LE MATEMATICHE
Vol. LXXIX (2024) – Issue II, pp. 327–370
doi: 10.4418/2024.79.2.5

SOME NEW ITERATIVE SCHEMES FOR SOLVING GENERAL QUASI VARIATIONAL INEQUALITIES

M.A. NOOR - K.I. NOOR

Several new classes of general quasi variational inequalities involving two arbitrary operators are introduced and considered in this paper. Some important cases are discussed, which can be obtained by choosing suitable and appropriate choice of the operators. It is shown that the implicit obstacle boundary value can be studied via these quasi variational inequalities. Projection technique is applied to establish the equivalent between the general quasi variational inequalities and fixed point problems. This alternative formulation is used to discuss the uniqueness of the solution as well as to propose a wide class of proximal point algorithms. Convergence criteria of the proposed methods is considered. Asymptotic stability of the solution is studied using the first order dynamical system associated with variational inequalities. Second order dynamical systems associated with general quasi variational inequalities are applied to suggest some inertial type methods. Some special cases are discussed as applications of the main results. Several open problems are indicated for future research work.

1. Introduction

Stampacchia [70] proved that the minimum of a differentiable convex function associated with obstacle problem in potential problems can be characterized by

Received on April 18, 2024

AMS 2010 Subject Classification: 49J40, 90C33

Keywords: Variational inequalities, projection method, Wiener-Hopf equations, Dynamical system, iterative methods, convergence, sensitivity analysis

an inequality, which is called the variational inequality. Motivated and inspired by these facts, Lions and Stampacchia [27] considered and studied the variational inequalities. They also mentioned that the Riesz-Frechet representation theorem and Lax-Milgram lemma are special cases of the variational inequalities. Variational inequality theory can be viewed as a novel extension and generalization of the variational principles, the origin of which can be traced back to Euler, Lagrange, Newton and Bernoulli's brothers. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas. In fact, the history of variational principles comprises of the following distinct stages: The basic search of solutions of variational problems, led through the work of Euler, Lagrange, Legendre, Jacobi and many others, to develop along the lines of differential and integral equations as well as functional analysis. The Hamiltonian-Jacobi theory represents a general framework for the mathematical description of the propagation of actions in nature and optimal modeling of control processes in daily life. Using the ideas and techniques of Hamiltonian-Jacobi theory in mechanics, Cartan introduced differential geometry and exterior calculus in the calculus of variations. Many basic equations of mathematical physics result from variational problems. It is known that the gauge fields theories are a continuation of Einstein's concept of describing physical effects mathematically in terms of differential geometry. These theories play a fundamental role in the modern theory of elementary particles and are right tool of building up a unified theory of elementary particles, which includes all kind of known interactions. For example, the *Weinberg-Salam theory* unifies weak and electromagnetic interactions. It is also known that the variational formulation of field theories allows for a degree of unification absent their versions in terms of differential equations. It is amazing that a wide class of unrelated problems can be studied in the general and unified framework of variational inequalities, which occur in various branches of pure and applied sciences. For more details, see [1, 7, 9, 13, 15–20, 23–25, 28–49, 51–56, 58, 59, 61–65, 67]. These methods have been extended and modified in numerous ways for solving the quasi variational inclusions and their variant forms., see [2,4,10,11, 23,24,29,38,40, 48,52, 53,55,56,57,70,76,77] and the references there in. If the set involved in the variational inequality depends upon the solution explicitly or implicity, then the variational inequality is called the quasi-variational inequality, introduced by Bensoussan and Lions [10] in the field of impulse control. Noor [39] proved that the quasi-variational inequalities are equivalent to the implicit fixed point problem using the projection lemma(best approximation). This equivalent formulation played an important role in studying the unique existence of the solution and developing numerical methods, dynamical systems, sensitivity analysis and other aspects of quasi-variational inequalities. One of the most difficult and important problems in quasi variational inequalities is the development of efficient numerical methods. Several numerical methods have been developed for solving the variational inclusions and their variant forms. These methods have been extended and modified in numerous ways for solving the quasi variational inclusions and their variant forms. Noor [38–42, 46] suggested and analyzed several three-step forward-backward splitting algorithms for solving variational inequalities and quasi variational inclusions by using the updating technique. These three-step methods are also known as Noor's iterations. It is noted that these forward-backward splitting algorithms are similar to those of Glowinski et al. [20], which they suggested by using the Lagrangian technique. Haubruge et al. [17] discussed the convergence analysis and applications of the Glowinski-Le Tallec splitting method. It is known that three-step schemes are versatile and efficient. These three-step schemes are a natural generalization of the splitting methods for solving partial differential equations. For applications of the splitting techniques to partial differential equations, see Ames [3] and the references therein. In recent years, considerable interest has been shown in developing various extensions and generalizations of Noor iterations, both for their own sake and for their applications. In passing, we point out that the tree-step iterative methods are also known Noor iterations, which contain Mann (one step)iteration, Ishikawa (two-step) iterations as special cases. It have shown the Noor orbit demonstrates that the boundary of the fixed point region is similar to natural features such as bird nests and certain types of peacock wing structures. This is demonstrated by geometrical and numerical analysis of composite Julia sets and composite Mandelbrot sets for the Noor iteration, see Negi et al. [32]. Recently, Noor iterations [38–40] have been generalized and extended in various directions using innovative ideas to explore their applications in fractal, chaos, images, signal recovery, polynomiography, fixed point theory, compress programming, nonlinear equations, compressive sensing, solar energy optimizations and image in painting. For novel applications, modifications and generalizations of the Noor iterations, see [6–8, 22, 26, 50, 55, 58–60, 64, 67] and the references therein.

Variational inequalities represent the optimality conditions for the differentiable convex functions on the convex sets in normed space. It is known that the properties of the solutions of the variational inequalities may not hold, in general, when the convex set is non-convex. In recent years, the concept of convexity has been generalized in several directions, see, for example [11, 12, 42, 43, 68] and the references therein. A significant generalization of the convex set is the introduction of the general (g-convex) convex set [38, 42, 49] and general (g-convex) functions [42, 43]. We would like to emphasize that the general convex set and general convex functions may not be convex sets and convex functions. Noor [45] had proved that the minimum of a differentiable general convex function on the general convex set can be characterized by a class of variational inequality, which is also called the general variational inequality.

Shi [68] considered the problem of solving the Wiener-Hopf equations the using the fixed point formulation for solving the system of equations associated with variational inequalities independently. Noor [39] proved that quasi variational inequalities are equivalent to the implicit Wiener-Hopf equations. This technique has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inequalities. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [19]. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. This equivalent formulation is useful in studying the asymptotic stability of the solution of the variational inequality applying the Lyapunov theory of functional differential equations. This dynamical system is a first order initial value problem. Discretizing the dynamical system and using the finite difference idea, Noor et. al. [54] have been shown that the dynamical system can be used to suggest some implicit iterative method for solving quasi variational inequalities. For the applications and numerical methods applying the dynamical systems, see [19, 39, 45–47, 51, 52, 54, 66, 67] and the references therein. Variational inequalities are being used as a mathematical programming tool in modeling various equilibria in economics, operations research, optimization,regional, machine learning and transportation sciences. The behavior of such problems solution as a result of changes in the problem data is always of concern.

Motivated and inspired by ongoing research in these dynamic and active areas, we consider some new classes of general quasi variational inequalities involving two arbitrary operators. For appropriate and suitable choice of the operators, convex set-valued set and the space, we can obtain the inverse quasi variational inequalities, quasi complementarity problems and variational inequalities as special cases. Making use of the best approximation result, we show that the general quasi variational inequalities are equivalent to the fixed point problems. We use this alternative formulation to discuss the unique existence of the solution. Several multi step proximal point methods are proposed and investigated for solving the general quasi variational inequalities applying the fixed point, Wiener-Hopf, auxiliary principle and dynamical system. These methods include the Mann (one-step) iteration, Ishikawa (two-step) iteration, Noor (three-step)iteration and forward-backward splitting methods for finding the approximate solution. Convergence criteria is investigated under suitable conditions. We also considered the second order boundary value problem related to the variational inequalities coupled with dynamical system. Using the finite difference forward and backward interpolation, proximal point methods are proposed. We have only investigated the theoretical aspect of the iterative methods. Developments of the numerical applicable methods need further research efforts and can be considered an important open problems. Since the general quasi variational inequalities include the general variational inequalities, quasi variational inequalities and complementarity problems as special cases, our result continue to hold for these problems. It is expected the techniques and ideas of this paper may be starting point for further research.

2. Formulations and basic facts

Let Ω be a nonempty closed set in a real Hilbert space $\mathcal{H}$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively.

First of all, we now show that the minimum of a differentiable general convex function on a general convex set Ω in H can be characterized by the general variational inequalities. For this purpose, we recall the following well known concepts and results [10, 27, 42, 43].

Definition 2.1. A set Ω_g is said to be a general convex set, if there exist a function $g : H \longrightarrow H$ such that

$$\mu + t(g(\nu) - \mu) \in \Omega_g, \quad \forall \mu . \nu \in \Omega_g, \quad t \in [0, 1].$$

Note that every convex set is a general convex set, but the converse is not true. It is worth mentioning that the general convex (g-convex) set is different than the E-convex set of Youness [77] and various general convex sets [14–16]. For the applications of the general convex sets in information technology, railway systems, computer aided design, digital vector optimization and comparison with other concepts, see Cristescu et al. [14–16]. If $g = I$, then the general convex set Ω_g is exactly the convex set Ω.

Definition 2.2. A function $F : \Omega_g \longrightarrow H$ is said to be a general convex, if there exists a function g such that

$$F(\mu + t(g(\nu) - \mu)) \leq (1 - t)F(\mu) + tF(g(\nu)), \quad \forall \mu, \nu \in \Omega_g, \quad t \in [0, 1].$$

Clearly every convex function is a general convex, but the converse is not true, see [48, 49].

Lemma 2.3. *Let $F : \Omega_g \longrightarrow H$ be a differentiable general convex function. Then $\mu \in \Omega_g$ is the minimum of general convex function F on Ω_g, if and only if, $\mu \in \Omega_g$ satisfies the inequality*

$$\langle F'(\mu), g(\nu) - \mu \rangle \geq 0, \quad \forall \in \Omega_g, \tag{1}$$

where $F'(\mu)$ is the differential of F at $\mu \in \Omega_g$.

