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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR PARABOLIC PROBLEMS OF FRACTIONAL TYPE AND SIGN-CHANGING MEASURE DATA

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We prove a new asymptotic behavior result (with respect to the time variable t) of entropy solutions for fractional parabolic problems, with Dirichlet boundary at infinity, whose model is

$$(\mathcal{P})\begin{cases} u_t + (-\Delta)^s_p u(x) = \mu \text{ in } (0,\infty)\times\mathbb{R}^N, \\ u(0,x) = u_0(x) \text{ in } \mathbb{R}^N, \end{cases}$$

where $(-\Delta)^s_p u$ is the fractional (s,p)-Laplace operator (with $ps < N$, $0 < s < 1$ and $p > 2 - \frac{s}{N}$), $u_0 \in L^1_{loc}(\mathbb{R}^N)$ and μ is a bounded, compactly supported Radon measure whose support is compactly contained in $Q := (0,\infty)\times\mathbb{R}^N$, $N \geq 2$ (not depending on time) which does not charge the sets of the fractional (s,p)-capacity.

Résumé. Soit Ω un ouvert borné de $\mathbb{R}^N$, $N \geq 2$ et $T > 0$, nous montrons un résultat de comportement asymptotique (selon la variable du temps t) des solutions entropiques pour un problème fractionnaire prabolique dont le modèle est $(\mathcal{P})$ où $(-\Delta)^s_p u$ est l'opérateur fractionnaire (s,p)-Laplacian (avec $ps < N$, $0 < s < 1$ et $p > 2 - \frac{s}{N}$), $u_0 \in L^1(\Omega)$ et $\mu \in \mathcal{M}^{s,p}_0(Q)$ est une mesure de Radon avec une variation totale bornée dans $Q := (0,\infty)\times\Omega$ (ne dépend pas du temps) qui ne prend pas en charge les parties de (s,p)-capacité nulle.

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1. Introduction

This paper is concerned with asymptotic behavior of solutions for some fractional parabolic problems. We point out the results proved in this context are new, even for regular data. To this aim, we consider model problems

$$\begin{cases} u_t + (-\Delta)^s_p u = \mu \text{ in } Q := (0,\infty)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega,\ u(t,x) = 0 \text{ on } (0,\infty)\times\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary $\partial\Omega$ and $u \mapsto (-\Delta)^s_p u$ is the *fractional p-Laplace* operator, which, up to renormalization factors, is defined as

$$\begin{aligned} (-\Delta)^s_p u(t,x) &:= \mathbf{P.V.}\int_{\mathbb{R}^N} \frac{|u(t,x)-u(t,y)|^{p-2}(u(t,x)-u(t,y))}{|x-y|^{N+ps}}dy \\ &= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(t,x)-u(t,y)|^{p-2}(u(t,x)-u(t,y))}{|x-y|^{N+ps}}dy, \end{aligned} \tag{2}$$

where $(t,x)\in\mathbb{R}^+\times\mathbb{R}^N$ and $\mathbf{P.V.}$ is commonly used abbreviation for "*in the principal value sense*", u_0 is a function in $L^1(\Omega)$ and $\mu \in \mathcal{M}^{s,p}_0(Q)$ is any measure with bounded variation over $Q := (0,\infty)\times\Omega$ which does not charge the sets of zero fractional-capacity. The purpose of this article is to give different properties of the *fractional capacity* in connection with Radon measures under which the asymptotic behavior of *entropy* solutions hold; namely, we characterize the fractional order Sobolev spaces for which the fractional capacity is defined and we show existence, regularity and asymptotic behavior results for generalized solutions under suitable assumptions on the data. In our study, we will be only concerned with *nonlocal* equations in the case $p \neq 2$ which makes the asymptotic behavior result more difficult since the operator turns out to be nonlinear and parabolic. However, we stress that, even when $p = 2$, while sufficiently regular data, the existence result for *duality* solutions is not trivial.

In order to better describe how far the results presented in this paper extend to the fractional setting those available in the classical case, we shall give a short review of some of the key point results of the classical local theory. An existence and regularity theory for general quasilinear equations involving measures has been established by *Boccardo* & *Gallouët* in a series of papers [16–18]. The authors deal with Dirichlet problems of the type $-\operatorname{div}(a(x,\nabla u)) = \mu$ in Ω, with Dirichlet boundary $u = 0$ on $\partial\Omega$, where the vector field $a(x,\nabla u)$ has p-growth and coercivity with respect to the gradient, and exhibits a measurable dependence on x. The main model case here is given by the p-Laplace operator with measurable coefficients. Under the optimal assumption $p > 2 - \frac{1}{N}$, *Boccardo* &

Gallouët introduce the notion of *SOLA* (Solution Obtained as the Limit of Approximations), that are defined as distributional solutions which have been obtained as limits (a.e. and in L^{p-1}) of a sequence of $W^{1,p}$-solutions (u_n) of problems $-\text{div}(a(x,\nabla u_n)) = \mu_n$ in Ω where the sequence $(\mu_n)_{n\in\mathbb{N}} \subset C^\infty(\Omega)$ converges to μ weakly in the sense of measures. The final outcome is the existence of a distributional solution u to the original problem satisfying $u \in W^{1,q}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$; in this case, the solutions are not in general energy solutions and that does not belong to $W^{1,p}_{loc}$. For this reason such solutions are often called *very weak solutions*; we remark that different notions of solutions have been proposed in [25, 38, 58**?**], also in order to prove unique solvability (which is still an open problem). Such definitions are all equivalent in the case of diffuses measures as eventually shown in [25]. Before giving the definition of solution we need to specify a few nonlocal objects that will be crucial in the subsequent analysis, we recall that nonlocal operators have attracted increasing attention over the last years. Apart from their theoretical interest, and the new mathematical phenomena they allow to observe, they intervene in a quantity of applications and models since they allow to catch more efficiently certain peculiar aspects of the modelled situations. For instance, we mention their use in quasi-geostrophic dynamics [31], nonlocal diffusion and modified porous medium equations [26, 82, 83], dislocation problems [27], phase transition models [9, 28], image reconstruction problems [48]. For this reason it is particularly important to study situations when such nonlocal operators are involved in equations featuring L^1 or regular data, as for instance those modelling source terms which are diffuse. This leads to study nonlocal equations having measures as data, that read, in the elliptic Dirichlet case, as $-\mathcal{L}_\Phi u = \mu$ in $\Omega \subset \mathbb{R}^N$, $N \geq 2$, where $-\mathcal{L}_\Phi$ is a nonlocal operator defined by

$$\langle -\mathcal{L}_\Phi u, \varphi\rangle := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Phi(u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)dxdy,$$

for every smooth function φ with compact support. It is assumed that μ belongs to $\mathcal{M}(\mathbb{R}^N)$, that is the space of diffuse measures with finite total mass on $\mathbb{R}^N$. The function $\Phi : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous, satisfying $\Phi(0) = 0$ together with a mononocity property and the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is assumed to be measurable, and satisfying an ellipticity/coercivity properties. Notice that, upon taking the special case $\Phi(t) = |t|^{p-2}t$, we recover the fractional p-Laplace operator with measurable coefficients (see [15, 40, 41]). On the other hand, in the case $\Phi(t) = t$ we cover the special case of linear fractional operators, with measurable coefficients, defined by

$$\langle -\mathcal{L}_\Phi u, \varphi\rangle := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)dxdy,$$

for these equations see also [12, 29, 30]. Finally, when $K(x,y) = |x-y|^{-(N+ps)}$, with $s \in (0,1)$ and $p > 2 - \frac{s}{N}$, we recover the case of the classical fractional Laplacian $(-\Delta)^s$.

