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ALMOST AUTOMORPHIC SOLUTIONS FOR LOTKA-VOLTERRA SYSTEMS WITH DIFFUSION AND TIME-DEPENDENT PARAMETERS

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In this work we study the response for a class of Lotka-Volterra prey-predator systems with diffusion and time-dependent parameters to a large class of oscillatory type functions, namely the pseudo almost automorphic type oscillations. To this end, using the exponential dichotomy approach and a fixed point argument, we propose to analyze a class of nonautonomous semilinear abstract evolution equation of the form $(\star)\, z'(h) = A(h)z(h) + g(h, z(h)),\ h \in \mathbf{R}$, where $A(h),\ h \in \mathbf{R}$ is a family of closed linear operators acting in a Banach space T, the nonlinear term g is μ-pseudo-almost automorphic in a weak sense (Stepanov sense) with respect to h and Lipschitzian in T with respect to the second variable. Therefore, according to the results obtained for equation $(\star)$ we establish the existence and uniqueness of μ-pseudo-almost automorphic solutions in the strong sense (Bohr sense) to a nonautonomous system of reaction-diffusion equations describing a Lotka-Volterra prey-predator model with diffusion and time-dependent parameters in a generalized almost automorphic environment.

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1. Introduction

As part of this work, we study the existence and uniqueness of a μ-pseudo-almost automorphic solutions for the following semilinear evolution equation :

$$z'(h) = A(h)z(h) + g(h, z(h)), \;\; h \in \mathbf{R}, \tag{1}$$

where $(A(h), D(A(h)))$, $h \in \mathbf{R}$ is a family of closed linear operators that generates a strongly continuous evolution family $(U(h,s))_{h \geq s}$ on a Banach space T. The nonlinearity $g : \mathbf{R} \times T \to T$ satisfies some suitable conditions with respect to the second variable and it's μ-pseudo almost automorphic in Stepanov sense in h for each $z \in T$. $(U(h,s))_{h \geq s}$ is an exponential dichotomy on $\mathbf{R}$. The topics in the qualitative theory of ordinary or functional differential equations concerning the existence and uniqueness of almost periodic, almost automorphic, pseudo-almost periodic and pseudo-almost automorphic solutions is one of the most attractive in the recent year due to thier applications in the physical sciences, mathematical biology and control theory. Almost automorphic functions are a natural generalization of almost-periodic functions in the sense of Bohr [7] and they was first introduced in the literature by S. Bochner [8]. For more details about almost automorphic functions and their applications, we refer to the books by N'Guérékata [22], Diangana [14], and the work done by K. Khalil et al in [16] where the authors gave an important overview about almost automorphic functions and their applications to differential equations. In [18], K. Ezzinbi and G.M. N'Guérékata studied the existence of almost automorphic solutions for partial and neutral functional differential equations. They proved that the existence of a bounded solution in $\mathbf{R}^+$ is enough to get an almost-automorphic solution. Other important contributions to the theory of almost automorphic functions include those from Zaki [26], N'Guérékata [22], and Shen and Yi [24].

The concept of pseudo-almost automorphic which was introduced in the literature a few years ago by Xiao et al. [25], comes from the perturbation of an almost automorphic fonction by an ergodic term. Since then, such a powerful concept has generated several developments and extensions, see for instance [13]. A little earlier, K. Ezzinbi et al. in [11] present $\mu-$pseudo-almost automorphic functions which is a new concept of pseudo-almost automorphic functions. Later on, K. Khalil et al in [2] studied Eq (1.1), in the parabolic context, that is when $(A, D(A))$ generates an analytic semigroup $(\mathcal{W}(h))_{h \geq 0}$ on a Banach space T which has an exponential dichotomy on $\mathbf{R}$ and g is Stepanov almost periodic of order $1 \leq p < \infty$ and Lipschitzian with respect to z. They proved the existence and uniqueness of almost periodic solutions for equation (1.1). After in [3] K. Khalil et al proved the existence and uniqueness of μ-pseudo-almost automorphic solutions Eq. (1.1) by using exponential dichotomy assuming that

g is just μ-pseudo-almost automorphic in Stepanov sense in h and Lipshitzian with respect to the second variable. Recently, Alan Chàvez et al. [4] introduced a new concept of almost automorphic functions in a compact space called Compact Almost Automorphic Solutions to Poisson's and Heat Equations.