Proof. Let $\mu \in \Omega_g$ be a minimum of a differentiable general convex function F on Ω_g. Then

$$F(\mu) \leq F(g(\nu)), \quad \forall \nu \in \Omega_g. \tag{2}$$

Since Ω_g is a general convex set, so, for all $\mu, \nu \in \Omega_g, t \in [0,1], \nu_t = \mu + t(g(\nu) - \mu) \in \Omega_g$. Setting $\nu = \nu_t$ in (2), we have

$$F(\mu) \leq F(\mu + t(g(\nu) - \mu)).$$

Dividing the above inequality by t and taking $t \longrightarrow 0$, we have

$$\langle F'(\mu), g(\nu) - \mu \rangle \geq 0, \quad \forall \nu \in \Omega_g,$$

which is the required result(1).
Conversely, let $\mu \in \Omega_g$ satisfy the inequality (1). Since F is a general convex function, $\quad \forall \mu, \nu \in \Omega_g \quad t \in [0,1], \quad \mu + t(g(\nu) - \mu) \in \Omega_g$ and

$$F(\mu + t(g(\nu) - \mu)) \leq (1-t)F(\mu) + tF(g(\nu)),$$

which implies that

$$F(g(\nu)) - F(\mu) \geq \frac{F(\mu + t(g(\nu) - \mu)) - F(\mu)}{t}.$$

Letting $t \longrightarrow 0$, and using (1), we have

$$F(g(\nu) - F(\mu) \geq \langle F'(\mu), g(\nu) - \mu \rangle \geq 0,$$

which implies that

$$F(\mu) \leq F(g(\nu)), \quad \forall \in \Omega_g$$

showing that $\mu \in \Omega_g$ is the minimum of F on Ω_g in H. □

The inequality of the type (1) is called the general variational inequality. It is known that the problem (1) may not arise as the optimality conditions of the differentiable convex functions. This motivated us to consider a more general problem of which the problem (1) is a special case. To be more precise, for given operators $\mathtt{T}, g : H \longrightarrow H$, consider the problem of finding $\mu \in \Omega \subseteq H$, such that

$$\langle \mathtt{T}\mu, g(\nu) - \mu \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{3}$$

which is called the general variational inequality, introduced and studied by Noor [38].

We now introduce the problem of general quasi variational inequality. Let $\Omega \subseteq \mathtt{H} \longrightarrow \mathtt{H}$ be a set-valued mapping which, for any element $\mu \in \mathtt{H}$, associates a convex-valued and closed set $\Omega(\mu) \subseteq \mathtt{H}$. we consider the problem of finding $\mu \in \Omega(\mu)$, such that

$$\langle \mathcal{T}\mu + \mu - g(\mu), g(\nu) - \mu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{4}$$

which is called the general quasi variational inequality.

Special Cases

We now point out some very important and interesting problems, which can be obtained as special cases of the problem (4).

(I). If $g(\mu) = \mu$, then problem (4) reduces to finding $\mu \in \Omega(\mu)$, such that

$$\langle \mathcal{T}(g(\mu)), g(\nu) - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{5}$$

is called the general quasi variational inequality.

(II). If $\mathcal{T} = I$, the identity operator, then problem (5)reduces to finding $\mu \in \Omega(\mu)$ such that

$$\langle \mu, g(\nu) - \mu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{6}$$

This inequality is called the inverse quasi variational inequality.

(III). If $g = I$, then the problem (4) collapses to finding $\mu \in \Omega(\mu)$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{7}$$

which is called quasi variational inequality, introduced by Bensoussan and Lions [7] in the impulse control theory. For the numerical analysis, sensitivity analysis, dynamical systems and other aspects of quasi variational inequalities and related optimization programming problems. see [7, 14, 18, 19, 24, 31, 33, 36, 40, 41, 44–49, 53, 54, 62] and the references therein.

(IV). If $\Omega^*(\mu) = \{\mu \in \mathcal{H} : \langle \mu, g(\nu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu)\}$ is a polar (dual) cone of a convex-valued cone $\Omega(\mu)$ in $\mathcal{H}$, then problem (4) is equivalent to finding $\mu \in \mathcal{H}$, such that

$$g(\mu) \in \Omega(\mu), \quad \mathcal{T}\mu \in \Omega^*(\mu) \quad \text{and} \quad \langle \mathcal{T}\mu, g(\mu)\rangle = 0, \tag{8}$$

which is known as the general quasi complementarity problems and appears to be a new one.

(V). For the polar cone $\Omega^* = \{\mu \in \mathcal{H} : \langle \mu, g(\nu)\rangle \geq 0, \forall \nu \in \Omega\}$, the problem (8) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \Omega, \quad \mathcal{T}\mu \in \Omega^* \quad \text{and} \quad \langle \mathcal{T}\mu, g(\mu)\rangle = 0, \tag{9}$$

is called the general complementarity problem. Obviously general quasi complementarity problems include the general complementarity problem nolinear, complementary problems and linear complementarity problems, which were introduced and studied in Cottle et al. [13] and Noor [37, 40]. This inter relations among these problems have played a major role in developing numerical results for these problems and their applications.

(VI). If $\Omega(\mu) = \Omega$, where Ω is a convex set in $\mathcal{H}$, then problem (4) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, g(\nu) - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{10}$$

which is the general variational inequalities (3).

(VII). If $\Omega(\mu) = \Omega$, then problem (5) reduces to finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}(g(\mu)), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{11}$$

is called the general variational inequality.

(VIII). If $g = I$, then problem (11) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{12}$$

is known as the variational inequality, which was introduced by Lions and Stampacchia [23], For the recent applications, generalizations, extensions, numerical results, dynamical systems, sensitivity and other aspects, see [17–20, 23–25, 28–49, 51–56, 58, 59, 61–65, 67] and the references therein.

Remark 2.4. It is worth mentioning that for appropriate and suitable choices of the operators $\mathcal{T}, g$, set-valued convex set $\Omega(\mu)$ and the spaces, one can obtain several classes of variational inequalities, complementarity problems and optimization problems as special cases of the nonlinear quasi-variational inequalities (4). This shows that the problem (4) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the general quasi-variational inequalities and their variants.

For the sake of completeness and to convey the main ideas, we include the following example, which is mainly due to Noor and Noor [45, 46].

Example 2.5. [45, 46]. To convey an idea of the applications of the general quasi variational inequalities, we consider the implicit second-order obstacle boundary value problem of finding μ such that

$$\left.\begin{array}{ll} -\mu'' \geq \phi(x) & \text{on } \Omega_1 = [a,b] \\ \mu \geq \mathcal{M}(\mu) & \text{on } \Omega_1 = [a,b] \\ [\mu'' + \phi(x)][\mu - \mathcal{M}(\mu)] = 0 & \text{on } \Omega_1 = [a,b] \\ \mu(a) = 0, \quad \mu(b) = 0. & \end{array}\right\} \tag{13}$$

where $\phi(x,\mu)$ is a continuous function and $\mathcal{M}(\mu)$ is the cost (obstacle) function. The prototype encountered is

$$\mathcal{M}(\mu) = \eta + \inf_i \{\mu^i\}. \tag{14}$$

In (14), η represents the switching cost. It is positive, when the unit is turned on and equal to zero when the unit is turned off. The operator $\mathcal{M}$ provides the coupling between the unknowns $\mu = (\mu^1, \mu^2, \ldots, \mu^i)$. We study the problem (13) in the framework of quasi variational inequality approach. To do so, we first define the set as

$$\Omega(\mu) = \{\nu : \nu \in \mathcal{H}_0^1(\Omega_1) : \nu \geq \mathcal{M}(\mu), \quad \text{on } \Omega_1\},$$

which is a closed convex-valued set in $\mathcal{H}_0^1(\Omega)$, where $\mathcal{H}_0^1(\Omega)$ is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (13) is

$$\begin{aligned} \mathcal{I}[\nu] &= -\int_a^b \left(\frac{d^2\nu}{dx^2}\right) \nu dx - 2\int_a^b \phi(x)\nu dx, \quad \forall \nu \in \Omega(\mu) \\ &= \int_a^b \left(\frac{d\nu}{dx}\right)^2 dx - 2\int_a^b \phi(x)\nu dx \\ &= \langle \mathcal{T}\nu, \nu\rangle - 2\langle \phi(x), \nu\rangle, \end{aligned} \tag{15}$$

where

$$\begin{aligned}\langle \mathcal{T}\mu, \nu\rangle &= -\int_a^b \left(\frac{d^2\mu}{dx^2}\right)\nu dx = \int_a^b \frac{d\mu}{dx}\frac{d\nu}{dx}dx \qquad (16)\\ \phi(\nu) &= \int_a^b \phi(x)\nu dx.\end{aligned}$$

It is clear that the operator $\mathcal{T}$ defined by (16) is linear, symmetric and positive. Noor and Noor [50] have shown that the minimum of the functional $\mathcal{I}[\nu]$ defined by (15) associated with the problem (13) on the closed convex-valued set $\Omega(\mu)$ can be characterized by the inequality of type

$$\langle \mathcal{T}\mu, \nu-\mu\rangle \geq \langle \phi, \nu-\mu\rangle, \quad \forall \nu \in \Omega(\mu), \qquad (17)$$

which is exactly the quasi variational inequality (7).

We also need the following result, known as the projection Lemma(best approximation), which plays a crucial part in establishing the equivalence between the general quasi variational inequalities and the fixed point problems. This result is used in the analysing the convergence analysis of the projective implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 2.6. *[36, 39] Let $\Omega(\mu)$ be a closed and convex-valued set in $\mathcal{H}$. Then, for a given $z \in \mathcal{H}$, $\mu \in \Omega(\mu)$ satisfies the inequality*

$$\langle \mu - z, \nu-\mu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \qquad (18)$$

if and only if,

$$\mu = \Pi_{\Omega(\mu)}(z),$$

where $\Pi_{\Omega(\mu)}$ is implicit projection of $\mathcal{H}$ onto the closed convex-valued set $\Omega(\mu)$.

It is well known that the implicit projection operator $\Pi_{\Omega(\mu)}$ is not nonexpansive, but it is required to satisfy the following assumption, which plays an important part in the derivation of the results..

Assumption 1.

$$\|\Pi_{\Omega(\mu)}\omega - \Pi_{\Omega(\nu)}\omega\| \leq \eta\|\mu-\nu\|, \forall \mu, \nu, \omega \in \mathcal{H}, \qquad (19)$$

where $\eta > 0$ is a constant.

Assumption 1 has been used to prove the existence of a solution of general quasi variational inequalities as well as in analyzing convergence of the iterative methods.
In many important applications, the convex-valued set $\Omega(\mu)$ can be written as

$$\Omega(\mu) = m(\mu) + \Omega,$$

is known as the moving convex set,where $m(\mu)$ is a point-point mapping and Ω is a convex set. In this case, we have

$$\Pi_{\Omega(\mu)}\omega = \Pi_{m(\mu)+\Omega} = m(\mu) + \Pi_{\Omega}[w - m(\mu)], \quad \forall \mu, w \in \Omega.$$

We note that, if $m(\mu)$ is a Lipschitz continuous mapping with constant $\gamma > 0$, then

$$\begin{aligned} \|\Pi_{\Omega(\mu)}w - \Pi_{\Omega(\nu)}w\| &= \|m(\mu) - m(\nu) + \Pi_{\Omega}[w - m(\mu)] - \Pi_{\Omega}[w - m(\nu)\| \\ &\leq 2\|m(\mu) - m(\nu)\| \leq 2\gamma\|\mu - \nu\|, \quad \forall \mu, \nu, w \in \Omega. \end{aligned}$$

which shows that Assumption 1 holds with $\eta = 2\gamma$.

Definition 2.7. [20, 38] An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:

(i). Strongly monotone, if there exist a constant $\alpha > 0$, such that

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}.$$

(ii). Lipschitz continuous, if there exist a constant $\beta > 0$, such that

$$\|\mathcal{T}\mu - \mathcal{T}\nu\| \leq \beta\|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$

(iii). Monotone, if

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

(iv). Pseudo monotone, if

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}\nu, \nu - \mu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

Remark 2.8. Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.

3. Projection Method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the general quasi variational inequalities.

Using Lemma 2.6, one can show that the general quasi variational inequalities are equivalent to the fixed point problems.