Of course, we shall introduce a natural function class allowing for solvability of the fractional problem; in particular, we shall introduce a suitable notion of solution to equations of the type (1). Such solutions, called entropy solutions and constructed via an approximation procedure with problems involving more regular data, do not in general lie in the natural energy space associated to the fractional operator $-\mathcal{L}_\Phi$, that is $W^{s,p}$, but exhibit a lower degree of integrability and differentiability, see Definition 3.1 below. This is in perfect analogy with what happens in the case of classical, local measure data problems. For such reasons these solutions should be considered as the analog of the *very weak solutions* usually considered in the classical case. To this aim, let us first recall what is already known in the literature: in the case $p = 2$, general structure conditions on *nonlocal* elliptic equations where given in [4, 53, 56] to ensure that the existence of *renormalized/duality* solutions holds whenever μ is a measure data (the corresponding parabolic results can be found in [56]). Their methods rely on different approaches; the authors in [4] first investigate under which condition the existence and uniqueness of *renormalized* solutions for problems with *maximal monotone graph* in $\mathbb{R}$ and $L^1(\mathbb{R}^N)$-data hold; in the second paper, they use some *duality* arguments in the sense of *Stampacchia* [78] to reduce the *nonlocal* problem $(-\Delta)^s u = \mu$, with μ being a bounded Radon measure whose support is compactly contained in $\mathbb{R}^N$, to the former "good" situation. This idea was a bit refined in [65] in order to deal with *integro-differential* equations including some cases of measure data. On the other hand, existence results for *nonlinear* operators, $p \neq 2$, must be handled with care since an easy counterexample, see [75, 77], show that it may fall. This is essentially due to the fact that the *regularization* of μ needs a proper functional setting to work; therefore a sufficient *renormalization* on the solutions has to be properly defined in the *weak* sense. This problem only occurs if we deal with *uniqueness* and if the data is irregular; indeed for sufficiently regular data, *uniqueness* results hold in more generality (see for instance [13, 42]). The problem of *existence/uniqueness* in the evolution case including generalized solutions was dealt in [24, 46, 76] (and in [5, 64] under more restrictive conditions) using an approach due to [45, 47] which relies on the decomposition of μ (with respect to the *capacity*) whenever the *entropy* argument should not work for general data expect in the framework of *Radon-measure valued* solutions, see [62, 73, 74, 81] (and also [72]). Different type of solutions were also proved in [54, 55] in the context of *SOLA* (*solutions obtained as limit of approximations*), *Viscosity* solutions and solutions obtained via *integration by parts/comparison principles* for nonlinear elliptic problems.

Indeed, to our knowledge, there aren't any general result including fractional *p*-Laplacian operator (2) in the literature giving existence results for generalized solutions to (1) when a singular measure is considered. Apart from some model cases contained in [2] where fractional elliptic *p-Laplace* equations with *weight* and general datum are considered, see also [52] for a class of nonlinear *degenerate* elliptic-parabolic problem with fractional time derivative and L^1-data and [3] for existence results of *SOLA/Entropy* solutions for parabolic problems with L^1-data.

In this article, we aim at establishing rather general asymptotic behavior results of *entropy* solutions when μ is a Radon measure which does not charges the sets of zero fractional capacity. To this purpose, we use alternatively either the approach of *Petitta* [66, 67] (see also [68]), based on a key result, namely the *comparison principle result* between suitable entropy sub- and super-solutions of the parabolic problem (1) (which is a natural extension of the one for the elliptic case, see [63]) coupled with a *compactness/convergence* approach mostly relying on some a priori estimates. We will also exploit one more idea consisting in the characterization of the data in terms of fractional parabolic *capacity* defined with fractional order *Sobolev* spaces. This method relies on the fundamental work [67] (re-adapted in [1]) which suggests the use of an adequate approximate problem to establish some a priori estimates for the sequence of approximate solutions and to derive a subsequence to obtain a limit function which is an *entropy* solution by virtue of the convergence results. We will use this later method as a key-point in order to get a *strong* asymptotic behavior result. We mention that, in [1], a part of the results of this paper was announced (and proved for a particular case of non-negative measures and non-negative initial datum) where we consider a fractional parabolic problem in presence of a nonnegative measure with bounded variation over Q which does not charge the sets of zero fractional (s,p)-capacity (i.e., $\mu \in \mathcal{M}_{0,+}^{s,p}(Q)$) and a nonnegative initial datum (i.e., $\mu_0 \in L^1_+(\Omega)$); nevertheless, we include here all the details, by using the positive and the negative parts of μ and u_0, and give a self-contained exposition for the sake of clarity. We confine ourselves to the case of *entropy* solution; in order not to mix different issues; however, we point out that same results, can be obtained by using *renormalized* solutions. A significant point that we wish to stress in our results is the role of the *fractional relative capacity*, by specifying how the *aymptotic* result would hold under more general data. Our specific interest in this question is clearly motivated by the study of the parabolic *capacity* and its properties (a reason which we feel sufficiently strong for us to choose the right-hand side as measure data). In addition, the complexity of this term allows us to observe interesting phenomena compared with other possible choices of data including measures *not charging sets* of fractional capacity; as an example, the model problem

$$u_t + (-\Delta)_p^s u = f - \mathrm{div}(G) \text{ in } Q, \tag{3}$$

with $f \in L^1(\Omega)$ and $G \in L^{p'}(\Omega)^N$. Let us also refer the reader to [70, 71] (see also [69]) for a totally different approach, based on *equidiffuse* properties and *without* using the strong convergence of truncatures, to the question of existence and uniqueness when a *zero order term* is present. However, the use of this approach could be suitable in our context, more precisely for *fractional-Laplace* problems and equi-diffuse measures.

The paper is organized as follows. In Section 2, we give some notations and some well-known results as they are used to obtain our main result; we recall the definition and some properties of the *fractional* Sobolev spaces in 2.1, and we define the *fractional capacity*, we give its properties and its relation to the *Radon* measure and we use the fractional capacity to characterize these measures and we give some necessary and sufficient conditions for the *decomposition* of μ to hold in Section 2.2. In Section 3, we introduce the entropy formulation to the *fractional-Laplacian* problem and we show that each of these solutions generate *compactness* estimates and several other convergence results. In Section 4, using the above mentioned formulation, we define *sub-super* problems, and we investigate the asymptotic behavior result of *entropy* solution to the fractional parabolic boundary value problem (1).

2. Preliminary results

2.1. Fractional Sobolev spaces and *fractional p-Laplace* operators

In this part, we collect some properties of fractional Sobolev spaces, we analyse the relations among some of their possible definitions and we collect some embedding results. Most of the results we present here are probably well knwon. We begun with the definition of these spaces. No prerequisite is needed, we just recall that $\Omega \subset \mathbb{R}^N$ is an open set whose boundary $\partial\Omega$ and $p \in [1, \infty)$. The first order Sobolev space

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \int_\Omega |\nabla u|^p dx < \infty\}, \tag{4}$$

is a *Banach* space endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} := (\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)})^{\frac{1}{p}}. \tag{5}$$

Denote

$$\widetilde{W}^{1,p}(\Omega) = \overline{W^{1,p}(\Omega)\cap C_c(\overline{\Omega})}^{W^{1,p}(\Omega)} \text{ and } W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}(\Omega)}. \tag{6}$$

Let us recall that $\widetilde{W}^{1,p}(\Omega)$ is a proper closed subspace of $W^{1,p}(\Omega)$ (see e.g., [60, 61]). Moreover, if Ω has the $W^{1,p}$*-extension property*, that is, if for every $u \in W^{1,p}(\Omega)$ there exists $w \in W^{1,p}(\mathbb{R}^N)$ such that $w_{|\Omega} = u$, then $\widetilde{W}^{1,p}(\Omega) = W^{1,p}(\Omega)$, see also [7, 8]. For $s \in (0,1)$ and $p \in [1,\infty)$, we denote by

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) : \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy < \infty\}. \tag{7}$$

The Sobolev space of fractional order is endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p dx + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{1/p}. \tag{8}$$

Similarly, denote

$$\widetilde{W}^{s,p}(\Omega) = \overline{W^{s,p}(\Omega)\cap C_c(\overline{\Omega})}^{W^{s,p}(\Omega)} \text{ and } W_0^{s,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)}. \tag{9}$$

Recall that $\widetilde{W}^{s,p}(\Omega)$ contains $W_0^{s,p}(\Omega)$ as a closed subspace and, by definition, $W_0^{s,p}(\Omega)$ is the smaller closed subspace of $W^{s,p}(\Omega)$ containing $\mathcal{D}(\Omega) = \mathbb{R}^N \times \mathbb{R}^N\setminus(\Omega^c\times\Omega^c)$ where $\Omega^c = \mathbb{R}^N\setminus\Omega$ (see e.g., [57, 59, 80]). In general $W^{1,p}(\Omega)$ is not a subspace of $W^{s,p}(\Omega)$, see [43, Example 9.1], but the following result holds true.

Proposition 2.1. *Let $p \in [1,\infty)$ and $s \in (0,1)$, let $\Omega \subset \mathbb{R}^N$ be an open set having the $W^{1,p}$-extension property. Then, there exists a constant $C = C(N,s,p) \geq 0$ such that for every $u \in W^{1,p}(\Omega)$,*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}. \tag{10}$$

Proof. See [84, Proposition 2.3]. □

The following result has been proved in [84, Lemma 2.4] under the assumption that $\varphi \in C^{0,1}(\overline{\Omega})\cap L^\infty(\Omega)$.