Our purpose of this paper is to prove the existence of $\mu-$pseudo-almost automorphic solutions of Eq. 1. Our results is based on the the exponential dichotomy approach and a fixed point argument. The organization of this work is as follows. In Section 2, we shall give some definitions and theorems of pseudo-almost automorphic functions. In Section 3, we shall study the existence and uniqueness of μ-pseudo-almost automorphic solutions of Eq. 1. In Section 4, to illustrate our abstract results, we shall give an application.

2. Preliminaries

To prove our main results, some lemmas and definitions will be presented in this section.

Let $A(h) : D(A(h)) \subset T \longrightarrow T$, $h \in \mathbf{R}$ be a family of closed linear operators in a Banach space T. In general $A(h),\ h \in \mathbf{R}$ are time-dependent suitable differential operators that corresponding to the following nonautonomous Cauchy problem:

$$\begin{cases} z'(h) = A(h)z(h),\ h \geq s, \\ z(s) = x \in T. \end{cases}$$

A solution (mild) for equation 2 can be expressed as $z(h) = U(h,s)x$ where $(U(h,s))_{h\geq s}$ is a two parameters family generated by $(A(h))_{h\in\mathbf{R}}$ on T that called strongly continuous evolution family, i.e., $(U(h,s))_{h\geq s} \subset \mathcal{L}(T)$ such that:

(i) $U(h,r)U(r,s) = U(h,s)$ and $U(h,h) = I$ for all $h \geq r \geq s$ and $h,r,s \in \mathbf{R}$.

(ii) The map $(h,s) \to U(h,s)x$ is continuous for all $x \in T$, $h \geq s$ and $h,s \in \mathbf{R}$,

see [17, 23] for more details. Unlike to semigroups, there is no general theory for existence of a corresponding evolution family. However, we can rely on several quite existence theorems corresponding on different contexts. In fact, in the hyperbolic case, we refer to [23] and [1] for the parabolic case.

We recall the Acquistapace and Terreni, conditions which are very important to solve problem (2) : Let $(A(h),D(A(h)))$, $h \in \mathbf{R}$ be a family of linear closed operators on a Banach space T that satisfies the following conditions: there exist constants $\omega \in \mathbf{R}, \theta \in (\frac{\pi}{2}, \pi), M > 0$ and $\eta, \nu \in (0,1]$ with $\eta + \nu > 1$ such that

$$\begin{cases} \Sigma_{\omega,\theta} := \{z \in \mathbf{C} : z \neq 0, \mid arg(z) \mid \leq \theta\} \subset \rho(A(h) - \omega) \\ \\ \|\lambda R(\lambda, A(h) - \omega)\|_{\mathcal{L}(T)} \leq L, \end{cases} \tag{2}$$

$$\|(A(h) - \omega)R(\lambda, A(h) - \omega)[R(\omega, A(h)) - R(\omega, A(s))]\|_{\mathcal{L}(T)} \leq \frac{M|h-s|^{\eta}}{|\lambda|^{\nu}} \tag{3}$$

for all $h \geq s$, $h, s \in \mathbf{R}$ and $\lambda \in \Sigma_{\omega,\theta}$. The domains $D(A(h))$ of the operators $A(h)$ may change with respect to h and do not required to be dense in T. The interpolation spaces for the operators $A(h)$, $h \in \mathbf{R}$. Let A be a sectorial operator, i.e., A satisfy 2 instead of $A(h)$ (it is well known that A generates an analytic semigroup $(H_A(h))_{h \geq 0}$ on T).