Lemma 3.1. [54] *The function $\mu \in \Omega(\mu)$ is a solution of the general quasi variational inequality (4), if and only if, $\mu \in \Omega(\mu)$ satisfies the relation*

$$\mu = \Pi_{\Omega(\mu)}[g(\mu) - \rho T\mu], \tag{20}$$

where $\Pi_{\Omega(\mu)}$ is the projection operator and $\rho > 0$ is a constant.

Proof. Let $\mu \in \Omega(\mu)$ be the problem (4). Then

$$\langle \rho T\mu + \mu - g(\mu), h(\nu) - \mu\rangle \geq 0, \quad \forall \nu \in \Omega_\mu.$$

Using Lemma 2.6, we have

$$\mu = \Pi_{\Omega(\mu)}[g(\mu) - \rho T\mu],$$

the required result. □

Lemma 3.1 implies that the general quasi variational inequality (4) is equivalent to the fixed point problem (20). This equivalent fixed point formulation (20) will play an important role in deriving the main results.

From the equation (20), we have

$$\mu = \Pi_{\Omega(\mu)}[g(\mu) - \rho T\mu].$$

We define the function F associated with (20) as

$$F(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho T\mu], \tag{21}$$

To prove the unique existence of the solution of the problem (4), it is enough to show that the map F defined by (21) has a fixed point.

Theorem 3.2. *Let the operators $\mathcal{T}, g$ be strongly monotone with constants $\alpha > 0, \sigma > 0$ and Lipschitz continuous with constants $\beta > 0, \zeta > 0$, respectively. If Assumption 1 holds and there exists a parameter $\rho > 0$, such that*

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{22}$$

where

$$\begin{aligned}
\theta &= \sqrt{(1-2\alpha\rho+\rho^2\beta^2)}+k \\
k &= \sqrt{1-2\sigma+\zeta^2}+\eta.
\end{aligned}$$

then there exists a unique solution of the problem (4).

Proof. From Lemma 3.1, it follows that problems (20) and (4) are equivalent. Thus it is enough to show that the map $F(\mu)$, defined by (21) has a fixed point. For all $\nu \neq \mu \in \Omega(\mu)$, we have

$$\begin{aligned}
\|F(\mu)-F(\nu)\| &= \Pi_{\Omega(\mu)}\|[g(\mu)-\rho T\mu]-\Pi_{\Omega(\nu)}[g(\nu)-\rho T\nu]\| \\
&= \|\Pi_{\Omega(\mu)}[g(\nu)-\rho\mathcal{T}\nu]-\Pi_{\Omega(\nu)}[g(\nu)-\rho\mathcal{T}\nu]\| \\
&+ \|\Pi_{\Omega(\nu)}[g(\nu)-\rho\mathcal{T}\nu]-\Pi_{\Omega(\mu)}[g(\mu)-\rho(\mathcal{T}\mu]\| \\
&\leq \|\mu-\nu-(g(\mu)-g(\nu))\|+\eta\|\nu-\mu\| \\
&\quad +\|\nu-\mu-\rho(\mathcal{T}\nu-\mathcal{T}\mu)\|. \qquad (23)
\end{aligned}$$

Since the operators $g,\mathcal{T}$ are strongly monotone with constant $\sigma>0,\alpha>0$ and Lipschitz continuous with constant $\zeta>0,\beta>0$, respectively, it follows that

$$\begin{aligned}
\|\mu-\nu-(g(\mu)-g(\nu))\|^2 &\leq \|\mu-\nu\|^2-2\langle g(\mu)-g(\nu),\mu-\nu\rangle+\zeta^2\|g(\mu)-g(\nu)\|^2 \\
&\leq (1-2\sigma+\zeta^2)\|\mu-\nu\|^2. \qquad (24)
\end{aligned}$$

and

$$\begin{aligned}
\|\mu-\nu-(\mathcal{T}\mu-\mathcal{T}\nu)\|^2 &\leq \|\mu-\nu\|^2-2\langle\mathcal{T}\mu-\mathcal{T}\nu,\mu-\nu\rangle+\zeta^2\|\mathcal{T}\mu-\mathcal{T}\nu\|^2 \\
&\leq (1-2\alpha\rho+\rho^2\beta^2)\|\mu-\nu\|^2. \qquad (25)
\end{aligned}$$

From (49), (24)and (25), we have

$$\begin{aligned}
\|F(\mu)-F(\nu)\| &\leq \left\{\sqrt{(1-2\sigma+\zeta^2)}+\eta+\sqrt{(1-2\alpha\rho+\rho^2\beta^2)}\right\}\|\mu-\nu\| \\
&= \theta\|\mu-\nu\|,
\end{aligned}$$

where

$$\theta = \sqrt{(1-2\alpha\rho+\rho^2\beta^2)}+k \qquad (26)$$

$$k = \sqrt{1-2\sigma+\zeta^2}+\eta. \qquad (27)$$

From (22), it follows that $\theta<1$, which implies that the map $F(u)$ defined by (21) has a fixed point, which is the unique solution of (4). ☐

The fixed point formulation (20) is applied to propose and suggest the iterative methods for solving the problem (4).

This alternative equivalent formulation (20) is used to suggest the following iterative methods for solving the problem (4) using the updating technique of the solution.

Algorithm 1. For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes

$$\begin{aligned} y_n &= \{\Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho \mathcal{T} \mu_n]\} && (28) \\ w_n &= \{\Pi_{\Omega(y_n)}[g(y_n) - \rho \mathcal{T} y_n]\} && (29) \\ \mu_{n+1} &= \{\Pi_{\Omega(w_n)}[(w_n) - \rho \mathcal{T} w_n]\}. && (30) \end{aligned}$$

Algorithm 2 is a three step forward-backward splitting algorithm for solving general quasi variational inequality (4). This method is very much similar to that of Glowinski et al. [20] for variational inequalities, which they suggested by using the Lagrangian technique.

We now study the convergence analysis of Algorithm 2, which is the main motivation of our next result.

Theorem 3.3. *Let the operators $\mathcal{T}, g$ satisfy all the assumptions of Theorem 3.2. If the condition (22) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 2 converges to the exact solution $\mu \in \Omega(\mu)$ of the general quasi variational inequality (4) strongly in $\mathcal{H}$.*

Proof. From Theorem 3.2, we see that there exists a unique solution $\mu \in \Omega(\mu)$ of the general quasi variational inequalities (4). Let $\mu \in \Omega(\mu)$ be the unique solution of (4). Then, using Lemma 3.1, we have

$$\begin{aligned} \mu &= \{\Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T} \mu]\} && (31) \\ &= \{\Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T} \mu]\} && (32) \\ &= \{\Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T} \mu]\}. && (33) \end{aligned}$$

From (30),(31) and Assumption (1), we have

$$\begin{aligned} \|\mu_{n+1} - \mu\| &= \Pi_{\Omega((w_n)}[g(w_n) - \rho \mathcal{T} w_n] - \Pi_{(\mu)}[\mu - \rho \mathcal{T} \mu\}\| \\ &\leq \|\Pi_{\Omega((w_n)}[w_n - \rho \mathcal{T} w_n] - \Pi_{\Omega(w_n)}[g(\mu_n) - \rho \mathcal{T} \mu\}\| \\ &+ \|\{\Pi_{(w_n)}[g(\mu_n) - \rho \mathcal{T} \mu] - \Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T} \mu\}\| \\ &\leq \|g(w_n) - g(\mu) - \rho(\mathcal{T} w_n - \mathcal{T} \mu)\| + \eta \|w_n - \mu\| \\ &\leq \|w_n - \mu_n - (g(w_n) - g(\mu_n))\| + \|w_n - \mu_n - \rho(\mathcal{T} w_n - \mathcal{T} \mu)\| \\ &\leq \theta \|w_n - \mu\|, && (34) \end{aligned}$$

where θ is defined by (26).
In a similar way, from (28) and (32), we have

$$\begin{aligned}\|w_n-\mu\| &\leq \theta\|y_n-\mu-(g(y_n)-g(\mu))\| \\ &\quad +\|y_n-\mu-\rho(Ty_n-T\mu)\|+\eta\|y_n-\mu\| \\ &\leq \theta\|y_n-\mu\|, \end{aligned} \tag{35}$$

where θ is defined by (26).
From (28) and (33), we obtain

$$\|y_n-\mu\| \leq \theta\|\mu_n-\mu\|. \tag{36}$$

From (35) and (36), we obtain

$$\|w_n-\mu\| \leq \theta\|\mu_n-\mu\|. \tag{37}$$

Form the above we equations, have

$$\|\mu_{n+1}-\mu\| \leq \theta\|\mu_n-\mu\|.$$

From (22), it follows that $\theta < 1$, Consequently the sequence $\{u_n\}$ converges strongly to μ. From (36), and (37), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

We now suggested and analyzed the three step scheme for solving the general quasi variational inequality (4). These three step schemes also are called the novel Noor iterations. For the applications of novel Noor iterations in signal recovery, polynomiography, fixed point theory, compress programming, nonlinear equations, compressive sensing and image in painting, see [3–6, 8, 22, 26, 55, 58, 59, 64] and the references therein.

Algorithm 2. For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n+\gamma_n\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho\mathcal{T}\mu_n] \\ w_n &= (1-\beta_n)y_n+\beta_n\Pi_{\Omega(y_n)}[g(y_n)-\rho\mathcal{T}y_n] \\ \mu_{n+1} &= (1-\alpha_n)w_n+\alpha_n\Pi_{\Omega(w_n)}[g(w_n)-\rho\mathcal{T}w_n]. \end{aligned}$$

Convergence analysis of Algorithm 2 can be studied using the techniques as developed in [7, 26, 55, 58]. For $\gamma_n = 0$, Algorithm 2 reduces to:

Algorithm 3. For a given $\mu_0 \in \Omega(\mu)$, compute $\{\mu_{n+1}\}$ by the iterative schemes

$$\begin{aligned} w_n &= (1-\beta_n)\mu_n+\beta_n\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= (1-\alpha_n)\mu_n+\alpha_n\Pi_{\Omega(\mu_n)}[g(w_n)-\rho\mathcal{T}w_n], \end{aligned}$$

which is known as the Ishikawa iterative scheme for the problem (4).
Note that for $\gamma_n = 0$ *and* $\beta_n = 0$, Algorithm 2 is called the Mann iterative method, that is.

Algorithm 4. For a given $\mu_0 \in \Omega(\mu)$, compute $\{\mu_{n+1}\}$ by the iterative schemes

$$\mu_{n+1} = (1-\beta_n)\mu_n + \beta \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho \mathcal{T} \mu_n].$$

We suggest another perturbed iterative scheme for solving the general quasi variational inequality (4).

Algorithm 5. For a given $\mu_o \in \mathcal{H}$, compute the approximate solution $\{\mu_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n + \gamma_n \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho T \mu_n] + \gamma_n h_n \\ w_n &= (1-\beta_n) y_n + \beta_n \Pi_{\Omega(y_n)}[g(y_n) - \rho \mathcal{T} y_n] + \beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n) w_n + \alpha_n \Pi_{\Omega(w_n)}[g(w_n) - \rho \mathcal{T} w_n] + \alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and $\Pi_{\Omega(\mu_n)}$ is the corresponding perturbed projection operator and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0 \le \alpha_n, \beta_n, \gamma_n \le 1; \quad \forall n \ge 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving general variational inequality (4).