Lemma 2.2. *Let $p \in [1,\infty)$ and $s \in (0,1)$, let $u \in W^{s,p}(\Omega)$ and $\varphi \in C^{0,1}(\overline{\Omega})\cap L^\infty(\Omega)$. Then, $\varphi u \in W^{s,p}(\Omega)$ and there is a constant $C > 0$ (depending on N, p, s and $\|\varphi\|_{L^\infty(\Omega)}$) such that*

$$\|\varphi u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)}. \tag{11}$$

We notice that Lemma 2.2 remains true if one replaced $W^{s,p}(\Omega)$ with the space $\widetilde{W}^{s,p}(\Omega)$. The linear space of *Lebesgue measurable* functions $u : \mathbb{R}^N \mapsto \mathbb{R}$ such that the quantity

$$\left(\int_{\Omega}|u(x)|^p dx + \iint_{\mathcal{D}(\Omega)} \frac{|u(x)-u(y)|}{|x-y|^{N+ps}} dxdy\right) < \infty, \tag{12}$$

is denoted by $X^{s,p}(\Omega)$. It is easy to see that $X^{s,p}(\Omega)$ is not trivial since it contains *bounded* and *Lipschitz* functions. Moreover, $X_0^{s,p}(\Omega)$ is defined as the space of functions $u \in W^{s,p}(\Omega)$ that vanish a.e. in Ω^c. For every function $u \in X_0^{s,p}(\Omega)$, it is clear that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} = \int_{\Omega}\int_{\Omega} &\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \\ &+ 2\int_{\Omega}|u(x)|^p \int_{\Omega^c} \frac{1}{|x-y|^{N+ps}} dydx.\end{aligned} \tag{13}$$

Recalling [44, Lemma 6.1], we have

$$\int_{\Omega^c} \frac{1}{|x-y|^{N+ps}} dy \geq C|\Omega|^{\frac{-sp}{N}}$$

where $C = C(N,p,s) > 0$. A simple computation, using *Poincaré* inequality, gives

$$\int_{\Omega}|u(x)|^p dx \leq C\int_{\mathcal{D}(\Omega)}|u(x)-u(y)|^p d\nu \text{ with } d\nu = \frac{dxdy}{|x-y|^{N+ps}}, \quad \forall p \geq 1. \tag{14}$$

Thus, we can endow $X_0^{s,p}(\Omega)$ with the equivalent norm

$$||u||_{W_0^{s,p}(\Omega)} := \left(\int_{\mathcal{D}(\Omega)}|u(x)-u(y)|^p d\nu\right)^{1/p}. \tag{15}$$

Observe that, since $X_0^{s,p}(\Omega)$ is a reflexive *Banach* space, and as similar to $W_0^{s,p}(\Omega)$, we have $X_0^{s,p}(\Omega) = \overline{W_0^{\infty}(\Omega)}^{X^{s,p}(\Omega)}$. Now, in order to make the paper clear as possible, we introduce the *fractional Laplace* operator $(-\Delta)^s u$ for $(p = 2)$ which resembles to the familiar *Laplace* operator, let $0 < s < 1$ and set

$$C_{N,s} = \frac{s2^{2s}\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)}, \tag{16}$$

where Γ denotes the usual *Gamma* function, we define the *functional Laplacian* $(-\Delta)^s u$ by the formula

$$\begin{aligned}(-\Delta)^s u(x) &= C_{N,s}\mathbf{P}.\mathbf{V}.\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy \\ &= \lim_{\varepsilon\downarrow 0} C_{N,s}\int_{\{y\in\mathbb{R}^N:|y-x|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy.\end{aligned} \tag{17}$$

Notice that, if $0 < s < \frac{1}{2}$ and u smooth (*Lipschitz continuous* for example), the integral in (17) is in fact not really singular near x. Note also that $(-\Delta)^s$ can be defined as a *pseudo-differential* operator by the *Fourier* transformation (with symbol $|\xi|^{2s}$) by the method of *bilinear Dirichlet* forms (a closed self auto-adjoint associated to a bilinear symmetric form) or by the *contraction semigroup* theory, see [22, 23, 50, 51] for more precise details. As concerned, we have to define the *generalization* of the fractional Laplace operator to the case $p \neq 2$, and to study the *existence* and the *regularity* of the fractional differential equation (1) associated with these *nonlocal* operators $(-\Delta)^s_p$. We proceed as follows, let $w \in W^{s,p}(\mathbb{R}^N)$ be an arbitrary function, we define

$$(-\Delta)^s_p w(x) := \mathbf{P.V.} \int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+ps}} dy, \tag{18}$$

we restrict the *integral Kernel* of the functional *p-Laplacian* to the open set $\Omega \subset \mathbb{R}^N$, and we define the functional $\langle (-\Delta)^s_p w, \cdot \rangle$ for all $w \in W^{s,p}(\mathbb{R}^N)$ as

$$\langle (-\Delta)^s_p w, v \rangle = \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))(v(x)-v(y)) d\nu, \tag{19}$$

for all $v \in W^{s,p}(\mathbb{R}^N)$. As similar to the case $W^{s,p}(\mathbb{R}^N)$, if $w \in X^{s,p}_0(\Omega)$ then

$$\langle (-\Delta)^s_p w, v \rangle = \frac{1}{2} \int_{\mathcal{D}(\Omega)} |w(x)-w(y)|^{p-2}(w(x)-w(y))(v(x)-v(y)) d\nu, \tag{20}$$

for all $v \in X^{s,p}(\Omega)$, and is called the *regional* functional *p-Laplacian*, see [49–51], and can also be defined as a *pseudo-differential* operator from $X^{s,p}_0(\Omega)$ onto its dual space $X^{s,p}_0(\Omega)^\star$. A simple calculation in the evolution case, gives that, if $w \in L^p(0,T;X^{s,p}_0(\Omega))$ then $(-\Delta)^s_p : L^p(0,T;X^{s,p}_0(\Omega)) \mapsto L^{p'}(0,T;X^{s,p}_0(\Omega)^\star)$ where $L^p(0,T;X^{s,p}_0(\Omega))$ is defined as the set of functions u such that $u \in L^p(Q)$ with

$$\|u\|_{L^p(0,T;X^{s,p}_0(\Omega))} = \left(\int_0^T \int_{\mathcal{D}(\Omega)} |u(t,x)-u(t,y)|^p d\nu dt \right)^{\frac{1}{p}} < \infty. \tag{21}$$

2.2. Fractional capacity and space of measures

The attempt to find a different formulation for (1), when μ is sufficiently regular, which could allow to have both *existence* and *uniqueness* have developed in [52] and in [3] where the notions of *entropy* and *renormalized* solutions have been, respectively, introduced. Both these definitions, which have been proved to be equivalent in [79] ask for solutions in the functional space

$$\begin{aligned} \mathcal{T}^{s,p}_0(\Omega) := \big\{ & u : [0,T] \times \mathbb{R}^N \mapsto \mathbb{R} \text{ measurable s.t.,} \\ & T_k(u) \text{ belongs to } L^p(0,T;X^{s,p}_0(\Omega)) \text{ for every } k > 0 \big\}, \end{aligned} \tag{22}$$

and use a *weak* formulation of the equation where nonlinear test functions depending on u are used in order to restrict the equation on subsets where u is bounded. Both these approaches are able to get *uniqueness* provided that μ belongs to $L^1(Q)+L^{p'}(0,T;X_0^{s,p}(\Omega)^*)$. In terms of measures, this restriction has a slightly relationship with the notion of *fractional parabolic p-capacity* as it was proved in [6] for the stationary case. In order to state this result, we need first to introduce the notion of the *fractional parabolic p-capacity* (for more details, see [84, Section 3] and references quoted therein, in particular [8, 33, 39], where more properties and estimates are presented).

