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ANALYTIC SEMIGROUP GENERATED BY AN ELLIPTIC OPERATOR WITH DISCONTINUOUS COEFFICIENTS

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We consider the generation of analytic semigroups by elliptic operators with discontinuous coefficients.

1. Introduction.

The semigroup approach in the study of parabolic equations is very well known. The basic step consists in proving a generation result in a suitable topology. This is achieved proving a particular estimate for an elliptic operator depending on a complex parameter λ

$$Lu - \lambda u = f \quad \text{in } \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ and proper boundary conditions are imposed on the solution u.

The point is an estimate of the type

$$|\lambda| \|u\|_{L^p(\Omega)} + \sqrt{|\lambda|} \|Du\|_{L^p(\Omega)} + \|D^2 u\|_{L^p(\Omega)} \le c \|f\|_{L^p(\Omega)}. \tag{1.1}$$

Usually (1.1) is proved assuming the coefficients a_{ij} of the leading part of the operator L continuous at least (see, for example [3] and [4]). Here we show a case in which (1.1) holds true with $a_{ij} \notin C^0(\Omega)$. The class to which the a_{ij}

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belong to is the Sarason class VMO introduced in [7] for other purposes. Later solvability of the Dirichlet problem with $a_{ij} \in VMO \cap L^\infty$ was proved (e.g. [1], [2], [8], [9] and [5]). In proving this, suitable *a priori* estimate was established.

We shall prove (1.1) using a classical method due to Agmon and an estimate for elliptic equations with discontinuous coefficients.

2. Notations and preliminary results.

We say that a locally integrable function f is in the space BMO if

$$\sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx = \|f\|_* < +\infty$$

where B ranges in the balls of $\mathbb{R}^n$ and f_B denotes the average of f over B.

If $f \in BMO$ and $r > 0$ we set

$$\sup_{\rho \le r} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx = \eta(r).$$

We say that a function $f \in BMO$ belongs to the space VMO if, in addition, $\lim_{r \to 0^+} \eta(r) = 0$. In the sequel we shall refer to $\eta(r)$ as the VMO modulus of f.

Let Ω be a bounded open subset of $\mathbb{R}^n$ ($n \ge 3$), with $\partial\Omega \in C^{1,1}$ and let be L the elliptic operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} \tag{2.1}$$

where

$$a_{ij}(x) \in VMO \cap L^\infty(\mathbb{R}^n) \qquad i, j = 1, \dots, n; \tag{2.2}$$

$$a_{ij}(x) = a_{ji}(x) \qquad i, j = 1, \dots, n \; a.e. \; x \in \Omega; \tag{2.3}$$

$$\exists \nu > 0 \; : \; \nu^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \le \nu|\xi|^2 \; a.e. \; x \in \Omega \quad \forall \xi \in \mathbb{R}^n. \tag{2.4}$$

Our estimate will be an easy consequence of the following result by Guidetti [5].

Let

$$Mu = -\sum_{i,j=1}^{n} b_{ij}(x) u_{x_i x_j} \tag{2.5}$$

with

$$b_{ij} : \Omega \to \mathbb{C}$$

bounded measurable.

We assume that

$$\exists \mu > 0 \ : \ |\sum_{i,j=1}^{n} b_{ij}\xi_i\xi_j| \geq \mu|\xi|^2 \ a.e.\ x \in \Omega \quad \forall \xi \in \mathbb{R}^n \tag{2.6}$$

Theorem 2.1. (see Proposition 3.1 in [5]) *Let Ω be an open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^{1,1}$. Consider the Dirichlet problem*

$$\begin{cases} Mu = f \quad \text{in } \Omega \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \ f \in L^p(\Omega) \text{ with } p \in]1, +\infty[\end{cases} \tag{2.7}$$

where the operator M satisfies (2.5) *and* (2.6). *Assume $b_{ij} \in VMO(\Omega)$. Then there exists a positive constant c such that for any solution of the problem* (2.7) *we have*

$$\|u\|_{W^{2,p}(\Omega)\cap W^{1,p}(\Omega)} \leq c(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}).$$

Theorem 2.2. (see Proposition 4.1 in [5]) *Let Ω be an open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^{1,1}$. Consider the Dirichlet problem*

$$\begin{cases} Mu = f \quad \text{in } \Omega \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \ f \in L^p(\Omega) \text{ with } p \in]1, +\infty[\end{cases} \tag{2.7}$$

where the operator M satisfies (2.5) *and* (2.6). *Assume $b_{ij} \in VMO(\Omega)$. Then the Dirichlet problem* (2.7) *has a unique solution u. Furthermore there exists a positive constant c such that*

$$\|u\|_{W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)} \leq c\|f\|_{L^p(\Omega)}.$$

Our main result is

Theorem 2.3. *Assume hypotheses* (2.2), (2.3) *and* (2.4) *be true and $p \in]1, +\infty[$. Then there exist two positive constants δ and c such that if $Re\,\lambda > \delta$ we have*

$$|\lambda|\|u\|_{L^p(\Omega)} + \sqrt{|\lambda|}\|Du\|_{L^p(\Omega)} + \|D^2u\|_{L^p(\Omega)} \leq c\|\lambda u - Lu\|_{L^p(\Omega)},$$

for every $u \in W^{2,p}(\Omega)$.