Algorithm 6. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho \mathcal{T} \mu_n], \quad n = 0, 1, 2, ...$$

which is known as the projection method and has been studied extensively.

Algorithm 7. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[g(\mu_n) - \rho \mathcal{T} \mu_{n+1}], \quad n = 0, 1, 2, ...$$

which is known as the implicit projection method and is equivalent to the following two-step method.

Algorithm 8. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho \mathcal{T} \mu_n] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[g(\mu_n) - \rho T \omega_n], \quad n = 0, 1, 2, ... \end{aligned}$$

We also propose the following iterative method.

Algorithm 9. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho \mathcal{T} \mu_{n+1}], \quad n = 0, 1, 2, ...$$

which is known as the modified projection method and is equivalent to the iterative method.

Algorithm 10. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho \mathcal{T} u_n] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[g(\omega_n) - \rho \mathcal{T} \omega_n], \quad n = 0, 1, 2, ... \end{aligned}$$

which is two-step predictor-corrector method for solving the problem (4). We can rewrite the equation (20) as:

$$\mu = \Pi_{\Omega(\mu)}\left[g\left(\frac{\mu + \mu}{2}\right) - \rho \mathcal{T}\right]. \tag{38}$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 11. [56]. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}\left[g\left(\frac{\mu_n + \mu_{n+1}}{2}\right) - \rho \mathcal{T} \mu_{n+1}\right]. \tag{39}$$

Applying the predictor-corrector technique, we suggest the following inertial iterative method for solving the problem (4) .

Algorithm 12. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho \mathcal{T} \mu_n] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}\left[g\left(\frac{\omega_n + \mu_n}{2}\right) - \rho \mathcal{T} \omega_n\right]. \end{aligned}$$

From equation (20), we have

$$\mu = \Pi_{\Omega(\mu)}\left[g(\mu) - \rho \mathcal{T}\left(\frac{\mu + \mu}{2}\right)\right]. \tag{40}$$

This fixed point formulation (40) is used to suggest the implicit method for solving the problem (4) as:

Algorithm 13. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}\left[g(\mu_n) - \rho\mathcal{T}\left(\frac{\mu_n + \mu_{n+1}}{2}\right)\right]. \tag{41}$$

We can use the predictor-corrector technique to rewrite Algorithm 13 as:

Algorithm 14. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned}\omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n],\\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}\left[g(\mu_n) - \rho\mathcal{T}\left(\frac{\mu_n + \omega_n}{2}\right)\right].\end{aligned}$$

is known as the mid-point implicit method for solving the problem (4).

We again use the above fixed formulation to suggest the following implicit iterative method.

Algorithm 15. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}\left[g(\mu_{n+1}) - \rho\mathcal{T}\left(\frac{\mu_n + \mu_{n+1}}{2}\right)\right]. \tag{42}$$

Using the predictor-corrector technique, Algorithm 15 can be written as:

Algorithm 16. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned}\omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n],\\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}\left[g(\omega_n) - \rho\mathcal{T}\left(\frac{\mu_n + \omega_n}{2}\right)\right],\end{aligned}$$

which appears to be new one.

It is obvious that the above Algorithms have been suggested using different variant of the fixed point formulations (20). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the problem (4) and related optimization problems.

One can rewrite (20) as

$$\mu = \Pi_{\Omega(\mu)}\left[g\left(\frac{\mu + \mu}{2}\right) - \rho\mathcal{T}\left(\frac{\mu + \mu}{2}\right)\right]. \tag{43}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (4).

Algorithm 17. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}\left[g\left(\frac{\mu_n+\mu_{n+1}}{2}\right) - \rho\mathcal{T}\left(\frac{\mu_n+\mu_{n+1}}{2}\right)\right]. \tag{44}$$

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 9 as the predictor and Algorithm 17 as corrector. Thus, we obtain a new two-step method for solving the problem (4).

Algorithm 18. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned}\omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n]\\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}\left[g(\frac{\omega_n+\mu_n}{2}) - \rho\mathcal{T}\left(\frac{\omega_n+\mu_n}{2}\right)\right],\end{aligned}$$

which is a new predictor-corrector two-step method.
For a parameter ξ, one can rewrite the (20) as

$$\mu = \Pi_{\Omega(u)}[g\big((1-\xi)\mu + \xi\mu)\big) - \rho\mathcal{T}\mu].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (4).

Algorithm 19. For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}[g\big((1-\xi)\mu_n + \xi\mu_{n-1}\big) - \rho\mathcal{T}\mu_n], \quad n = 0,1,2,....$$

It is noted that Algorithm 19 is equivalent to the following two-step method.

Algorithm 20. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the inertial iterative scheme

$$\begin{aligned}\omega_n &= (1-\xi)u_n + \xi u_{n-1}\\ \mu_{n+1} &= \Pi_{\Omega(\mu_n)}[g(\omega_n) - \rho\mathcal{T}\mu_n].\end{aligned}$$

Using this idea, we can suggest the following iterative methods for solving general quasi variational inequalities.

Algorithm 21. For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \xi_n(u_n - u_{n-1}) + \alpha_n\Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n], \quad n = 0,1,2,...$$

which is called the inertial proximal point method and appears to be new one.

Here $\alpha_n, \xi_n \geq 0$ are constants and term $\xi_n(u_n - u_{n-1})$ is called the inertial term.

Algorithm 22. For a given $u_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\xi)u_n + \xi u_{n-1} \\ u_{n+1} &= \Pi_{\Omega(y_n)}[g(y_n) - \rho T y_n)], \quad n = 0,1,2,.... \end{aligned}$$

We now suggest multi-step inertial methods for solving the general quasi variational inequalities (4).

Algorithm 23. For given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}) \\ y_n &= (1-\beta_n)\omega_n + \beta_n \Pi_{(\omega_n)}\left[g\left(\frac{\omega_n + \mu_n}{2}\right) - \rho\mathcal{T}\left(\frac{\omega_n + \mu_n}{2}\right)\right], \\ \mu_{n+1} &= (1-\alpha_n)y_n + \alpha_n \Pi_{\Omega(y_n)}\left[g\left(\frac{\omega_n + \mathtt{y}_\mathtt{n}}{2}\right) - \rho\mathcal{T}\left(\frac{\mathtt{y}_n + \omega_n}{2}\right)\right], \end{aligned}$$

where $\beta_n, \alpha_n, \Theta_n \in [0,1], \forall n \geq 1$.

Algorithm 23 is a three-step modified inertial method for solving general quasi variational inclusion(4).
Similarly a four-step inertial method for solving the general quasi variational inequalities (4) is suggested.

Algorithm 24. For given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}), \\ t_n &= (1-\gamma_n)\omega_n + \gamma_n \Pi_{(\omega_n)}\left[g\left(\frac{\omega_n + \mu_n}{2}\right) - \rho\mathcal{T}\left(\frac{\omega_n + \mu_n}{2}\right)\right], \\ y_n &= (1-\beta_n)t_n + \beta \Pi_{\Omega(\mu_n)}\left[g\left(\frac{t_n + \omega_n}{2}\right) - \rho\mathcal{T}\left(\frac{t_n + \omega_n}{2}\right)\right], \\ \mu_{n+1} &= (1-\alpha_n)y_n + \alpha_n \Pi_{\Omega(y_n)}\left[g\left(\frac{y_n + t_n}{2}\right) - \rho\mathcal{T}\left(\frac{y_n + t_n}{2}\right)\right], \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1], \quad \forall n \geq 1$.

Using the technique of Noor et al. [38] and Jabeen et al [22], one can investigate the convergence analysis of these inertial projection methods.

4. Wiener-Hopf Equations Technique

In this section, we discuss the Wiener-Hopf equations associated with the quasi variational inequalities. It is worth mentioning that the Wiener-Hopf equations associated with variational inequalities were introduced and studied by Shi [68]. Noor [42] proved that the quasi variational inequalities are equivalent to the implicit Wiener-Hopf equations.

We now consider the problem of solving the Wiener-Hopf equations related to the general quasi variational inequalities. Let $\mathcal{T}$ be an operator and $\mathcal{R}_{\Omega(\mu)} = \mathcal{I} - \Pi_{\Omega(\mu)}$, where $\mathcal{I}$ is the identity operator and $\Pi_{\Omega(\mu)}$ is the projection operator. We consider the problem of finding $z \in \mathcal{H}$ such that

$$\mathcal{T}\Pi_{\Omega(\mu)}z + \rho^{-1}\mathcal{R}_{\Omega(\mu)}z = 0. \tag{45}$$

The equations of the type (45) are called the implicit Wiener-Hopf equations. It have been shown that the implicit Wiener-Hopf equations play an important part in the developments of iterative methods, sensitivity analysis and other aspects of the variational inequalities and related optimization problems.

Lemma 4.1. *The element $\mu \in \Omega(\mu)$ is a solution of the quasi variational inequality* (4)*, if and only if, $z \in \mathcal{H}$ satisfies the resolvent equation* (45)*, where*

$$\mu = \Pi_{\Omega(\mu)}z, \tag{46}$$
$$z = g(\mu) - \rho\mathcal{T}\mu, \tag{47}$$

where $\rho > 0$ is a constant.

From Lemma 4.1, it follows that the general quasi variational inequalities (4) and the implicit Wiener-Hopf equations (45) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving the strongly nonlinear quasi variational inequalities and related optimization problems.
We use the Wiener-Hopf equations (45) to suggest some new iterative methods for solving the nonlinear quasi variational inequalities. From (46) and (47),

$$z = g(\Pi_{\Omega(\mu)}z) - \rho\mathcal{T}g(\Pi_{\Omega(\mu)}z).$$

Thus, we have

$$g(\mu) = \rho T\mu + g(\mu) - \rho Tg(\Pi_{\Omega(\mu_n)}[g(\mu) - \rho\mathcal{T}\mu]).$$

implies that

$$\rho T\mu - \rho Tg(\Pi_{\Omega(\mu_n)}[g(\mu) - \rho T\mu]) = 0.$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned}\mu &= (1-\alpha_n)\mu + \alpha_n\{\rho T g(\Pi_{\Omega(\mu_n)})[g(\mu) - \rho T\mu] - \rho\mathcal{T}\mu\} \\ &= (1-\alpha_n)\mu + \alpha_n\Pi_{\Omega(\mu)}\{\rho\mathcal{T}\omega - \rho\mathcal{T}\mu\}, \end{aligned} \tag{48}$$

where

$$\omega = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu]. \tag{49}$$

Using (48) and (49), we can suggest the following new predictor-corrector method for solving the quasi variational inequalities.

Algorithm 25. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned}\omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Pi_{(\omega_n)}\left\{\rho\mathcal{T}\omega_n - \rho\mathcal{T}\mu_n\right\}.\end{aligned}$$

If $\alpha_n = 1$, then Algorithm 25 reduces to

Algorithm 26. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned}\omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[\rho\mathcal{T}\omega_n - \rho\mathcal{T}\mu_n],\end{aligned}$$

which appears to be a new one.

In a similar way, we can suggest and analyse the predictor-corrector inertial method for solving the quasi variational inequalities (4), which only involve only one projection.