Definition 2.3. For $s \in (0,1)$ and $p \in (1,\infty)$, a *Choquet capacity* on a topological space is defined as the mapping $\mathcal{C} : \mathcal{D}(T)$ (the power set of T) $\mapsto [0,\infty)$ satisfying

$(\mathcal{C}_0)$ $\mathcal{C}(\emptyset)=0$,

$(\mathcal{C}_1)$ $A \subset B \subset \mathcal{T}$ implies $\mathcal{C}(A) \subseteq \mathcal{C}(B)$,

$(\mathcal{C}_2)$ $(A_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ an *increasing* sequence implies

$$\lim_{n\to\infty} \mathcal{C}(A_n) = \mathcal{C}((\cup_{n=1}^{\infty} A_n),$$

$(\mathcal{C}_3)$ $(K_n)_n \subset \mathcal{T}$ a *decreasing* sequence, K_n compact, implies

$$\lim_{n\to\infty} \mathcal{C}(K_n) = \mathcal{C}((\cap_{n=1}^{\infty} K_n).$$

Following the lines of the previous definition for Choquet capacities, here we want to give some basic knowledge on what has been done, up to known, about the classical *Bessel capacity* of order (s,p) denoted by $\text{cap}_{(s,p)}$, see [8, 60] for details. It is defined for any *open* set $U \subset \mathbb{R}^N$ by

$$\text{cap}_{(s,p)}(U) = \inf\left\{ \|u\|^p_{W^{s,p}(\mathbb{R}^N)} : \ u \in W^{s,p}(\mathbb{R}^N), \ u \neq 1 \text{ a.e. on } U \right\}. \qquad (23)$$

For an *arbitrary* set $E \subset \mathbb{R}^N$,

$$\text{cap}_{(s,p)}(E) = \inf\left\{ \text{cap}_{(s,p)}(U) : \ U \text{ is an open set in } \mathbb{R}^N \text{ containing } E \right\}, \quad (24)$$

and where, as usual, we use the convention that $\inf \emptyset = +\infty$; then one can extend this definition by regularity to any *Borel* subset of Q. Let us recall that a function $u \in W^{s,p}(\mathbb{R}^N)$ is said to be *$cap_{(s,p)}$-quasi-continuous* (*$cap_{(s,p)}$-q.c*) if for every $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}^N$ such that $\text{cap}_{(s,p)}(U) \leq \varepsilon$ and u is continuous in $\mathbb{R}^N \backslash U$. It is well known that every *Bessel* capacity $\text{cap}_{(s,p)}$ is a *Choquet* capacity, see [8, Section 2.2], and that every function $u \in W^{s,p}(\mathbb{R}^N)$ admits a unique (up to a polar set) $cap_{(s,p)}$-*q.c* function $\widetilde{u} : \mathbb{R}^N \mapsto \mathbb{R}$ such that $\widetilde{u} = u$ $\mathrm{cap}_{(s,p)}$-q.e. on $\mathbb{R}^N$. Thanks to this fact it is also possible to prove the following: for any capacity set $K \subset \mathbb{R}^N$, we have

$$\mathrm{cap}_{(s,p)}(K) = \inf\left\{\|u\|^p_{W^{s,p}(\mathbb{R}^N)} : u \in W^{s,p}(\mathbb{R}^N) \cap C_c(\mathbb{R}^N),\ u \geq 1 \text{ on } K\right\}. \quad (25)$$

Moreover, if $B \subset \mathbb{R}^N$ is a Borel set, we have

$$\mathrm{cap}_{(s,p)}(B) = \sup\left\{\mathrm{cap}_{(s,p)}(K) : K \subseteq B \subset \mathbb{R}^N \text{ compact}\right\}. \quad (26)$$

Further results on the relationship between the classical *Bessel* capacity $\mathrm{cap}_{(s,p)}$ and the related *Hausdorff* measures can be found in [8, 60]. Now, we recall the required functional parabolic spaces and fractional capacity associated to our problem (for further details, see [10, 11, 14, 20, 21, 34–36, 85]).

Definition 2.4. Let $Q = (0,T) \times \Omega$ for any fixed $T > 0$, and let us recall that

$$W^{s,p}(Q) = \left\{u \in L^p(0,T;W^{s,p}(\Omega));\ u_t \in L^{p'}(0,T;(W^{s,p}(\Omega)^*)\right\}, \quad (27)$$

(resp., $\widetilde{W}^{s,p}(Q)$ the corresponding definition for the space $\widetilde{W}^{s,p}(\Omega)$). So, if $U \subset \overline{Q}$ is a *relatively* open set (with respect to the relative topology of $\overline{Q}$), we define the (*relatively*) *functional parabolic capacity* of U (with respect to Q) as

$$\mathrm{cap}^{\overline{Q}}_{(s,p)}(U) := \inf\left\{\|u\|^p_{W^{s,p}(Q)} : u \in L^p(0,T;\widetilde{W}^{s,p}(\Omega)),\ u \geq 1 \text{ a.e. on } U\right\}, \quad (28)$$

where as usual we set $\inf \emptyset = +\infty$, then for any *arbitrary* set $E \subset \overline{Q}$ we define

$$\mathrm{cap}^{\overline{Q}}_{(s,p)}(E) = \inf\left\{\mathrm{cap}^{\overline{Q}}_{(s,p)}(U) :\ U \text{ relatively open in } \overline{Q} \text{ containing } E\right\}. \quad (29)$$

Let $K \subset \overline{Q}$ be a compact set, then

$$\mathrm{cap}_{(s,p)}(K) = \inf\left\{\|u\|^p_{W^{s,p}(Q)} :\ u \in W^{s,p}(Q) \cap C_c(\overline{Q}),\ u \geq 1 \text{ on } K\right\}, \quad (30)$$

and, for any Borel set $B \subset \overline{Q}$, we have

$$\mathrm{cap}^{\overline{Q}}_{(s,p)}(B) = \sup\left\{\mathrm{cap}^{\overline{Q}}_{(s,p)}(K) :\ K \subseteq B \subset \overline{Q} \text{ compact}\right\}.$$

This second definition of capacity, that enjoys the *Choquet*-properties as well as the first we gave, will turn out to be very useful to our aim since it allows to extend the notion of *Bessel* capacity to sets with respect to any open set contained in Q.

Proposition 2.5. *Let E be an arbitrary set of $\overline{Q}$. Then*

$$cap^{\overline{Q}}_{(s,p)}(E) = cap_{(s,p)}(E). \tag{31}$$

As mentioned before, let us recall some fundamental properties extended directly both to the case of relatively functional capacity.

Proposition 2.6. *Some properties are in order to be given:*

(i) *A set $E \subset \overline{Q}$ is called relatively polar if $cap^{\overline{Q}}_{(s,p)}(E) = 0$.*

(ii) *A property $\mathcal{P}(t,x)$ is said to hold on a set $F \subset \overline{Q}$ relatively quasi everywhere (r.q.e.) if there exists a relatively polar set $E \subset F$ such that the property holds everywhere on $F \backslash E$.*

(iii) *A function $u : \overline{Q} \mapsto \mathbb{R}$ is said to be relatively quasi-continuous (r.q.c.) if for every $\varepsilon > 0$ there exists a relatively open set $U \subset \overline{Q}$ such that $cap^{\overline{Q}}_{(s,p)}(U) < \varepsilon$ and $u_{|\overline{Q}\backslash U}$ is continuous.*

(iv) *For any function in $\widetilde{W}^{s,p}(Q)$, there exists a unique (up to a relatively polar set) relatively quasi-continuous representative (r.q.c.r).*

(v) *Let u_n be a sequence of r.q.c. functions in $\widetilde{W}^{s,p}(Q)$ which converges to a r.q.c. function $u \in \widetilde{W}^{s,p}(Q)$. Then, there exists a subsequence which converges r.q.e. to u on $\overline{Q}$.*

(vi) *Assume that Q has the $W^{s,p}$-extension property, that is, for every element $w \in L^p(0,T;W^{s,p}(\Omega))$ there exists a function $U \in L^p(0,T;W^{s,p}(\mathbb{R}^N))$ with $U_{|Q} = u$. Then, $cap_{(s,p)}$ and $cap^{\overline{Q}}_{(s,p)}$ are equivalent.*

Now, let us define the concept of the fractional elliptic (s,p)-capacity associated to our problem. If $K \subset \Omega$ is a capacity set then $cap_{(s,p)}(K)$ can be defined by

$$cap^{e}_{(s,p)}(K) = \inf\{\|u\|^p_{W^{s,p}(\Omega)} : u \in W^{s,p}(\Omega) \cap C_c(\mathbb{R}), u \geq 1 \text{ on } K\}.$$

If $U \subset \Omega$ is an open set, we define the elliptic (s,p)-capacity of U as follows

$$cap^{e}_{(s,p)}(U) := \inf\left\{\|u\|^p_{W^{s,p}(\Omega)} : u \in W^{s,p}(\Omega),\ u \geq 1 \text{ a.e. on } U\right\}.$$

