functions in $L^{\sigma_i(y)}(Q)$ that approximate the datum μ, by (29) we infer that

$$\sum_{i=1}^{N} \int_Q |\partial_i v_\varepsilon|^{p_i(y)} dydt \le \sum_{i=1}^{N} \int_Q |\partial_i T_{k_0}(v_\varepsilon)|^{p_i(y)} dydt + \sum_{i=1}^{N} \int_{\{|v_\varepsilon| \ge k_0\}} |\partial_i v_\varepsilon|^{p_i(y)} dydt$$
$$\le C + \sum_{i=1}^{N} \sum_{h=1}^{\infty} \int_{\{|v_\varepsilon| \ge h\}} \frac{|f_\varepsilon|}{h^{s_i^-}} dydt$$
$$\le C + \sum_{i=1}^{N} \sum_{h=1}^{\infty} \sum_{j=h}^{\infty} \int_{\{j \le v_\varepsilon < j+1\}} \frac{|f_\varepsilon|}{h^{1 - \frac{(p_i^\star)^+}{(\sigma_i^\star)^+}}} dydt \tag{33}$$
$$\le C + \sum_{i=1}^{N} \sum_{j=0}^{\infty} \int_{\{j \le v_\varepsilon < j+1\}} |f_\varepsilon| \sum_{h=0}^{j} \frac{1}{(1+h)^{1 - \frac{(p_i^\star)^+}{(\sigma_i^\star)^+}}} dydt.$$

As $\sum_{h=0}^{j} \frac{1}{(1+h)^{s_i^-}} \leq C(1+j)^{1-s_i^+}$ with $0 < s_i(y) < 1$, by Hölder's inequality and the Sobolev embedding theorem, we can easily determine that

$$\begin{aligned}\sum_{i=1}^{N} \Big(\int_Q |v_\varepsilon|^{p_i^\star(y)} dydt \Big)^{\frac{p_i^-}{(p_i^\star)^-}} &\leq \sum_{i=1}^{N} \int_Q |\partial_i v_\varepsilon|^{p_i(y)} dydt \\ &\leq C \sum_{i=1}^{N} \Big[1 + \int_Q |f_\varepsilon| \Big(1+|v_\varepsilon|\Big)^{\frac{p_i^\star(y)}{(\sigma_i^\star)^-}} dydt \Big] \\ &\leq C \Big[\sum_{i=1}^{N} \Big(\int_Q (|v_\varepsilon + N|)^{p_i^\star(y)} dydt \Big)^{\frac{1}{(\sigma_i^\star)^-}} + N \Big]\end{aligned}$$

where $\frac{p_i^-}{(p_i^\star)^-} > \frac{1}{(\sigma_i^\star)^-}$. Then, we simply find an a priori estimate of v_ε in $\bigcap_{i=1}^{N} L^{p_i^-}(0,T;W_0^{1,p_i(y)}(\Omega))$.

Step 1 : If $1 - \frac{p_i^\star(y)}{\sigma_i^\star(y)} < s_i(y)$. Using $\Psi_k\big(|v_\varepsilon|^{s_i(y)} v_\varepsilon\big) = T_1\Big(|v_\varepsilon|^{s_i(y)} v_\varepsilon - T_k\big(|v_\varepsilon|^{s_i(y)} v_\varepsilon\big)\Big)$ as test function in the weak formulation of $(\mathcal{P}_\varepsilon)$ and reminding that $s_i(y) > 1 - \frac{p_i^\star(y)}{\sigma_i^\star(y)}$, we get, for a. e. $t \in [0,T]$, that

$$\begin{aligned}&\int_0^T \big\langle (v_\varepsilon)_t, T_1(|v_\varepsilon|^{s_i(y)} v_\varepsilon - T_k(|v_\varepsilon|^{s_i(y)} v_\varepsilon)) \big\rangle dt \\ &\qquad + \sum_{i=1}^{N} (s_i^- + 1) \int_{\{k \leq |v_\varepsilon|^{s_i(y)+1} < k+1\}} |v_\varepsilon|^{s_i(y)} a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon) \cdot \partial_i v_\varepsilon dydt \\ &\qquad\qquad \leq \int_{\{|v_\varepsilon|^{s_i(y)+1} \geq k\}} |f_\varepsilon| dydt\end{aligned}$$

Thus, by setting $\Theta_1^{s_i(y)}(\ell) = \int_0^\ell T_1(|\omega|^{s_i(y)} \omega - T_k(|\omega|^{s_i(y)} \omega)) d\omega$ and applying the integration by parts formula, we obtain

$$\begin{aligned}&\int_0^T \big\langle (v_\varepsilon)_t,\ T_1(|v_\varepsilon|^{s_i(y)} v_\varepsilon - T_k(|v_\varepsilon|^{s_i(y)} v_\varepsilon)) \big\rangle dt \\ &\qquad = \int_\Omega \Theta_1^{s_i(y)}(v_\varepsilon)(T) dy - \int_\Omega \Theta_1^{s_i(y)}(v_\varepsilon)(0) dy,\end{aligned}$$

As $\Theta_1^{s_i(y)}(\ell) \leq |\ell|$ and $\Theta_1^{s_i(y)}(v_\varepsilon)(T) \geq 0$, then the first member is positive,

$$\sum_{i=1}^{N} \left(s_i^- + 1\right) \int_{\{k \leq |v_\varepsilon|^{s_i(y)+1} < k+1\}} |v_\varepsilon|^{s_i(y)} a_i(t,y,v_\varepsilon,\partial_i v_\varepsilon) \cdot \partial_i v_\varepsilon dydt$$
$$\leq \int_{\{|v_\varepsilon|^{s_i(y)+1} \geq k\}} |f_\varepsilon| dydt + C \int_\Omega \Theta_1^{s_i(y)}(v_\varepsilon)(0)dy.$$