Algorithm 27. For given $u_0, u_1 \in \Omega(\mu)$, compute u_{n+1} by the iterative scheme

$$\begin{aligned}\omega_n &= \mu_n - \xi_n(\mu_n - \mu_{n-1}) \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Pi_{(\omega_n)}\left\{\rho\mathcal{T}\omega_n - \rho\mathcal{T}\omega_n\right\}\end{aligned}$$

One can study the convergence of the Algorithm 27 using the technique in [45, 51, 55].

Remark 4.2. We have only given some glimpse of the technique of the Wiener-Hopf equations for solving the quasi variational inequalities. One can explore the applications of the Wiener-Hopf equations in developing efficient numerical methods for variational inequalities and related nonlinear optimization problems.

5. Auxiliary principle technique

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can be applied for suggesting the iterative methods for solving the several nonlinear variational inequalities and equilibrium problems. To overcome these drawbacks, one usually applies the auxiliary principle technique, which is mainly due to Glowinski et al [20] as developed in [48, 51, 52, 56], to suggest and analyze some proximal point methods for solving general quasi variational inequalities (4). We apply the auxiliary principle technique involving an arbitrary operator for finding the approximate solution of the problem (4).

For a given $\mu \in \Omega(\mu)$ satisfying (4), find $w \in \Omega(\mu)$ such that

$$\langle \rho T(w+\eta(\mu-w)), g(\nu)-w\rangle + \langle M(w)-M(\mu), \nu-w\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{50}$$

where $\rho > 0, \eta \in [0,1]$ are constants and M is an arbitrary operator. The inequality (78) is called the auxiliary general quasi variational inequality.
If $w=\mu$, then w is a solution of (4). This simple observation enables us to suggest the following iterative method for solving (4).

Algorithm 28. For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\begin{aligned} &\langle \rho T(\mu_{n+1}+\eta(\mu_n-\mu_{n+1})), g(\nu)-\mu_{n+1}\rangle \\ &\quad + \langle M(\mu_{n+1})-M(\mu_n), \nu-\mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu). \end{aligned} \tag{51}$$

Algorithm 28 is called the hybrid proximal point algorithm for solving the general quasi variational inequalities (4).

Special Cases

We now discuss some special cases are discussed.

(I). For $\eta = 0$, Algorithm 28 reduces to

Algorithm 29. For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho T\mu_{n+1}, g(\nu)-\mu_{n+1}\rangle + \langle M(\mu_{n+1})-M(\mu_n), \nu-\mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{52}$$

is called the implicit iterative methods for solving the problem (4).

(II). If $\eta = 1$, then Algorithm 28 collapses to

Algorithm 30. For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho T\mu_n, g(\nu) - \mu_{n+1}\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

is called the explicit iterative method.

(III). For $\eta = \frac{1}{2}$, Algorithm 28 becomes:

Algorithm 31. For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\left\langle \rho T\left(\frac{\mu_{n+1} + \mu_n}{2}\right), g(\nu) - \mu_{n+1}\right\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

is known as the mid-point proximal method for solving the problem (4).

For the convergence analysis of Algorithm 29, we need the following concepts.

Definition 5.1. An operator T is said to be pseudomontone with respect to the operator g, if

$$\langle T\mu, g(\nu) - \mu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

implies that

$$-\langle T\nu, g(\mu) - \nu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu).$$

Theorem 5.2. *Let the operator T be a pseudo-monotone with respect to the operator g. Let the approximate solution μ_{n+1} obtained in Algorithm 29 converges to the exact solution $\mu \in \Omega(\mu)$ of the problem (4). If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi \|\mu_{n+1} - \mu_n\| \leq \zeta \|\mu - \mu_n\|. \tag{53}$$

Proof. Let $\mu\Omega(\mu)$ be a solution of the problem (4). Then,

$$-\langle \rho(T\nu, \mu - g(\nu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{54}$$

since the operator T is a pseudo-monotone with respect to the operator g.
Takin $\nu = \mu_{n+1}$ in (54), we obtain

$$-\langle \rho T\mu_{n+1}, \mu - g(\mu_{n+1})\rangle \geq 0. \tag{55}$$

Setting $v = \mu$ in (79), we have

$$\langle \rho T\mu_{n+1}, \mu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \geq 0. \quad (56)$$

Combining (56) and (55), we have

$$\langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \quad \geq \quad -\langle \rho T\mu_{n+1}, \mu - g(\mu_{n+1})\rangle \geq 0. \quad (57)$$

From the equation (57), we have

$$\begin{aligned} 0 &\leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n + \mu_n - u_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle + \langle M(\mu_{n+1} - M(\mu_n), \mu_n - \mu_{n+1}\rangle, \end{aligned}$$

which implies that

$$\langle M(\mu_{n+1} - M(\mu_n), \mu_{n+1} - \mu_n\rangle \leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle.$$

Now using the strongly monotonicity with constant $\xi > 0$ and Lipschitz continuity with constant ζ of the operator M, we obtain

$$\xi\|\mu_{n+1} - \mu_n\|^2 \leq \zeta\|\mu_{n+1} - \mu_n\|\|\mu_n - \mu\|.$$

Thus

$$\xi\|\mu_n - \mu_{n+1}\| \leq \zeta\|\mu_n - \mu\|,$$

the required result (53). □

Theorem 5.3. *Let H be a finite dimensional space and all the assumptions of Theorem 5.2 hold. Then the sequence $\{\mu_n\}_0^\infty$ given by Algorithm 29 converges to the exact solution $\mu \in \Omega(\mu)$ of (4).*

Proof. Let $\mu \in \Omega(\mu)$ be a solution of (4). From (53), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\xi \sum_{n=0}^{\infty} \|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu_n - \mu\|,$$

which implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0. \quad (58)$$

Let $\hat{\mu}$ be the limit point of $\{\mu_n\}_0^\infty$; whose subsequence $\{\mu_{n_j}\}_1^\infty$ of $\{\mu_n\}_0^\infty$ converges to $\hat{\mu} \in \Omega(\mu)$. Replacing w_n by μ_{n_j} in (79), taking the limit $n_j \longrightarrow \infty$ and using (58), we have

$$\langle \rho T\hat{\mu}, g(\nu) - \hat{\mu}\rangle \geq 0, \qquad \forall \nu \in \Omega(\mu),$$

which implies that $\hat{u}$ solves the problem (4) and

$$\|\mu_{n+1}-\mu\| \le \|\mu_n-\mu\|.$$

Thus, it follows from the above inequality that $\{\mu_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty}(\mu_n) = \hat{\mu},$$

which is the required result. □

In recent years, some inertial type iterative methods have been applied to find the approximate solutions of variational inequalities and related optimizations. We again apply the auxiliary approach to suggest some hybrid inertial proximal point schemes for solving the general quasi variational inequalities.

For a given $\mu \in \Omega(\mu)$ satisfying (4), find $w \in \Omega(\mu)$ such that

$$\begin{aligned}&\langle \rho T(w+\eta(\mu-w)), g(\nu)-w\rangle \\ &\quad +\langle M(w)-M(\mu)+\alpha(\mu-\mu), \nu-w\rangle \ge 0, \quad \forall \nu \in \Omega(\mu), \quad (59)\end{aligned}$$

where $\rho > 0, \eta, \alpha \in [0,1]$ are constants and M is a nonlinear operator.
Clearly $w=\mu$, implies that w is a solution of (4). This simple observation enables us to suggest the following iterative method for solving (4).

Algorithm 32. For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\begin{aligned}&\langle \rho T(\mu_{n+1}+\eta(\mu_n-\mu_{n+1})), g(\nu)-\mu_{n+1}\rangle \\ &\quad +\langle M(\mu_{n+1})-M(\mu_n)+\alpha(\mu_n-\mu_{n-1}), \nu-\mu_{n+1}\rangle \ge 0, \quad \forall \nu \in \Omega(\mu)\end{aligned}$$

Algorithm 32 is called the hybrid proximal point algorithm for solving the general quasi variational inequalities (4). For $\alpha = 0$, Algorithm 32 is exactly Algorithm 28. Using the technique and ideas of Theorem 5.2 and Theorem 5.3, one can analyze the convergence of Algorithm 32 and its special cases.
For $M = I$, the identity operator, Algorithm 32 reduces to the following inertial method for solving the problem (4).

Algorithm 33. For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\begin{aligned}&\langle \rho T(\mu_{n+1}+\eta(\mu_n-\mu_{n+1})), g(\nu)-\mu_{n+1}\rangle \\ &\quad +\langle \mu_{n+1}-\mu_n+\alpha(\mu_n-\mu_{n-1}), \nu-\mu_{n+1}\rangle \ge 0, \quad \forall \nu \in \Omega(\mu)\end{aligned}$$

which is called the hybrid proximal point method.

For different and suitable values of the parameters η, α, operators and set-valued convex sets, one can suggest and investigate several new and known methods for solving the general quasi variational inequalities and related non-convex programming problems. For the implementable numerical methods need further research efforts.

6. Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving quasi variational inequalities. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [19] . This dynamical system is a first order initial value problem. This implies that the numerical methods for solving initial value and boundary value can be used to develop numerical methods for solving variational inequalities. Consequently, variational inequalities, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. For the applications of dynamical systems, see [19, 52, 54, 66, 67]. We consider some iterative methods for solving the general quasi variational inequalities. We investigate the convergence analysis of these new methods involving only the monotonicity of the operators.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - \mu\}. \tag{60}$$

Invoking Lemma 3.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem(4), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{61}$$

We now consider a dynamical system associated with the general quasi variational inequalities (4). Using the equivalent formulation (3.1), we suggest a class of projection dynamical systems as

$$\frac{d\mu}{dt} = \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}u] - \mu\}, \quad \mu(t_0) = \alpha, \tag{62}$$

where λ is a parameter. The system of type (72) is called the projection dynamical system associated with the problem (4). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (4) can be studied.

We note that $\mu \in \Omega(\mu)$ is a solution of the general quasi variational inequality (4), if and only if, $\mu \in \Omega(\mu)$ is an equilibrium point.

Definition 6.1. [19] The dynamical system is said to converge to the solution set S^* of (72), if , irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} \text{dist}(\mu(t), S^*) = 0, \tag{63}$$

where

$$\text{dist}(\mu, S^*) = \inf_{\nu\in S^*} \|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (63) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 6.2. The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies

$$\|\mu(t) - \mu^*\| \le u_1\|\mu(t_0) - \mu^*\| exp(-\eta(t-t_0)), \quad \forall t \ge t_0,$$

where u_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 6.3. *(Gronwall Lemma)[19] Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \le t_0\}$ and let $\alpha(t) = \alpha_0(|t-t_0|)$, where α_0 is a monotone increasing function. If, for $t \ge t_0$,*

$$\hat{\mu} \le \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \le \alpha(t) exp\left\{ \int_{t_0}^{t} \hat{\nu}(s)ds \right\}.$$

We now show that the trajectory of the solution of the projection dynamical system (72) converges to the unique solution of the general quasi variational inequality (4). The analysis is in the spirit of Noor and Noor [52] and Xia and Wang [73, 74].