Since $cap^{e}_{(s,p)}$ is a choquet capacity, we have that for every Borel set $B \subset \mathbb{R}^N$

$$cap^{e}_{(s,p)}(B) = sup\{cap^{e}_{(s,p)}(K) : K \subseteq B \subset \Omega \text{ compat}\}. \tag{32}$$

Now, we want to recall some feature about spaces of measures; more precisely, the most important tools to deal with the evolution functional problem (1). By $\mathcal{M}^{s,p}(Q)$ we denote the space of *finite* Radon measures on Q, and by $\mathcal{M}^{s,p}_{0,+}(Q) \subset \mathcal{M}^{s,p}(Q)$ the cone of *nonnegative* (finite Radon) measures on Q. For any $\mu \in \mathcal{M}^{s,p}(Q)$, we set $\|\mu\|_{\mathcal{M}^{s,p}(Q)} := |\mu|(\overline{Q})$ where $|\mu|$ denotes the *total variation* of μ. The space of continuous functions with compact support in Q will be denoted by $C_c(Q)$, and the space of *continuous* functions in $\overline{Q}$ by $C(\overline{Q})$; we also set

$$C_0(\overline{Q}) := \left\{ \varphi \in C(\overline{Q}) \text{ s.t. } \varphi(t,x) = 0 \text{ on } (0,T) \times \partial\Omega \right\}. \tag{33}$$

The duality map $\langle \cdot, \cdot \rangle$ between the spaces $\mathcal{M}^{s,p}(Q)$ and $C_c(Q)$, namely $\langle \mu, \varphi \rangle = \int_Q \varphi(t,x) d\mu(t,x)$, can be extended to functions $\varphi \in C_0(\overline{Q})$. Let us denote with $\mathcal{M}^{s,p}_0(\Omega)$ the set of all measures with bounded variation over Ω that does not charge the sets of zero elliptic (s,p)-capacity, that is, if $\mu \in \mathcal{M}^{s,p}_0(Q)$ then $\mu(E) = 0$ for all $E \in \Omega$ such that $\text{cap}^e_{(s,p)}(E) = 0$. Analogously we define $\mathcal{M}^{s,p}_0(Q)$ the set of all measures with bounded variation over Q that does not charge the sets of parabolic (s,p)-capacity, that is, if $\mu \in \mathcal{M}^{s,p}_0(Q)$ then $\mu(E) = 0$ for all $E \in Q$ such that $cap_{(s,p)}(E) = 0$. Moreover we suppose that μ does not depend on the time variable t (i.e., there exists a bounded Radon measure ν on Ω such that, for every Borel set $B \subseteq \Omega$, and $0 < t_0, t_1 < T$, we have $\mu(B \times (t_0,t_1)) = (t_1 - t_0)\nu(B)$. Actually, we investigate the limit as T tends to infinity of the solution $u(T,x)$ of problem (1). Since we want to deal with problem (1) where μ is a measure which does not charge sets of null capacity, this means that we consider measure data (not depending on the time variable t) which can be splitted in Ω.

Lemma 2.7. *If $\mu \in \mathcal{M}^{s,p}_0(\Omega)$, then there exist $G \in L^{p'}(\Omega)^N$ and $f \in L^1(\Omega)$ such that $\mu = f - \text{div}(G)$, in the sense that,*

$$\int_Q \varphi d\mu = \int_\Omega f dx + \int_\Omega G \cdot \nabla \varphi dxdt \tag{34}$$

for every $\varphi \in C^\infty_c(\Omega)$.

Proof. The previous proof remains the same also for measures that are zero on the sets of zero fractional (s,p)-capacity (i.e., the capacity defined starting from $W^{s,p}_0(\Omega)$, $s > 1$), since these measures can be decomposed as $h\gamma$, with h a Borel function and γ a measure of $W^{-s,p'}(\Omega)$. Thus, it is possible to prove that every signed measure on Ω which is zero on the sets of zero (s,p)-capacity can be decomposed in the sum of an element in $W^{-s,p'}(\Omega)$ and of a function in $L^1(\Omega)$, and *vice versa*. □

Moreover, we will deal with functions that may not belong to the fractional Sobolev spaces, so that we need to give a suitable definition of gradient for functions that enjoy some properties. To this purpose, if $k > 0$, we define

$$T_k(s) = \max(-k, \min(k; s)) \quad \forall s \in \mathbb{R}, \tag{35}$$

the truncature at levels k and $-k$, and $\Theta_k(s) = \int_0^s T_k(\tau)d\tau$. One has $\Theta_k(s) \geq 0$. The truncations will be very useful for defining good class of solutions, as in [13].

Definition 2.8. Let u be a measurable function on Q such that $T_k(u)$ belongs to $L^p(0,T;W_0^{s,p}(\Omega))$ for every $k > 0$. Then, see [13, Lemma 2.1], there exists a unique measurable function $v : Q \to \mathbb{R}^N$ such that

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \text{ a.e. in } Q \text{ for every } k > 0. \tag{36}$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$. If u belongs to $L^1(0,T,W_0^{1,1}(\Omega))$, the gradient coincides with the usual gradient in distributional sense.

The following lemma (see e.g. [78]) which is of analytic nature will be useful in deriving an a priori estimate of *entropy* solutions.

Lemma 2.9. *Let $G : \mathbb{R} \mapsto \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$. Then, for every $u \in L^p(0,T;W_0^{s,p}(\mathcal{D}))$, $\mathcal{D}$ is any bounded open subset of $\mathbb{R}^N$, we have $G(u) \in L^p(0,T;W_0^{s,p}(\mathcal{D}))$ and $\nabla G(u) = G'(u)\nabla u$ a.e. in $(0,T) \times \mathcal{D}$.*

The following result contains a generalization of the *integration by parts* formula when the time derivative is less regular (its proof can be found in [32, 46]).

Lemma 2.10. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a continuous C^1-function such that $f(0) = 0$ and f' is zero away from a compact set of $\mathbb{R}$; let us denote $F(s) = \int_0^s f(\tau)d\tau$. If $u \in L^p(0,T;W_0^{s,p}(\Omega))$ is such that $u_t \in L^{p'}(0,T;W^{-s,p'}(\Omega)) + L^1(Q)$ and if $\psi \in C^\infty(\overline{Q})$, then we have*

$$\begin{aligned} &\int_0^T \langle u_t, f(u)\psi\rangle dt \\ &= \int_Q F(u(T))\psi(T)dx - \int_\Omega F(u(0))\psi(0)dx - \int_\Omega \psi_t F(u)dxdt. \end{aligned} \tag{37}$$

3. Entropy formulation and main result

In this section we introduce some properties of entropy solutions and we give some intermediary results in order to prove our asymptotic behavior result. It is well known that, if dealing with L^1 or measure data, the concept of solution in the sense of distributions of problem like (1) is not strong enough to give uniqueness of solution. We need to introduce the definition of *entropy solution* of (1).

Definition 3.1. Let $\mu \in \mathcal{M}_0^{s,p}(\Omega)$ and let $u_0 \in L^1(\Omega)$. A measurable function $u \in C([0,\infty],L^1(\Omega))$ is an *entropy solution* of (1) if, for all $k>0$, $T_k(u) \in L^p(0,T;X_0^{s,p}(\Omega)$ for every $k>0$, and it holds

$$\begin{aligned}&\int_\Omega \Theta_k(u-\varphi)(T)dx - \int_\Omega \Theta_k(u_0-\varphi(0))dx + \int_0^T \langle \varphi_t, T_k(u-\varphi)\rangle dt\\&+\frac{1}{2}\int_0^T\int_{D(\Omega)} U(t,x,y)[T_k(u(t,x)-\varphi(t,x))-T_k(u(t,x)-\varphi(t,y))]d\nu dt\\&\quad\le \int_Q T_k(u-\varphi)d\mu \quad \forall \varphi \in L^p(0,T;X_0^{s,p}(\Omega))\cap L^\infty(Q)\cap C([0,T];L^1(\Omega))\end{aligned} \tag{38}$$

with $\varphi_t \in L^{p'}(0,T;X^{-s,p'}(\Omega))$ (here $U(t,x,y)=|u(t,x)-u(t,y)|^{p-2}(u(t,x)-u(t,y))$).

An analogous definition will be given in the elliptic case following [13, 75]. To our aim, it suffices to give the definition in the case of measures which does not charge sets of zero elliptic (s,p)-capacity.