Moreover, in the case of a constant domain, i.e., $D := D(A(h))$, $h \in \mathbf{R}$, we can replace assumption 3 with the following: There exist constants $\omega \in \mathbf{R}$, $L \geq 0$ and $0 < \mu \leq 1$ such that

$$\|(A(h) - A(s))R(\omega, A(r))\| \leq L|h-s|^{\mu} \quad \text{for } t, s, r \in \mathbf{R}. \tag{4}$$

A sufficient condition ensuring 4 is the following:

$$\|(\omega - A(h))R(\omega, A(s)) - I_T\| \leq L_0|h-s|^{\mu_0} \quad \text{for } h, s \in \mathbf{R} \tag{5}$$

for some $\omega \in \mathbf{R}$, $L_0 \geq 0$ and $0 < \mu_0 \leq 1$. For more details see [1].
An evolution family $(U(h,s))_{s \leq h}$ on a Banach space T is called has an exponential dichotomy (or hyperbolic) in $\mathbf{R}$ if there exists a family of projections $P(t) \in \mathcal{L}(T)$, $h \in \mathbf{R}$, being strongly continuous with respect to h, and constants $\delta, M > 0$ such that

(i) $U(h,s)P(s) = P(h)U(h,s)$;

(ii) $U(h,s) : Q(s)T \to Q(h)T$ is invertible with the inverse $\tilde{U}(h,s)$;

(iii) $\|U(h,s)P(s)\| \leq Me^{-\delta(h-s)}$ and $\|\tilde{U}(h,s)Q(h)\| \leq Me^{-\delta(h-s)}$

for all $h, s \in \mathbf{R}$ with $s \leq h$, where, $Q(h) = I - P(h)$.

Note that, exponential dichotomy is a classical concept in the study of long-time behaviour of evolution equations. If $P(h) = I$ for $h \in \mathbf{R}$, then $(U(h,s))_{s \leq h}$ is exponential stable. For more details we refer [17]. Hence, for given a hyperbolic evolution family $(U(h,s))_{s \leq h}$, we define its associated Green's function by:

$$\Upsilon(h,s) = \begin{cases} U(h,s)P(s), & h, s \in \mathbf{R}, s \leq h \\ \tilde{U}(h,s)Q(s), & h, s \in \mathbf{R}, s > h. \end{cases}$$

2.1. μ-pseudo-almost automorphic functions

Notations : Let $(T, \|\cdot\|)$ be any Banach space. We denote by $L^p_{loc}(\mathbf{R}, T)$ with $1 \leq p < \infty$, the space of functions $g : \mathbf{R} \longrightarrow T$ measurable such that

$$\left(\int_{[a,b]} \|g(s)\|^p ds\right)^{\frac{1}{p}} < \infty$$

for all $a < b$ in $\mathbf{R}$. $BC(\mathbf{R}, T)$ equipped with the supremum norm is the Banach space of bounded continuous functions from $\mathbf{R}$ into T. Let $1 \leq p < \infty$ and q denotes its conjugate exponent defined by $\frac{1}{p} + \frac{1}{q} = 1$.

In the following, we give properties of μ-pseudo-almost automorphic functions in the classical sense and in Stepanov sense.

Definition 2.1. (H. Bohr) [9] A continuous function $g : \mathbf{R} \to T$ is said to be almost periodic if for every $\varepsilon > 0$, there exists $l_\varepsilon > 0$, such that for every $a \in \mathbf{R}$, there exists $\tau \in [a, a + l_\varepsilon]$ satisfying:

$$\|g(h+\tau) - g(h)\| < \varepsilon \text{ for all } h \in \mathbf{R}.$$

The space of all such functions is denoted by AP($\mathbf{R}, T$).

Definition 2.2. (S. Bochner) [6] A continuous function $g : \mathbf{R} \to T$ is called almost automorphic if for every sequence $(s'_n)_{n\geq 0}$ of real numbers, there exists a subsequence $(s_n)_{n\geq 0} \subset (s'_n)_{n\geq 0}$ and a measurable function $u : \mathbf{R} \to T$, such that

$$u(h) = \lim_{n\to\infty} g(h+s_n) \text{ and } g(h) = \lim_{n\to\infty} u(h-s_n) \text{ for all } h \in \mathbf{R}.$$

The space of all such functions is denoted by $AA(\mathbf{R}, T)$.