Theorem 6.4. *Let the operator $\mathcal{T} : H \longrightarrow H$ be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$. Let the operator g be Lipschitz continuous with constant $\zeta > 0$. If $\lambda\{(1+\eta+\zeta+\rho\beta\} < 1$ and Assumption 1 then, for each $\mu_0 \in \Omega\mu$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (72) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - \mu\}, \quad \forall \mu \in H.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$, For $\forall \mu, \nu \in H$, and using (25), we have

$$\begin{aligned}
&\|G(\mu) - G(\nu)\| \\
\leq\ & \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}\nu]\| + \|\mu - \nu\|\} \\
=\ & \lambda\{\|g(\mu) - g(\nu)\| + \|\Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}\mu] - \Pi_{\Omega(\mu)}[\nu - \rho\mathcal{T}\nu]\| \\
& + \|\Pi_{\Omega(\mu)}[\nu - \rho\mathcal{T}\nu] - \Pi_{\Omega(\nu)}[\nu - \rho\mathcal{T}\nu]\|\} \\
\leq\ & \lambda\{\|\mu - \nu\| + \eta\|\mu - \nu\| + \|g(\mu) - g(\nu) - \rho(\mathcal{T}\mu - \mathcal{T}\nu)\} \\
\leq\ & \lambda\{\|\mu - \nu\| + \eta\|\mu - \nu\| + \{\zeta + \rho\beta\}\|\mu - \nu\|\} \\
\leq\ & \lambda\{(1+\eta+\zeta+\rho\beta\}\|\mu - \nu\|.
\end{aligned}$$

where have used the fact that g is Lipschitz continuous with a constant ζ and the operator $\mathcal{T}$ is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$,respectively. This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant $\lambda\{(1+\eta+\zeta+\rho\beta\} < 1$ and for each $\mu \in \Omega(\mu)$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (72), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider , for any $\mu \in \Omega(\mu)$,

$$\begin{aligned}
\|G(\mu)\| &= \|\frac{d\mu}{dt}\| \\
&= \lambda\|[g(u) - \rho\mathcal{T}\mu] - \mu\| \\
&\leq \lambda\{\|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - \Pi_{\Omega(\mu)}[0]\| + \|\Pi_{\Omega(\mu)}[0] - \mu\|\} \\
&\leq \lambda\{\delta\|\{g(\mu) - \rho T\mu\| + \|\Pi_{\Omega(\mu)}[g(\mu)] - \Pi_{\Omega(\mu)}[0]\| \\
&\quad + \|\Pi_{\Omega(\mu)}[0] - \mu\|\} \\
&\leq \lambda\{(\rho\beta + 2 + \zeta)\|\mu\| + \|\Pi_{\Omega(\mu)}[0]\|\}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|\mu(s)\| ds \\
&\leq (\|\mu_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\| ds,
\end{aligned}$$

where $k_1 = \lambda\|\Pi_{\Omega(\mu)}[0]\|$ and $k_2 = \rho\beta + 2 + \zeta$. Hence by the Gronwall Lemma 6.3, we have

$$\|\mu(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 6.5. *If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\sigma > 0$ and $\zeta > 0$, then the dynamical system (72) converges globally exponentially to the unique solution of the general quasi variational inequality (4).*

Proof. Since the operator g is Lipschitz continuous, it follows from Theorem 6.4 that the dynamical system (72) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in H$. Let $\mu(t)$ be a solution of the initial value problem (72). For a given $\mu^* \in H$ satisfying (4), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad u(t) \in \Omega(\mu). \tag{64}$$

From (72) and (64), we have

$$\begin{aligned}
\frac{dL}{dt} &= 2\lambda\left\langle \mu(t) - \mu^*, \frac{d\mu}{dt}\right\rangle \\
&= 2\lambda\langle \mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - \mu(t)\rangle \\
&= 2\lambda\langle \mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - \mu^* + \mu^* - \mu(t)\rangle \\
&= -2\lambda\langle \mu(t) - \mu^*, \mu(t) - \mu^*\rangle \\
&\quad +2\lambda\langle \mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - \mu^*\rangle \\
&\leq -2\lambda\langle \rho(\mathcal{T}\mu(t) - \mathcal{T}\mu^*), g(\mu(t)) - g(\mu^*)\rangle \\
&\quad +2\lambda\langle \mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] \\
&\quad -\Pi_{\Omega(\mu)}[g(\mu^*(t)) - \rho\mathcal{T}\mu^*(t)]\rangle, \\
&\leq -2\lambda\sigma\|\mu(t) - \mu^*\|^2 + \lambda\|\mu(t) - \mu^*\|^2 \\
&\quad +\lambda\|\Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] \\
&\quad -\Pi_{\Omega(\mu)}[g(\mu^*(t)) - \rho\mathcal{T}\mu^*(t)]\|^2
\end{aligned} \tag{65}$$

Using the Lipschitz continuity of the operators $\mathcal{T}, g$, we have

$$\begin{aligned}
&\|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - \Pi_{\Omega(\mu)}[g(\mu^*) - \rho\mathcal{T}\mu^*]\| \\
&\quad \leq \delta\|g(\mu) - g(\mu^*) - \rho(\mathcal{T}\mu - \mathcal{T}\mu^*)\| \leq \delta(\zeta + \rho\beta)\|\mu - \mu^*\|.
\end{aligned} \tag{66}$$

From (65) and (66), we have

$$\frac{d}{dt}\|\mu(t) - \mu^*\| \leq 2\xi\lambda\|\mu(t) - \mu^*\|,$$

where

$$\xi = (\delta(\zeta + \rho\beta) - 1).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t) - \mu^*\| \leq \|\mu(t_0) - \mu^*\| e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (72) converges globally exponentially to the unique solution of the general quasi variational inequality (4). □

We use the projection dynamical system (72) to suggest some iterative for solving the quasi variational inequalities (4). These methods can be viewed in the sense of Korpelevich [25] and Noor et al [52, 54] involving the double projections.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (72) becomes

$$\frac{d\mu}{dt} + \mu = \Pi_{\Omega(\mu)}\left[g(\mu) - \rho\mathcal{T}u\right], \quad \mu(t_0) = \alpha. \tag{67}$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (6), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_n = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_{n+1}], \tag{68}$$

where h is the step size.

Now, we can suggest the following implicit iterative method for solving the general quasi variational inequality (4).

Algorithm 34. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}\left[g(\mu_n) - \rho\mathcal{T}\mu_{n+1} - \frac{\mu_{n+1} - \mu_n}{h}\right],$$

This is an implicit method. Algorithm 34 is equivalent to the following two-step method.

Algorithm 35. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}\left[g(\mu_n) - \rho\mathcal{T}\omega_n - \frac{\omega_n - \mu_n}{h}\right], \end{aligned}$$

Discretizing (6), we now suggest an other implicit iterative method for solving (4).

$$\frac{\mu_{n+1}-\mu_n}{h_1}+\mu_n=\Pi_{\Omega(g(\mu_{n+1}))}[g(\mu_{n+1})-\rho\mathcal{T}\mu_{n+1}], \tag{69}$$

where h_1 is the step size.

This formulation enables us to suggest the two-step iterative method.

Algorithm 36. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned}\omega_n &= \Pi_{\omega(\mu_n)}[g(\mu_n)-\rho\mathcal{T}\mu_n]\\ \mu_{n+1} &= \Pi_{\Omega(g(\omega_n}\left[g(\omega_n)-\rho\mathcal{T}\omega_n-\frac{\omega_n-\mu_n}{h_1}\right].\end{aligned}$$

Discretizing (6), we have

$$\frac{\mu_{n+1}-\mu_n}{h}=-\mu_n+\Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1})-\rho\mathcal{T}\mu_{n+1}], \tag{70}$$

where h is the step size.

For $h=1$, this helps us to suggest the following implicit iterative method for solving the problem (4).

Algorithm 37. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned}\omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho\mathcal{T}\mu_n]\\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}\left[g(\omega_n)-\rho\mathcal{T}\omega_n\right].\end{aligned}$$

Discretizing (6), we propose another implicit iterative method.

$$\frac{\mu_{n+1}-\mu_n}{h}+\mu_n=\Pi_{\Omega(\mu_{n+1})}[g(\mu_n)-\rho\mathcal{T}\mu_{n+1}],$$

where h is the step size.

For $h=1$, we can suggest an implicit iterative method for solving the problem (4).

Algorithm 38. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1}=\Pi_{\Omega(\mu_{n+1})}[g(\mu_n)-\rho\mathcal{T}\mu_{n+1}].$$

Algorithm 38 is an implicit iterative method in the sense of Korpelevich.

From (6), we have

$$\frac{d\mu}{dt}+\mu=\Pi_{\Omega((1-\alpha)\mu+\alpha\mu)}[g((1-\alpha)\mu+\alpha\mu))-\rho\mathcal{T}((1-\alpha)\mu+\alpha\mu)], \tag{71}$$

where $\alpha \in [0,1]$ is a constant.

Discretization (71) and taking $h = 1$, we have

$$\mu_{n+1} = \Pi_{\Omega((1-\alpha)\mu_n+\alpha\mu_{n-1})}\big[g((1-\alpha)\mu_n+\alpha\mu_{n-1}) - \rho\mathcal{T}((1-\alpha)\mu_n+\alpha\mu_{n-1})\big],$$

which is an inertial type iterative method for solving the general quasi variational inequality (4). Using the predictor-corrector techniques, we have

Algorithm 39. For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative schemes

$$\begin{aligned} \omega_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1} \\ y_n &= \Pi_{\Omega(\omega_n)}\big[g(\omega_n) - \rho\mathcal{T}\omega_n\big] \\ \mu_{n+1} &= \Pi_{\Omega(y_n)}\big[y_n - \rho(\mathcal{T}g(\omega_n) + \mathcal{T}y_n)\big], \end{aligned}$$

which is known as the three-step inertial iterative method.

Remark 6.6. For appropriate and suitable choice of the operators $\mathcal{T}, g$, convex valued set, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving general quasi variational inequalities and related optimization problems. Using the techniques and ideas of Noor et al [49], one can discuss the convergence analysis of the proposed methods. It is an interesting problem to discuss the comparison of these proposed methods with the recent iterative methods in [2, 4, 22, 23, 28, 46, 47, 51, 54–56, 76] and the references therein.

We use this dynamical system to suggest and investigate some inertial proximal methods for solving the general quasi variational inequalities (4). These inertial implicit methods are constructed using the central finite difference schemes and its variant forms.

To be more precise, we consider the problem of finding $\mu \in \Omega(\mu)$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\mu)], \quad \mu(a) = \alpha, \mu(b) = \beta, \tag{72}$$

where $\gamma \geq 0$, $\quad \eta \geq 0$ and $\rho > 0$ are constants. Problem (72) is called second order dynamical system, which is a second boundary value problem.

We discretize the second-order dynamical systems (72) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \eta\frac{\mu_n - \mu_{n-1}}{h} + \mu_n = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\mu_{n+1})], \tag{73}$$

where h is the step size.

If $\gamma = 1, h = 1$, then, from equation(73) we have

Algorithm 40. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(g(\mu_{n+1})], \quad n = 0, 1, 2, \ldots.$$

which is the extragradient method of Korpelevich [25] for solving the quasi variational inequalities.

Algorithm 40 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the two-step inertial method.

Algorithm 41. For given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(y_n)], \quad n = 0, 1, 2, \ldots \end{aligned}$$

where $\theta_n \in [0.1]$ is a constant.

Similarly, we suggest the following iterative method.