Definition 3.2. Let $\mu \in \mathcal{M}_0^{s,p}(\Omega)$ and let $u_0 \in L^1(\Omega)$. A measurable function v is an *entropy solution* of problem

$$\begin{cases}(-\Delta)_p^s v(x) = \mu \text{ in } \Omega,\\ v(t,x)=0 \text{ on } \partial\Omega\end{cases} \tag{39}$$

if v is finite a.e. in Ω, $T_k(v)$ belongs to $X_0^{s,p}(\Omega)$ for every $k>0$, and

$$\int\int_{R_h} |v(x)-v(y)|^{p-2} d\nu \to 0 \text{ as } h\to 0 \tag{40}$$

where

$$\begin{aligned}R_h =&\{(x,y)\in\Omega\times\Omega : h+1 \le \max\{|v(x)|,|v(y)|\}\\&\text{with } \min\{|v(x)|,|v(y)|\}\le h \text{ or } v(x)v(y)<0\}\end{aligned}$$

and it holds

$$\begin{aligned}&\frac{1}{2}\int\int_{D_\Omega} |v(x)-v(y)|^{p-2}(v(x)-v(y))\cdot[T_k(v(x)-\varphi(x))-T_k(v(y)-\varphi(y))]d\nu\\&\le \int_\Omega T_k(v(x)-\varphi(x))d\mu\end{aligned} \tag{41}$$

for every $k>0$ and every $\varphi \in X_0^{s,p}(\Omega)\cap L^\infty(\Omega)$.

In order to prove some a priori estimates and further properties of entropy solutions, we will need a few technical ingredients

Lemma 3.3. *Let B be a Borel set in Ω, and $0 \leq t_0 \leq t_1 \leq T$. Then*

$$cap_{(s,p)}(B \times (t_0,t_1)) = 0 \textit{ if and only if } cap^e_{(s,p)}(B) = 0 \tag{42}$$

Proof. ($\Rightarrow$) Suppose that $cap^e_{(s,p)}(B) = 0$, then there exists an open subset U_ε, with $0 < \varepsilon < 1$ and $B \subset U_\varepsilon$, such that $cap^e_{(s,p)}(U_\varepsilon) < \varepsilon$ then we can choose $v_\varepsilon \in W_0^{(s,p)}(\Omega)$ such that $\chi_{U_\varepsilon} \leq v_\varepsilon \leq 1$ a.e. in Ω and $\|v_\varepsilon\|_{W_0^{s,p}(\Omega)} \leq \varepsilon$, and that $\|v_\varepsilon\|_{L^2(\Omega)} \leq C(\varepsilon)$ with $C(\varepsilon)$ tends to zero as ε goes to zero; that is the term $\|v_\varepsilon\|_{W_0^{(s,p)}(\Omega))} + \|v_\varepsilon\|_{L^2(\Omega)} \leq C(\varepsilon)$. By using the definition of the fractional capacity of $]t_0,t_1[\times U_\varepsilon$ with $u(t,x) = v_\varepsilon(x)$ we obtain that $cap_{(s,p)}(]t_0,t_1[\times U_\varepsilon) \leq C(\varepsilon)$; then, as ε goes to zero, we finally obtain that $cap(]t_0,t_1[\times B) = 0$.

($\Leftarrow$) Suppose that $cap_{(s,p)}(]t_0,t_1[\times B) = 0$, hence there exists an open set A_ε such that $(]t_0,t_1[\times B) \subset A_\varepsilon$ and $cap_{(s,p)}(A_\varepsilon) < \varepsilon$. Now, we choose t_0', t_1' such that $t_0 < t_0' < t_1' < t_1$, then $[t_0',t_1'] \times \{x\}$, with $x \in B$, is a compact subset of A_ε; hence, there exists an open subset $U_\varepsilon \subset \Omega$ such that $]t_0',t_1'[\times \{x\} \subset]t_0',t_1'[\times U_\varepsilon \subset A_\varepsilon$, which implies that $B \subset U := U_{x\in B} U_x \subset \Omega$ and $]t_0',t_1'[\times U \subset A_\varepsilon$; we deduce that $cap_{(s,p)}(]t_0',t_1'[\times U) \leq cap_{(s,p)}(A_\varepsilon) \leq \varepsilon$. Now, we choose $u_\varepsilon \in W^{s,p}$ such that $\chi_{]t_0',t_1'[\times U} \leq u_\varepsilon$ and $\|u_\varepsilon\|_{W^{s,p}} \leq \varepsilon$ and $v_\varepsilon = \frac{1}{t_1'-t_0'} \int_{t_0'}^{t_1'} u_\varepsilon dt$, it is clear that $v_\varepsilon \in W_0^{s,p}(\Omega)$, $\chi_U \leq v_\varepsilon$ a.e. in Ω and $\|v_\varepsilon\|_{W_0^{s,p}(\Omega)} \leq C(\varepsilon)$, which implies, by the arbitrariness of ε, that $cap^e_{(s,p)}(B) = 0$. □

Remark 3.4. Thanks to the result we derive that measures of $\mathcal{M}_0^{s,p}(Q)$ (which does not depend on time) can actually be identified with a measure in $\mathcal{M}_0^{s,p}(\Omega)$. So, if B is a Borel set in Ω of zero elliptic (s,p)-capacity; then, thanks to Lemma 3.3, we deduce that $cap_{(s,p)}(B \times (0,T)) = 0$ and so $\mu(B \times (0,T)) = 0$; and

$$0 = \mu(B \times (0,T)) = T\nu(B) \tag{43}$$

where $\nu \in \mathcal{M}_0^{s,p}(\Omega)$, and so $\nu(B) = 0$, thus $\mu \in \mathcal{M}_0^{s,p}(\Omega)$. Hence, in our case we can always identify μ and ν.

Now, following [1, 5, 64] and for $p > 2 - \frac{s}{N}$, we have u is bounded in the *Marcinkiewicz* space $\mathcal{M}^{p-1+\frac{ps}{N}}$ and $|\nabla u|$ is bounded in *Marcinkiewicz* space $\mathcal{M}^{p-\frac{N}{N+s}}$ (if $p < N$), and u is bounded in the *Marcinkiewicz* space $\mathcal{M}^q(Q)$ for every $q < \infty$ and $|\nabla u|$ is bounded in the *Marcinkiewicz* space $\mathcal{M}^r(Q)$ for every $r < N$ (if $p = N$). On the other hand, if v is a solution of the elliptic problem

(39), we have $v \in C([0,\infty],L^1(\Omega))$ and such a solution turns out to be an entropy solution of the parabolic problem (1) with initial datum $u_0(x) = v(x)$ since

$$\begin{aligned}
&\int_\Omega \Theta_k(v-\varphi)(T)dx - \int_\Omega \Theta_k(v-\varphi)(0)dx \\
&= \int_Q \frac{d}{dt}\Theta_k(v-\varphi)dxdt = \int_0^T \langle (v-\varphi)_t, T_k(v-\varphi)\rangle_{X^{-s,p'}(\Omega),X_0^{s,p}(\Omega)}dt \qquad (44)\\
&= -\int_0^T \langle \varphi_t, T_k(v-\varphi)\rangle_{X^{-s,p'}(\Omega),X_0^{s,p}(\Omega)}dt
\end{aligned}$$

that can be canceled out with the analogous term in (38) getting the right formulation (41) for v.

Now, let us introduce the notion of entropy sub- and super-solutions needed in the comparison principle result and in the proof of the asymptotic behavior result.

Definition 3.5. A function $\underline{u}(t,x) \in C([0,\infty],L^1(\Omega))$ is an *entropy sub-solution* of problem (1) if $T_k(\underline{u}) \in L^p(0,T;X_0^{s,p}(\Omega))$ for every $k>0$, and

$$\begin{cases} \underline{u}_t(t,x) + (-\Delta)_p^s \underline{u}(t,x) \geq \mu \text{ in } (0,\infty)\times\Omega, \\ \underline{u}(0,x) = \underline{u}_0(x) \geq u_0(x) \text{ in } \Omega,\ \underline{u}(t,x) \geq 0 \text{ on } (0,\infty)\times\partial\Omega \end{cases} \qquad (45)$$