3. Proof of Theorem 2.3.

In this last section we outline the proof of our result.

Let us define the operator

$$L_\theta := L + e^{i\theta} D_{tt}, \quad x \in \overline{\Omega},\ t \in \mathbb{R},\ |\theta| < \frac{\pi}{2}.$$

It is easy to see that L_θ satisfies the ellipticity condition (2.6) with constant $\mu = \frac{1}{2}\min(\nu, 1)$.

Let $\phi \in C_0^\infty(\mathbb{R})$ be a cut off function such that $\phi(t) = 1$ if $|t| \le \frac{1}{2}$, $\phi(t) = 0$ if $|t| \ge 1$. Set

$$v(x,t) := \phi(t)\, e^{irt}\, u(x), \quad x \in \Omega, t \in \mathbb{R},$$

with $u \in W^{2,p}(\Omega)$ and $r > 0$. Then

$$L_\theta v(x,t) = -\sum_{i,j=1}^{n} a_{i,j} v_{x_i x_j}(x,t) + e^{i\theta} D_{tt} v(x,t) =$$

$$= \phi(t) e^{irt} \left[L\, u - r^2 e^{i\theta} u\right] + e^{i(\theta + rt)} \left[\phi''(t) + 2ir\phi'(t)\right] u.$$

Using Theorem 2.1, we have

$$\|v\|_{W^{2,p}(\Omega\times\mathbb{R})} \le c\left(\|v\|_{L^p(\Omega\times\mathbb{R})} + \|L_\theta v\|_{L^p(\Omega\times\mathbb{R})}\right).$$

Now, recalling that

$$\|v\|_{L^p(\Omega\times\mathbb{R})} = \|u\|_{L^p(\Omega)} \|\phi\|_{L^p(\mathbb{R})},$$

and

$$\|L_\theta v\|_{L^p(\Omega\times\mathbb{R})} \le \|\phi\|_{L^p(\mathbb{R})} \|(L - r^2 e^{i\theta})u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \|\phi'' + 2ir\phi'\|_{L^p(\mathbb{R})},$$

we easily obtain

$$\begin{aligned}(3.1) \qquad \|v\|_{W^{2,p}(\Omega\times\mathbb{R})} &\le c\{\|u\|_{L^p(\Omega)}\|\phi\|_{L^p(\mathbb{R})} + \\ &+ \|\phi\|_{L^p(\mathbb{R})}\|(L - r^2 e^{i\theta})u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\|\phi'' + 2ir\phi'\|_{L^p(\mathbb{R})}\} \le \\ &\le c\{\|u\|_{L^p(\Omega)}[\|\phi\|_{L^p(\mathbb{R})} + 2r\|\phi'\|_{L^p(\mathbb{R})} + \|\phi''\|_{L^p(\mathbb{R})}] + \\ &+ \|(L - r^2 e^{i\theta})u\|_{L^p(\Omega)}\|\phi\|_{L^p(\mathbb{R})}\} \le \\ &\le c_1\{\|u\|_{L^p(\Omega)}(1 + r) + \|L\, u - r^2 e^{i\theta} u\|_{L^p(\Omega)}\},\end{aligned}$$

where $c_1 \equiv c \max\{\|\phi\|_{L^p(\mathbb{R})}, 2\|\phi'\|_{L^p(\mathbb{R})}\}$.

On the other hand, since $\phi \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$, we have

$$\|v\|_{W^{2,p}(\Omega\times]-\frac{1}{2},\frac{1}{2}[)} = \int_{\Omega\times]-\frac{1}{2},\frac{1}{2}[} \sum_{\alpha\leq 2} |D^\alpha(u(x)e^{irt})|^p \, dx dt = \tag{3.2}$$

$$= \int_\Omega \left\{ |u|^p(1 + r^p + r^{2p}) + (1 + 2r^p) \sum_{j=1}^n |u_{x_j}|^p + \sum_{i,j=1}^n |u_{x_i x_j}|^p \right\} dx \geq$$

$$\geq r^{2p}\|u\|^p_{L^p(\Omega)} + r^p\|D u\|^p_{L^p(\Omega)} + \|D^2 u\|^p_{L^p(\Omega)}.$$

From (3.1) and (3.2) we have

$$2r^2\|u\|_{L^p(\Omega)} + r\|D u\|_{L^p(\Omega)} + \|D^2 u\|_{L^p(\Omega)} \leq$$

$$\leq 2c_1\{\|u\|_{L^p(\Omega)}(1 + r) + \|L u + r^2 e^{i\theta} u\|_{L^p(\Omega)}\}.$$

Choosing r in such way that

$$2r^2 - 2c_1(1 + r) = r^2$$

and taking $\lambda = r^2 e^{i\theta}$ we have

$$|\lambda| \, \|u\|_{L^p(\Omega)} + \sqrt{|\lambda|}\|D u\|_{L^p(\Omega)} + \|D^2 u\|_{L^p(\Omega)} \leq \|L u + \lambda u\|_{L^p(\Omega)}.$$

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