Algorithm 42. For given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}[g(\mu_{n+1}) - \rho\mathcal{T}(\mu_{n+1})], \quad n = 0, 1, 2, \ldots$$

which is known as the double projection method, introduced and studied by Noor [42, 45] and can be written as

Algorithm 43. For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta\mu_{n-1} \\ \mu_{n+1} &= \Pi_{\Omega(\mu_n)}[g(y_n) - \rho\mathcal{T}(y_n)], \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is called the two-step inertial iterative Noor method.
Problem (72) can be rewritten as

$$\begin{aligned} \gamma\ddot{\mu} + \dot{\mu} + \mu &= \Pi_{\Omega((1-\theta_n)\mu+\theta_n u))}[g(1-\theta_n)\mu + \theta_n\mu) - \rho\mathcal{T}((1-\theta_n)\mu + \theta_n\mu)], \\ & \mu(a) = \alpha, \mu(b) = \beta, \end{aligned} \tag{74}$$

where $\gamma > 0, \theta_n \geq 0$ and $\rho > 0$ are constants.

Discretising the system (74), we have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + \mu_n$$
$$= \Pi_{\Omega((1-\theta_n)\mu_n+\theta_n\mu_{n-1})}[g(1-\theta_n)\mu_n + \theta_n\mu_{n-1}) - \rho\mathcal{T}((1-\theta_n)\mu_n + \theta_n\mu_{n-1})]$$

from which, for $\gamma = 0, \quad h = 1$, we have

Algorithm 44. For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega((1-\theta_n)\mu+\theta_n\mu))}[g(1-\theta_n)u_n + \theta_n\mu_{n-1}) - \rho\mathcal{T}((1-\theta_n)\mu_n + \theta_n\mu_{n-1})]$$

or equivalently

Algorithm 45. For a given $\mu_0, \mu_1 \in \Omega(\mu)$ compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \Pi_{\Omega(y_n)}[g(y_n) - \rho\mathcal{T}y_n] \end{aligned}$$

which is called the new inertial iterative method for solving the general quasi variational inequality.

We discretize the second-order dynamical systems (72) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + \mu_{n+1} = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\mu_{n+1})],$$

where h is the step size.

Using this discrete form, we can suggest the following an iterative method for solving the general quasi variational inequalities (4).

Algorithm 46. For given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}\left[g(\mu_n) - \rho\mathcal{T}(\mu_{n+1}) - \frac{\gamma\mu_{n+1} - (2\gamma - h)\mu_n + (\gamma - h)\mu_{n-1}}{h^2}\right].$$

Algorithm 46 is called the inertial proximal method for solving the quasi variational inequalities and related optimization problems. This is a new proposed method.

We note that, for $\gamma = 0$, Algorithm 46 reduces to the following iterative method for solving quasi variational inequalities (4).

Algorithm 47. For given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}\left[g(\mu_n) - \rho\mathcal{T}\mu_{n+1} - \frac{\mu_n - \mu_{n-1}}{h}\right], \quad n = 0, 1, 2, \ldots.$$

We again discretize the second-order dynamical systems (72) using central difference scheme and forward difference scheme to suggest the following inertial proximal method for solving (4).

Algorithm 48. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}\left[g(\mu_{n+1}) - \rho\mathcal{T}(\mu_{n+1}) - \frac{(\gamma + h)\mu_{n+1} - (2\gamma + h)\mu_n + \gamma\mu_{n-1}}{h^2}\right].$$

Algorithm 48 is quite different from other inertial proximal methods for solving the quasi variational inequalities.
If $\gamma = 0$, then Algorithm 48 collapses to:

Algorithm 49. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}\left[g(\mu_{n+1}) - \rho\mathcal{T}(\mu_{n+1}) - \frac{\mu_{n+1} - \mu_n}{h}\right].$$

Algorithm 48 is an proximal method for solving the quasi variational inequalities. Applying the technique and ideas of Noor and Noor [52], one can study the convergence criteria of these Algorithms with some modifications and adjustment. Such type of proximal methods were suggested by Noor[45] using the fixed point problems. In brief, by suitable descritization of the second-order dynamical systems (72), one can construct a wide class of explicit and implicit method for solving quasi variational inequalities and their variant forms.

Remark 6.7. We would like to emphasize that the proposed Algorithms 4.1-4.3 and Algorithms 6.1-.6.3 are new ones and can be viewed as significant extensions of the results obtained in [2,4,19, 23,24, 29, 52,53,57,70,77]. It is an interesting problem to compare these methods numerically and explore their applications in various branches of pure and applied sciences.

7. Applications and future research

In this section, we show that the general quasi variational inequality (4) reduces to the extended general variational inequalities, which were introduced and studied by Noor [48].

In many applications, the convex-valued set $\Omega(\mu)$ is of the form:

$$\Omega(\mu) = m(\mu) + \Omega, \tag{75}$$

where Ω is a convex set and m is a point-to-point mapping.

Let $\mu \in \Omega(\mu)$ be a solution of the problem (4). Then, from Lemma 3.1, it follows that $\mu \in \Omega(\mu)$ such that

$$\mu = \Pi_{\Omega(\mu)}\left[g(\mu) - \rho\mathcal{T}\mu\right]. \tag{76}$$

Combining (75) and (76), we obtain

$$\begin{aligned} \mu &= \Pi_{\Omega(\eta(\mu)+\Omega)}\left[g(\mu) - \rho\mathcal{T}\mu\right] \\ &= m(\mu) + \Pi_{\Omega}\left[g(\mu) - m(\mu) - \rho\mathcal{T}u\right]. \end{aligned}$$

Consequently, we obtain

$$\mu - m(\mu) = \Pi_{\Omega}\left[g(\mu) - m(\mu) - \rho \mathcal{T} u\right],$$

that is,

$$G(\mu) = \Pi_{\Omega}\left[H(\mu) - \rho \mathcal{T} u\right],$$

where $G(\mu) = \mu - m(\mu)$ and $H(\mu) = g(\mu) - m(\mu)$.

Thus the problem (4) is equivalent to finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}\mu + G(\mu) - H(\mu), H(\nu) - G(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega. \tag{77}$$

The inequality of the type (77) is called the extended general variational inequality, investigated by Noor [48]. Our results in this paper continue to hold for extended general quasi variational inequalities (77) with suitable modifications and adjustment.

We would like to mention that some of the results obtained and presented in this paper can be extended for more multi-valent general quasi variational inequalities. To be more precise, let $C(H)$ be a family of nonempty compact subsets of H. Let $T, V : H \longrightarrow C(H)$ be the multi-valued operators. For a given nonlinear bifunction $N(.,.) : H \times H \longrightarrow H$ and operators $g, h : H \longrightarrow H$, consider the problem of finding
$u \in \Omega(\mu), w \in T(\mu), y \in V(\mu)$ such that

$$\langle N(w,y) + h(\mu) - g(\nu), g(\nu) - h(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{78}$$

which is called the multivalued general quasi variational inequality. We would like to mention that one can obtain various classes of general quasi variational inequalities for appropriate and suitable choices of the bifunction $N(.,.)$, the operators g, h, and convex-valued set $\Omega(\mu)$.

Note that, if $N(w,y) = T\mu, \quad h = I$, then the problem (78) is equivalent to find $\mu \in \Omega(\mu)$, such that

$$\langle Tu + \mu - g(\mu), g(\nu) - u)\rangle \geq 0 \quad \forall \nu \in \Omega(\mu),$$

which is also called the general quasi variational inequality and is different than the problem (4).

Using Lemma 3.1, one can prove that the problem (78) is equivalent to finding $u \in \Omega(u)$ such that

$$h(u) = \Pi_{\Omega(\mu)}[g(\mu) - \rho N(w,y)] \tag{79}$$

which can be written as

$$\mu = \mu - h(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho N(w,y)].$$

Thus one can consider the mapping F associated with the problem (78) as

$$F(\mu) = \mu - h(\mu) + \Pi_{\Omega(u)}[g(\mu) - \rho N(w,y)],$$

which can be used to discuss the uniqueness of the solution of the problem (78). From (78) and (79, it follows that the multivalued general quasi variational inequalities are equivalent to the fixed problems. Consequently, all results obtained for the problem (4) continue to hold for the problem (78) with suitable modifications and adjustments. The development of efficient implementable numerical methods for solving the multivalued general quasi variational inequalities and non optimization problems requires further efforts. Despite the research activates, very few results are available. The development of efficient implementable numerical methods for solving the general quasi variational inequalities and non optimizations problems requires further efforts.

Conclusion.

In this paper, we have introduced and studied some new classes of general quasi variational inequalities for two arbitrary operators. By inter changing the roles of these operators, one can obtain a number of new classes of quasi variational inequalities and complementarity problems. It is shown that implicit obstacle second order boundary value problems can be studied via the general quasi variational inequalities. Applying the projection technique, we have established the equivalence between the fixed point problems and quasi variational inequalities. This equivalence formulation is used to study the unique existence of solution. Several hybrid multi-step iterative methods for solving the quasi variational inequalities are suggested applying the fixed point, the Wiener-Hopf equations and dynamical systems. We have also consider the second order boundary value problems associated with the general quasi variational inequalities. These new methods include extragradient method, modified double projection methods and inertial type methods. Convergence analysis of the proposed method is discussed under suitable weaker conditions. It is an open problem to compare these proposed methods with other methods. We have shown that the general quasi variational inequalities are equivalent to the extended general variational inequalities with suitable conditions of the convex-valued set. Applications of the fuzzy set theory, stochastic, quantum calculus, fractal, fractional and random can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research and variational inequalities. One may explore these aspects of the general quasi variational inequality and its variant forms in these areas. Using the ideas and techniques of this paper, one can explore the applications of these methods for solving the complementarity problems and related mathematical programming problems.

Contributions of the authors:

All the authors contributed equally in writing, editing, reviewing and agreed for the final version for publication.

Data availability:

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study

Conflict of Interest:

All authors have no conflict of interest.

Acknowledgement:

The authors wish to express their deepest gratitude to their respected teachers, students, colleagues, collaborators and friends, who have directly or indirectly contributed in the preparation of this paper.

REFERENCES

[1] F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, SIAM J. Optim., 14(2)(2003), 773-782.

[2] A. A. Alshejari, M. A. Noor and K. I. Noor, Some recent advances in general quasi variational inequalities, Inter. J. Anal. Appl. 22(2024), 84, pp. 1-39.

[3] W. F. Ames, Numerical Methods for Partial Differential Equations, Academic Press, New York, 1992

[4] A. S. Antipin, M. Jacimovic and N. Mijajlovi, Extragradient method for solving quasi variational inequalities. Optimization, 67(2018), 103-112

[5] K. Ashish, M. Rani and R. Chugh, Julia sets and Mandelbrot sets in Noor orbit, Appl. Math. Comput., 228(1)(2014), 615-631.

[6] Ashish, R. Chugh and M. Rani, Fractals and Chaos in Noor Orbit: a four-step feedback approach, Lap Lambert Academic Publishing, Saarbrucken, Germany, 2021.

[7] Ashish, J. Cao and M. A. Noor, Stabilization of fixed points in chaotic maps using Noor orbit with applications in cardiac arrhythmia, J. Appl. Anal. Computation, 13(5)(2023), 2452-2470. doi: 10.11948/20220350

[8] C. Chairatsiripong and T. Thianwan, Novel Noor iterations technique for solving nonlinear equations, AIMS Math. 7(6)(2022),10958-10976.

[9] A. Barbagallo and S. G. Lo Bainco, A random elastic traffic equilibrium problem via stochastic quasi-variational inequalities, Commun. Nonlin. Sci. Numer. Simul. 131(2024), 107798.