On the other hand, $\overline{u} \in C(0,\infty;L^1(\Omega))$ is an *entropy super-solution* of problem (1) if $T_k(\overline{u}) \in L^p(0,T;X_0^{s,p}(\Omega))$ for every $k>0$, and

$$\begin{cases} \overline{u}_t(t,x) + (-\Delta)_p^s \overline{u}(t,x) \leq \mu \text{ in } (0,\infty)\times\Omega, \\ \overline{u}(0,x) = \overline{u}_0(x) \leq u_0(x) \text{ in } \Omega,\ \overline{u}(t,x) \leq 0 \text{ on } (0,\infty)\times\partial\Omega \end{cases} \qquad (46)$$

where both (45) and (46) are understood in their entropy sense, i.e., $\underline{u}(t,x)$ satisfies

$$\begin{aligned}
&\int_\Omega \Theta_k(\underline{u}-\varphi)^+(T)dx - \int_\Omega \Theta_k(\underline{u}_0-\varphi(0))^+dx + \int_0^T \langle \varphi_t, T_k(\underline{u}-\varphi)^+\rangle dt \\
&+\frac{1}{2}\int_0^T\int_{D(\Omega)} \underline{U}(t,x,y)[T_k(\underline{u}(t,x)-\varphi(t,x)) - T_k(\underline{u}(t,x)-\varphi(t,y))]d\nu dt \qquad (47)\\
&\geq \int_Q T_k(\underline{u}-\varphi)d\mu
\end{aligned}$$

for every $\varphi \in C(0,T;L^1(\Omega)) \cap L^p(0,T;X_0^{s,p}(\Omega)) \cap L^\infty(Q)$, $\varphi \geq 0$ a.e. in Q such that $\varphi_t \in L^p(0,T;X^{-s,p'}(\Omega))$. And $\overline{u}(t,x)$ satisfies

$$\int_\Omega \Theta_k(\overline{u}-\varphi)^-(T)dx - \int_\Omega \Theta_k(\overline{u}_0-\varphi(0))^- dx + \int_0^T \langle \varphi_t, T_k(\overline{u}-\varphi)^-\rangle dt$$
$$+\frac{1}{2}\int_0^T \int_{D(\Omega)} \overline{U}(t,x,y)[T_k(\overline{u}(t,x)-\varphi(t,x)) - T_k(\overline{u}(t,x)-\varphi(t,y))]d\nu dt \quad (48)$$
$$\leq \int_Q T_k(\overline{u}-\varphi)d\mu$$

for every $\varphi \in C(0,T;L^1(\Omega)) \cap L^p(0,T;X_0^{s,p}(\Omega)) \cap L^\infty(Q)$, $\varphi \geq 0$ a.e. in Q such that $\varphi_t \in L^p(0,T;X^{-s,p'}(\Omega))$

Now, we are able to state our comparison principle result.

Lemma 3.6. *Let $\mu \in \mathcal{M}_0^{s,p}(\Omega)$, and let $\underline{u}$ and $\overline{u}$ be, respectively, the entropy sub- and super-solutions of problems (1), and let u be the unique entropy solution of the same problem. Then, for every $t > 0$*

$$\underline{u}(t) \leq u(t) \leq \overline{u}(t) \text{ a.e. in } \Omega. \quad (49)$$

Proof. See [1, Lemma 3.2]. □

Our main result is the following

Theorem 3.7. *Let $\mu \in \mathcal{M}_0^{s,p}(\Omega)$ be independent of the time variable t, $p > \frac{2N+s}{N+s}$, $u_0 \in L^1(\Omega)$ be a function, and let $u(t,x)$ be the entropy solution of problem (1). Then*

$$\lim_{t\to\infty} u(t,x) = v(x) \quad (50)$$

where $v(x)$ is the entropy solution of the corresponding elliptic problem (39).

4. Proof of the main result

Now, we are able to prove our main result and we will prove it in few steps.

Proof of Theorem 3.7. Let us first suppose that $v^\oplus$ and $v^\ominus$ are, respectively, the entropy solutions of the elliptic problems

$$\begin{cases} (-\Delta)_p^s u = \mu^+ \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases} \qquad \begin{cases} (-\Delta)_p^s v = -\mu^- \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases}$$

according to the comparison principle result [63], we have both

$$v^\ominus(x) \leq 0 \leq v^\oplus(x) \text{ and } v^\ominus(x) \leq v(x) \leq v^\oplus(x) \qquad \forall x \in \Omega. \quad (51)$$

On the other hand, we prove that $v^{\oplus}$ and $v^{\ominus}$ are also entropy solutions of the corresponding parabolic problems with themeselves as initial data.

Step.1 : A priori estimates ($u_0 = v^{\oplus}$). Let $u_n(t,x)$ be the entropy solution of the fractional parabolic problem[1]

$$\begin{cases} (u_n)_t(t,x) + (-\Delta)^s_p u_n(x) = \mu \text{ in } (0,1)\times\Omega, \\ u_n(0,x) = u(n,x) \text{ in } \Omega,\ u_n(t,x) = 0 \text{ on } \Sigma := (0,1)\times\partial\Omega \end{cases} \tag{52}$$

with $n \in \mathbb{N}\cup 0$ and $u(0,x) = v^{\oplus}$, since μ does not depend on time, u_n turns out to be the time translation (of lenght n) of the solution u with initial datum $v^{\oplus}$. Lemma 3.6 ensures that $u(t,x) \le v^{\oplus}$ for every $(t,x) \in Q$. So, by using again the comparison principle result between the solution $u(t+s,x)$ (with $s > 0$ a positive parameter) with $u_0 = u(s,x)$ as initial datum and the solution $u(t,x)$ with $u_0 = v^{\oplus}$ as initial datum, we have that

$$u(t+s,x) \le u(t,x) \quad \text{a.e. in } \Omega, \quad \forall s,t \ge 0.$$

We recall that $u \in C^{\infty}((0,\infty), L^1(\Omega))$, then by [1, Lemma 2.19] we also have

$$\begin{cases} u_n(t,x) \in L^1(\Omega) \\ \displaystyle\int_0^T \int_{D_\Omega} |T_k(u_n(t,x) - T_k(u_n(t,y)))|^p dxdt \le Ck. \end{cases} \tag{53}$$

Hence, u_n is uniformly bounded in the Marcinkiewicz space $M^{p+1+\frac{ps}{N}}(Q)$ which implies, since in particular $p > \frac{2N+s}{N+s}$, that u_n is uniformly bounded in $L^m(Q)$ for every $1 \le m < p+1+\frac{ps}{N}$. We also have that $|\nabla u_n|$ is equipped in $M^{\gamma}(Q)$ with $\gamma = p - \frac{N}{N+s}$ which implies that, since $p > \frac{2N+s}{N+s}$, $|\nabla u_n|$ is uniformly bounded in $L^s(Q)$ with $1 \le s < p - \frac{N}{N+s}$. Thus, there exist a function $\overline{u} \in L^q(0,T;W_0^{s,q}(\Omega))$ for every $q < p-1+\frac{ps}{N}$ such that u_n converges to $\overline{u}$ weakly in $L^q(0,T;W_0^{s,q}(\Omega))$. Observe that, obviously we have $\overline{u} = u$ a.e. in Q and also $(u_n)_t \in L^1(Q) + L^{\beta'}(0,T;W^{-s,\beta'}(\Omega))$ uniformly with respect to n where $\beta' = \frac{q}{p-1}$ for every $q < p-1+\frac{ps}{N}$, which imply by estimate (53), *Aubin Simon* type result an the fact the function $T_k(s)$ is bounded that

$$\begin{cases} u_n \to \overline{u} \text{ in } L^1(Q), \\ T_k(u_n) \rightharpoonup T_k(\overline{u}) \text{ weakly in } L^p(0,1;W_0^{s,p}(\Omega)), \\ T_k(u_n) \to T_k(\overline{u}) \text{ strongly in } L^p(Q), \\ \nabla u_n \to \nabla\overline{u} \text{ a.e. in } Q. \end{cases}$$

[1] In the following we will indicate the parabolic cylinder $Q_1 := (0,1)\times\Omega$ by Q and the boundary $\Sigma_1 : (0,1)\times\Omega$ by Σ.