Remark 2.3. An almost automorphic function may not be uniformly continuous. Indeed, the real function $g(h) = \sin\left(\dfrac{1}{2+\cos(h)+\cos(\sqrt{2}h)}\right)$ for $h \in \mathbf{R}$, belongs to $AA(\mathbf{R}, \mathbf{R})$, but is not uniformly continuous. Hence, g does not belongs to $AP(\mathbf{R}, \mathbf{R})$.

Then, we have the following inclusions:

$$AP(\mathbf{R}, T) \subset AA(\mathbf{R}, T) \subset BC(\mathbf{R}, T).$$

Definition 2.4. A continuous function $G : \mathbf{R} \times \mathbf{R} \to T$ is said to be bi-almost automorphic if for every sequence $(s'_n)_{n\geq 0}$ of real numbers, there exist a subsequence $(s_n)_{n\geq 0} \subset (s'_n)_{n\geq 0}$ and a measurable function $H : \mathbf{R} \to T$, such that

$$H(h,s) = \lim_{n\to\infty} G(h+s_n, s+s_n) \text{ and } G(h,s) = \lim_{n\to\infty} H(h-s_n, s-s_n) \text{for all} h, s \in \mathbf{R}.$$

The space of all such functions is denoted by $bAA(\mathbf{R}, T)$.

Example 2.5. [20] $G(h,s) = \sin(h)\cos(s)$ is bi-almost automorphic function from $\mathbf{R}\times\mathbf{R}$ to $\mathbf{R}$ as

$$G(h+2\pi, s+2\pi) = G(h,s), \text{ for all } h,s \in \mathbf{R}.$$

Definition 2.6. [15] Let $1 \leq p < \infty$. A function $g \in L^p_{loc}(\mathbf{R},T)$ is said to be bounded in the sense of Stepanov if

$$\sup_{h\in\mathbf{R}}\left(\int_{[h,h+1]}\|g(s)\|^p ds\right)^{\frac{1}{p}} = \sup_{h\in\mathbf{R}}\left(\int_{[0,1]}\|g(h+s)\|^p ds\right)^{\frac{1}{p}} < \infty.$$

The space of all such functions is denoted by $BS^p(\mathbf{R},XT)$ and is provided with the following norm:

$$\begin{aligned} \|g\|_{BS^p} &:= \sup_{h\in\mathbf{R}}\left(\int_{[h,h+1]}\|g(s)\|^p ds\right)^{\frac{1}{p}} \\ &= \sup_{h\in\mathbf{R}}\|g(h+\cdot)\|_{L^p([0,1],T)}. \end{aligned}$$

Then, the following inclusions hold:

$$BC(\mathbf{R},T) \subset BS^p(\mathbf{R},T) \subset L^p_{loc}(\mathbf{R},T). \tag{6}$$

Now, we give the definition of almost automorphy in the sense of Stepanov.

Definition 2.7. [5] Let $1 \leq p < \infty$. A function $g \in L^p_{loc}(\mathbf{R},T)$ is said to be almost automorphic in the sense of Stepanov (or S^p-almost automorphic), if for every sequence $(\sigma_n)_{n\geq 0}$ of real numbers, there exists a subsequence $(s_n)_{n\geq 0} \subset (\sigma_n)_{n\geq 0}$ and a measurable function $u \in L^p_{loc}(\mathbf{R},T)$, such that:

$$\lim_n \left(\int_h^{h+1}\|g(s+s_n)-u(s)\|^p ds\right)^{\frac{1}{p}} = 0$$

and

$$\lim_n \left(\int_h^{h+1}\|u(s-s_n)-g(s)\|^p ds\right)^{\frac{1}{p}} = 0,\ h \in \mathbf{R}.$$

The space of all such functions is denoted by $AAS^p(\mathbf{R},T)$.

Remark 2.8. [5]
(i) Every almost automorphic function is S^p-almost automorphic for $1 \leq p < \infty$.
(ii) For all $1 \leq p_1 \leq p_2 < \infty$, if g is S^{p_2}-almost automorphic, then g is S^{p_1}-almost automorphic.