[10] A. Bensoussan, J. L. Lions, Application des Inqualities Variationnelles en Contrl Stochastique, Paris: Bordas(Dunod), 1978.

[11] S. Y. Cho, A. A. Shahid, W. Nazeer and S. M. Kang, Fixed point results for fractal generation in Noor orbit and s-convexity, Springer Plus, 5(2006): 1843, DOI10.1186/s40064 - 016 - 3530 - 5.

[12] R. Chugh, M. Rani and Ashish, On the convergence of logistic map in Noor orbit, Inten. J. Computer Appl. 43(18)(2012),1-5, DOI: 10.5120/6200-8739.

[13] R. W. Cottle , J.-S. Pang and R. E. Stone, The Linear Complementarity Problem, SIAM Publ. Phildelphia, PA, USA., 2009.

[14] G. Cristescu and L. Lupsa, Non-Connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[15] G. Cristescu and G. Mihail, Detecting the non-convex sets with Youness and Noor types convexities, Bulet. Sttintif. Al Universit. Polytechnica, Timisoara, Math. Pizica, 54(68)(2009): 2

[16] G. Cristescu and G. Mihail, Shape properties of Noors g-convex sets, Proc. Twelfth Symp. Math. Appl., Timisoara, Romania, (2009), 1-13.

[17] S. Dafermos, Sensitivity Analysis in Variational Inequalities, Math. Oper. Research, 13(1988), 421434.

[18] S. Dey and S. Reich, A dynamical system for solving inverse quasi-variational inequalities, Optimization,72(2023): DOI: 10.1080/02331934.2023.21735

[19] P. Dupuis and A. Nagurney, Dynamical systems and variational inequalities, Annals Oper. Research, 44(1993), 7-42.

[20] R. Glowinski, J. L. Lions, and R. Tremolieres, Numerical Analysis of Variational Inequalities, NorthHolland, Amsterdam, (1981).

[21] R. Glowinski and P. Le Tallec, Augmented Lagrangian and Operator Splitting Methods in Nonlinear Mechanics, SIAM, Philadelphia, Pennsylvania, USA, 1989.

[22] S. Jabeen, B. B.Mohsin , M. A. Noor and K. I. Noor. Inertial projection methods for solving general quasi-variational inequalities, AIMS Math, 6(2)(2021), 1075-1086.

[23] A. G. Khan, M. A. Noor and K. I. Noor, Dynamical systems for generla quasi variational inequalities, Annal. Funct. Analysis, 6(1)(2015), 193-209.

[24] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, SIAM, Philadelphia, 2000

[25] G. M. Korpelevich, The extragradient method for finding saddle points and other problems, Ekonomika Mat. Metody, 12(1976), 747-756, 1976.

[26] Y. C. Kwuni, A. A. Shahid, W. Nazeer, S. I. Butt, M. Abbas and S. M. Kang, Tricorns and multicorns in Noor orbit with s-convexity, IEEE Access, 7(2019): DOI - 10.1109/ACCESS.2019.2928796.

[27] J. Lions and G. Stampacchia, Variational inequalities, Commun. Pure Appl. Math. 20(1967), 493519.

[28] N. Mijajlovic, J. Milojica and M. A. Noor, Gradient-type projection methods for quasi variational inequalities, Optim. Lett., 13(2019), 18851896, 2019.

[29] A. Nagurney and D. Zhang, Projected dynamical systems and variational inequalities with applications, Kluwer Academic Publishers, Boston, Dordrecht, London 1996.

[30] K. Nammanee, M. A. Noor and S. Suantai, Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 314(2006), 320-334.

[31] S. K. Natarajan and D. Negi, Green innovations utiling fractal and power for solar panel optimization, in: Green Innovations for Industrial Development and Business Sustainability (edited by R. Sharma, G. Rana and S. Agarwal), CRC Press, Florida, Boca Raton, USA, (2024), 146-152.

[32] D. Negi, A. K. Saini, N. Pandey, S. C. Wariyal and R. Sharma, An analysis of Julia sets and Noor iterations using a complex Mandelbrot iteration scheme,Preprint. 2016.

[33] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications, Springer-Verlag, New York, 2018.

[34] M. A. Noor, Riesz-Frechet Theorem and Monotonicity, MS Thesis, Queen's University, Kingston, Ontario, Canada, 1971.

[35] M. A. Noor, On Variational Inequalities, PhD Thesis, Brunel University, London, U. K. 1975.

[36] M. A. Noor, An iterative scheme for a class of quasi variational inequalities, J. Math. Anal. Appl., 110(1985), 463468.

[37] M. A. Noor, The quasi-complementarity problem, J. Math. Anal. Appl. 130(1998), 344-353.

[38] M. A. Noor, General variational inequalities, Appl. Math. Lett.,1(1988), 119-121.

[39] M. A. Noor, Quasi variational inequalities, Appl. Math. Lett., 1(4)(1988), 367-370.

[40] M. A. Noor, Fixed point approach for complementaritty problems, J. Math. Anal. Appl. 33(1988) , 437-448.

[41] M. A. Noor, General algorithm for variational inequalities, J. Optim. Theory Appl. 73(1992), 409413.

[42] M. A. Noor, Sensitivity analysis for quasi variational inequalities, J. Optim. Theory Appl. f95(2)(1997), 399-407, 1997.

[43] M. A. Noor, Variational inequalities for fuzzy mappings (III), Fuzzy Sets Syst. 110(1) (2000), 101108.

[44] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251(2000), 217-230.

[45] M. A. Noor, Some dvelopments in general variational inequalites, Appl. Math. Comput. 152(2004) , 199-277.

[46] M. A. Noor, Three-step iterative algorithms for multivalued quasi variational inclusions, J. Math. Anal., Appl. 255 (2)(2001), 589-604

[47] M. A. Noor, Merit functions for quasi variational inequalities, J. Math. Inequal., 1(2007), 259-269.

[48] M. A. Noor, Extended general variational inequalities, Appl. Math. Lett., 22(2009), 182-185.

[49] M. A. Noor, Differentiable non-convex functions and general variational inequalities, Appl. Math. Comput. 199(2)(2008), 623-630.

[50] M. A. Noor and K. I. Noor, Sensitivity analysis for quasi-variational inclusions, J. Math. Anal. Appl. 236 (1999), 290-299.

[51] M. A. Noor and K. I. Noor, Some novel aspects of quasi variational inequalities, Earthline J. Math. Sci., 10(1)(2022), 1-66

[52] M. A. Noor and K. I. Noor, Dynamical system technique for solving quasi variational inequalities, U.P.B. Sci. Bull., Series A, 84(4)(2022), 55-66.

[53] M. A. Noor and K. I. Noor, Some novel aspects and applications of Noor iterations and Noor orbits, J. Advanc. Math. Stud. 17(3)(2024)

[54] M. A. Noor, K. I. Noor and A. G. Khan, Dynamical systems for quasi variational inequalities, Annal. Funct. Anal. 6(1)(2015), 193-209.

[55] M. A. Noor, K. I. Noor, A. G. Khan and A. Ghiura, Iterative algorithms for solving a class of quasi variational inequalities, U.P.B. Sci. Bull., Series A, 72(1)(2010), 2-13.

[56] M. A. Noor. K.I. Noor and B. B. Mohsen, Some new classes of general quasi variational inequalities, AIMS Math. 6(6)(2021), 6406-6421. 6406-6421.

[57] M. A. Noor, K. I. Noor and A. Bnouhachem, Some new iterative methods for variational inequalities, Canad. J. Appl. Math. 3(1)(2021), 1-17.

[58] M. A. Noor, K. I. Noor, and M. T. Rassias, New trends in general variational inequalities, Acta Appl. Math. 170(1)(2020),981-1046.

[59] M. A. Noor, K. I. Noor and Th. M. Rassias, Some aspects of variational inequalities, J. Comput. Appl. Math. 47(1993), 285-312.

[60] M. A. Noor, K. I. Noor and Th. M. Rassias, Parametric general quasi variational inequalities, Math. Commun., 15(1)(2010), 205-212.

[61] M. A. Noor and W. Oettli, On general nonlinear complementarity problems and quasi equilibria, Le Matemat,(Catania), 49(1994), 313331.

[62] S. Paimsang, D. Yambangwai and T. Thainwan, A novel Noor iterative method of operators with property (E) as concerns convex programming applicable in signal recovery and polynomiography, Math. Methods Appl. Sci. (2024), 1-18, https : //doi.org/10.1002/mma.10083.

[63] M. Patriksson, Nonlinear Programming and Variational Inequalities: A Unified Approach, Kluwer Acadamic publishers, Drodrecht, 1998.

[64] J. Pecaric, F. Proschan and Y. L. Tong,Convex Functions, Partial Orderings and Statistical Applications, Academic Press, New York, 1992.

[65] W. Phuengrattana and S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SPiterations for continuous functions on an arbitrary interval, J. Comput. Appl. Math. 235(9)(2011) , 3006-3014.

[66] K. Rattanaseeha, S. Imnang, P. Inkrong and T. Thianwan, Novel Noor iterative methods for mixed type asmptotically nonexpansive mappings from the perpestive of convex programming in hyperbolic spaces, Int. J. Innov. Comput.Inform. Cont., 19(6)(2023), 1717-1734.

[67] K. Sanaullah, S. Ullah and N. M. Aloraini, A self adaptive three-step numerical scheme for variational inequalities, Axioms, 13(2024): 57.

[68] P. Shi, Equivalence of variational inequalities with WienerHopf equations, Proc. Amer. Math. Soc. 111(1991), 339346, 1991.

[69] Y. Shehu, A. Gibali and S. Sagratella, Inertial projection-type method for solving quasi variational inequalities in real Hilbert space, J. Optim. Theory Appl., 184(2020), 877894, 2020.

[70] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris, 258(1964), 4413-4416.

[71] S. Suantai, M. A. Noor, K. Kankam and P. Cholamjiak, Novel forward-backward algorithms for optimization and applications to compressive sensing and image in painting, Advan. Differ. Eqs. 2021(2021): Id-265, DOI: 10.1186/s13662-021-03422-9.

[72] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, SIAM J. Control Optim. 38(2000), 431-446.

[73] Y. S. Xia and J. Wang, A recurrent neural network for solving linear projection equations, Neural Network, 13(2000), 337-350.

[74] Y. S. Xia and J. Wang, On the stability of globally projected dynamical systems, J. Optim. Theory Appl., 106(2000), 129-150.

[75] A. Yadav and K. Jha, Parrondo's paradox in the Noor logistic map, Inter. J. Adanv. Research Eng. Technology, 7(5)(2016), 01-06, ID : IJARET0705001.

[76] Y. Yao, L. O. Jolaoso and Y. C. Shehu, FISTA type projection algorithm for quasi-variational inequalities. Numer. Algor. (2024). https://doi.org/10.1007/s11075-024-01852-6

[77] E. A. Youness, E-convex sets, E-convex functions and E-convex programming, J. Optim. Theory Appl., 102(2)(1999), 439-450.

M.A. NOOR
Mathematics,
COMSATS University Islamabad,
Islamabad, Pakistan.
e-mail: `noormaslam@gmail.com`

K.I. NOOR
Mathematics,
COMSATS University Islamabad,
Islamabad, Pakistan.
e-mail: `khalidan@gmail.com`