Hence, from (3.1) and (23), we get

$$\sum_{i=1}^{N} \int_{\{k \leq |v_\varepsilon|^{s_i(y)+1} < k+1\}} |\partial_i(|v_\varepsilon|^{s_i(y)} v_\varepsilon)|^{p_i(y)} dydt \leq C \int_{\{|v_\varepsilon|^{s_i(y)+1} \geq k\}} |f_\varepsilon| dydt$$
$$+ C \int_{\{|v_\varepsilon|^{s_i(y)+1} \geq k\}} |v_0| dy \ \ for\ all\ k \geq k_1 = m_0^{s_i^- + 1}.$$

Therefore, we get, as in [5, Theorem 3], the estimate of $|v_\varepsilon|^{s_i(y)+1}$ belngs to

$$L^{\frac{(N(p_i(y)-1)+p_i(y))\sigma_i(y)}{N+p_i(y)-\sigma_i(y)p_i(y)}}$$

and the desired (higher) summability of

$$v_\varepsilon \in L^{\delta_i(y)}(Q) \text{ where } \delta_i(y) = \frac{(N(p_i(y)+1)+p_i(y))\sigma_i(y)(s_i(y)+1)}{N+p_i(y)-\sigma_i(y)p_i(y)}$$

.

Step 2 : If $0 < s_i(y) < 1 - \dfrac{p_i^\star(y)}{\sigma_i^\star(y)}$.

Let $q_i(y) < 2$, $t \in [0,T]$ and $\Gamma_i(y)$ be a function such that $\Gamma_i(y) < 1 - s_i(y)$. Using the previous procedure, and taking the supremum for t in $(0,T)$, we obtain

$$C\|v_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \sum_{i=1}^{N} \int_Q |\partial_i v_\varepsilon|^{q_i(y)} dydt$$
$$\leq \Big(\sum_{h=1}^{\infty} \sum_{j=h}^{\infty} \int_{\{j \leq |v_\varepsilon| < j+1\}} \frac{|f_\varepsilon|}{h^{\Gamma_i(y)+s_i(y)}} dydt \Big)^{\frac{q_i^-}{p_i^-}} \Big(\int_Q (1+|v_\varepsilon|)^{\frac{\Gamma_i(y) q_i(y)}{p_i(y)-q_i(y)}} dydt \Big)^{\frac{p_i^- - q_i^-}{p_i^-}}$$
$$\leq C \Big(1 + \int_Q |f_\varepsilon| \Big(1+|v_\varepsilon|\Big)^{1-(s_i(y)+\Gamma_i(y))} dydt \Big)^{\frac{q^-}{p^-}}$$
$$\Big(\int_Q (1+|v_\varepsilon|)^{\frac{\Gamma_i(y) q_i(y)}{p_i(y)-q_i(y)}} dydt \Big)^{\frac{p_i^- - q_i^-}{p_i^-}}$$
$$\leq C\|f_\varepsilon\|_{L^{\sigma_i(y)}(Q)}^{\frac{q_i^-}{p_i^-}} \Big(\int_Q \Big(1+|v_\varepsilon|\Big)^{\big(1-(s_i(y)+\Gamma_i(y))\big)\sigma_i^\star(y)} dydt \Big)^{\frac{q^-}{p_i^-(\sigma_i^\star)^-}}$$
$$\times \Big(\int_Q (1+|v_\varepsilon|)^{\frac{\Gamma_i^- q_i^-}{p_i^- - q_i^+}} dydt \Big)^{\frac{p_i^- - q_i^-}{p_i^-}}.$$

Here, we choose $\Gamma_i^- = \frac{N(p_i^- - q_i^-)}{N - q_i^+}$ where

$$q_i^- = \frac{(N(p_i^- + 1) + p_i^-)\sigma_i^-(s_i^- + 1)}{N + p_i^-(1 - \sigma_i^+)},$$

to get

$$\frac{\Gamma_i^- q_i^-}{p_i^- - q_i^+} = \frac{N q_i^-}{N - q_i^+} = (q_i^\star)^-,$$

which gives that

$$\|v_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C,$$

and

$$\sum_{i=1}^{N} \Big(\int_Q |v_\varepsilon|^{q_i^*(y)} dydt \Big)^{\frac{q_i^-}{(q_i^\star)^-}} \leq \sum_{i=1}^{N} \int_Q |\partial_i v_\varepsilon|^{q_i(y)} dydt$$

$$\leq C \Big(\int_Q \Big(1 + |v_\varepsilon|\Big)^{q_i^\star(y)} dydt \Big)^{\frac{p_i^- - q_i^-}{p_i^-}}.$$

Therefore, we arrive at the desired estimates of v_ε in $\bigcap_{i=1}^{N} L^{q_i^-}(0,T;W_0^{1,q_i(y)}(\Omega))$ for each

$$q_i^- < \frac{N\sigma_i^-(s_i^- + 1)}{N - \sigma_i^-(1 - s_i^-)}.$$

□

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A. SABIRY
Laboratory LMACS
Sultan Moulay slimane University
e-mail: abdo.Sabiry@gmail.com

G. ZINEDDAINE
Laboratory LMACS
Sultan Moulay slimane University
e-mail: ghizlane.zineddaine@usms.ma

A. KASSIDI
Laboratory LMACS
Sultan Moulay slimane University
e-mail: a.Kassidi@usms.ma

L. S. ZHADLI
Laboratory LMACS
Sultan Moulay slimane University
e-mail: chadli@fstbm.ac.ma