To prove that we can pass to the limit in the entropy formulation

$$\begin{aligned} &\int_\Omega \Theta_k(u_n-\varphi)dx - \int_\Omega \Theta_k(u_n(0,x)-\varphi(0))dx + \int_0^T \langle \varphi_t, T_k(u_n-\varphi)\rangle dt \\ &+\int_0^T \langle (-\Delta)^s_p u_n, T_k(u_n-\varphi)\rangle dt \le \int_0^T \int_\Omega T_k(u_n-\varphi)d\mu \end{aligned} \tag{54}$$

for every $\varphi \in L^p(0,T;X_0^{s,p}(\Omega)) \cap L^\infty(Q) \cap C([0,1];L^1(\Omega))$ such that φ_t belongs to $L^{p'}(0,T;X^{-s,p'}(\Omega))$. Recalling that $\mu = f - \operatorname{div}(G)$ with $f \in L^1(\Omega)$ and $G \in L^{p'}(\Omega)^N$, $T_k(u_n-\varphi)$ converges to $T_k(\overline{u}-\varphi)$ *-weakly in $L^\infty(Q)$, and $T_k(u_n-\varphi)$ converges to $T_k(\overline{u}-\varphi)$ weakly in $L^p(0,T;X_0^{s,p}(\Omega))$ we get

$$\int_Q T_k(u_n-\varphi)d\mu = \int_Q T_k(\overline{u}-\varphi)d\mu + \omega(k).$$

On the other hand, using the a.e. convergence of the gradients and *Fatou*'s lemma, we have

$$\begin{aligned} &\int_0^T \langle (-\Delta)^s_p u_n, T_k(u_n-\varphi)\rangle \\ &= \int_0^T \langle (-\Delta)^s_p u_n - (-\Delta)^s_p \varphi, T_k(u_n-\varphi)\rangle dt + \int_0^T \langle (-\Delta)^s_p \varphi, T_k(u_n-\varphi)\rangle dt \\ &\le \liminf_{n\to\infty} \int_0^T \langle (-\Delta)^s_p u_n - (-\Delta)^s_p \varphi, T_k(u_n-\varphi)\rangle dt \\ &= \int_0^T \langle (-\Delta)^s_p \varphi, T_k(\overline{u}-\varphi)\rangle dt + \omega(n). \end{aligned}$$

So, being $u(t,x)$ is monotone nondecreasing in t and using (51), there exists a function such that $u(t,x)$ converges to w a.e. in Ω as t tends to infinity, and satisfying

$$v(x) \le w(x) \le u(t,x) \le v^{\oplus}(x),$$

which implies, by *Dominated convergence theorem* and the fact that w does not depend on time, that

$$u(t,x) \to u \text{ in } L^1(\Omega).$$

Our aim to check that $\overline{u} = v$ a.e. in Ω. To do that, it suffices to pass to the limit in (54) (observe that $\overline{u}$ does not depend on time, i.e., $\overline{u}(t,x) = w(x)$ and $u_n(t,x) = u(t+n,x)$). Thus,

$$\begin{aligned} &\int_\Omega \Theta_k(u_n-\varphi)(1)dx - \int_\Omega \Theta_k(u_n(0,x)-\varphi(0,x))dx + \int_0^T \langle \varphi_t, T_k(u_n-\varphi)\rangle dt \\ &= \omega(n), \end{aligned}$$

which implies that $\bar{u}$ satisfies the entropy formulation for the elliptic problem (39). In fact

$$\begin{aligned}
&\int_\Omega \Theta_k(u_n-\varphi)(1)dx - \int_\Omega \Theta_k(u_n(0,x)-\varphi(0,x))dx\\
&= \int_\Omega \Theta_k(w(x)-\varphi(1))dx - \int_\Omega \Theta_k(w(x)-\varphi(0))dx + \omega(n)\\
&= \int_\Omega\int_0^1 \frac{d}{dt}\Theta(w(x)-\varphi)dtdx\\
&= \int_0^1 \langle w(x)-\varphi)_t, T_k(w(x)-\varphi)\rangle_{X^{-s,p'}(\Omega),X_0^{s,p}(\Omega)}dt.
\end{aligned}$$

Since $T_k(u_n-\varphi)$ converges to $T_k(u-\varphi)$ weakly in $L^p(0,T;X_0^{s,p}(\Omega))$, we have

$$\begin{aligned}
&\int_0^1 \langle \varphi_t, T_k(u_n-\varphi)\rangle_{X^{-s,p'}(\Omega),X_0^{s,p}(\Omega)}dt\\
&= \omega(n) + \int_0^1 \langle \varphi_t, T_k(w-\varphi)\rangle_{X^{-s,p'}(\Omega),X_0^{s,p}(\Omega)}dt
\end{aligned}$$

we deduce that (since w does not depend on time) that

$$\begin{aligned}
&\int_\Omega (u_n-\varphi)(1)dx - \int_\Omega \Theta_k(u_n(0,x)-\varphi(0,x))dx + \int_0^T \langle \varphi_t, T_k(u_n-\varphi)\rangle dt\\
&= \omega(n) + \int_0^1 \langle w_t, T_k(w-\varphi)\rangle_{X^{-s,p'}(\Omega),X_0^{s,p}(\Omega)}dt = \omega(n),
\end{aligned}$$

which implies that $w(x) = v(x)$. Similarly, using the same arguments we can prove that the solution of (1) with $v^\ominus$ as initial data converges to v in $L^1(\Omega)$. Hence, by Lemma 3.6, we conclude that the right holds to any solution $u(t,x)$ of (1) with u_0 such that $v^\ominus \le u_0 \le v^\oplus$.

Step.2 : The case $v^{\ominus,\tau} \le u_0 \le v^{\oplus,\tau}$ with $\tau > 1$. For any fixed $\tau > 1$ and recalling that $\mu^\pm$ can be decomposed as $\mu^\pm = f^\pm - \operatorname{div}(G^\pm)$ with $f^\pm \ge 0$ in $L^1(\Omega)$ and $G^\pm \in L^{p'}(\Omega)^N$ (see [19]) we readapt the idea of [66, 67] (see also [1]) to prove that the result holds true for every initial data satisfying $v^{\ominus,\tau} \le u_0 \le v^{\oplus,\tau}$ where $v^{\oplus,\tau}$ and $v^{\ominus,\tau}$ are solutions of the elliptic problems (39) with, respectively,

$$\mu^{\oplus,\tau} = \begin{cases} \tau\mu^+ & \text{if } f^+ = 0\\ \tau f^+ - (\operatorname{div}(G^+) & \text{if } f^+ \neq 0 \end{cases}$$

and

$$\mu^{\ominus,\tau} = \begin{cases} -\tau\mu^- & \text{if } f^- = 0\\ -\tau f^- - (\operatorname{div}(G^-) & \text{if } f^- \neq 0 \end{cases}$$

Step.3 : The case $u_0 \in L^1(\Omega)$ and $\mu \neq 0$. *Step.2* ensures, if u_τ with $\tau > 1$ is the entropy solution of problem (1) with $u_{0,\tau}(x)$ as initial data with, that

$$u_{0,\tau} = \begin{cases} \min(u_0, v^{\oplus,\tau}) & \text{if } u_0 \geq 0, \\ \max(u_0, v^{\ominus,\tau}) & \text{if } u_0 < 0. \end{cases}$$

Then, u_τ converges, as t tends to ∞, to v a.e. in Ω. On the other hand

$$T_k(u_\tau(t,x)) \underset{t\to\infty}{\rightharpoonup} T_k(v) \text{ weakly in } X_0^{s,p}(\Omega), \quad \forall k > 0.$$

Thus, by [67, Lemma 3.4], we have

$$u_{0,\tau} \to u_0 \text{ in } L^1(\Omega) \text{ as } \tau \text{ tends to infinity.}$$

So, by using the stability of the solution, we get

$$T_k(u_\tau(t,x)) \underset{\tau\to\infty}{\to} T_k(u(t,x)) \text{ strongly in } L^p(0,T;X_0^{s,p}(\Omega)).$$

Now, using the same calculations used in [79], to get the uniqueness of the solutions applied to u and u_τ we have

$$\int_\Omega \Theta_k(u-u_\tau)(t)dx \leq \int_\Omega \Theta_k(u_0 - u_{0,\tau})dx,$$

dividing by k and passing to the limit as k tends to zero, we obtain

$$\|u(t,x) - u_\tau(t,x)\|_{L^1(\Omega)} \leq \|u_0(x) - u_{0,\tau}(x)\|_{L^1(\Omega)},$$

i.e.,

$$\|u(t,x) - v(x)\|_{L^1(\Omega)} \leq \|v(t,x) - v(t,x)\|_{L^1(\Omega)} + \|u_\tau(t,x) - v(x)\|_{L^1(\Omega)},$$

it suffices to choose $\tau = \overline{\tau}$ large enough to get

$$\|u(t,x) - u_{\overline{\tau}}(t,x)\| \leq \frac{\varepsilon}{2}.$$

On the other hand, by *Step.2*, there exists $\bar{t} > 0$ such that

$$\|u_{\overline{\tau}}(t,x) - v(x)\|_{L^1(\Omega)} \leq \frac{\varepsilon}{2} \qquad \forall t > \bar{t},$$

we finally obtain that

$$u_{\overline{\tau}}(t,x) \to v(x) \text{ in } L^1(\Omega),$$

which concludes the proof of Theorem 3.7.

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