In this section we recall some properties of μ-ergodic and μ-pseudo-almost automorphic functions. In the sequel, we denote by $\mathcal{B}(\mathbf{R})$ the Lebesgue σ-field of $\mathbf{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}(\mathbf{R})$ satisfying $\mu(\mathbf{R}) = +\infty$ and $\mu([a,b]) < +\infty$ for all $a,b \in \mathbf{R}$ with $(a \leq b)$. We assume the following hypothesis.

(M) For all $\tau \in \mathbf{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a+\tau : a \in A\}) \leq \beta\mu(A) \qquad \text{where} A \in \mathcal{B}(\mathbf{R}) \text{ and } A \cap I = \emptyset.$$

Definition 2.9. [11] Let $\mu \in \mathcal{M}$. A continuous bounded function $g : \mathbf{R} \longrightarrow T$ is called μ-ergodic, if

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|g(h)\| d\mu(h) = 0.$$

The space of all such functions is denoted by $\mathcal{E}(\mathbf{R}, T, \mu)$.

Example 2.10.
(1) In [27], the author defined an ergodic function as a μ-ergodic function in the particular case where the measure μ is the Lebesgue measure.
(2) In [21], the authors considered the space of bounded continuous functions $g : \mathbf{R} \longrightarrow T$ satisfying

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{[-r,r]} \|g(h)\| dt = 0 \quad \text{and} \quad \lim_{N \to +\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \|g(n)\| = 0.$$

This space coincides with the space of μ-ergodic functions where μ is defined in $\mathcal{B}(\mathbf{R})$ by the sum $\mu(A) = \mu_1(A) + \mu_2(A)$ with μ_1 is the Lebesgue measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ and

$$\mu_2(A) = \begin{cases} card(A \cap \mathbf{Z}) & \text{if } A \cap \mathbf{Z} \text{ is finite} \\ \infty & \text{if } A \cap \mathbf{Z} \text{ is infinite.} \end{cases}$$

Definition 2.11. [11] Let $\mu \in \mathcal{M}$. A continuous function $g : \mathbf{R} \longrightarrow T$ is said to be μ-pseudo almost automorphic if g is written in the form:

$$g = u + \varphi,$$

where $u \in AA(\mathbf{R}, T)$ and $\varphi \in \mathcal{E}(\mathbf{R}, T, \mu)$.
The space of all such functions is denoted by $PAA(\mathbf{R}, T, \mu)$.

Now, we give the definition and the important properties of μ-S^p-pseudo-almost automorphic functions.

Definition 2.12. [15] Let $\mu \in \mathcal{M}$. A function $g \in BS^p(\mathbf{R}, T)$ is said to be μ-ergodic in the sense of Stepanov (or μ-S^p-ergodic) if

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_{[h,h+1]} \|g(s)\|^p ds \right)^{\frac{1}{p}} d\mu(h) = 0. \tag{7}$$

The space of all such functions is denoted by $\mathcal{E}^p(\mathbf{R}, T, \mu)$.

Remark 2.13. We obtain by relation (7) that,

$$g \in \mathcal{E}^p(\mathbf{R}, T, \mu) \text{ if and only if } g^b \in \mathcal{E}(\mathbf{R}, L^p([0,1], T), \mu).$$

Proposition 2.14. *[15] Let $\mu \in \mathcal{M}$. Then, for all $1 \leq p < \infty$, $(\mathcal{E}^p(\mathbf{R}, T, \mu), \|\cdot\|_{BS^p})$ is a Banach space.*

Proposition 2.15. *[15] Let $\mu \in \mathcal{M}$ satisfy **(M)**. Then, the following hold:*
(i) *$\mathcal{E}^p(\mathbf{R}, T, \mu)$ is translation invariant.*
(ii) *$\mathcal{E}(\mathbf{R}, T, \mu) \subset \mathcal{E}^p(\mathbf{R}, T, \mu)$.*

2.2. Uniformly μ-pseudo-almost automorphic functions

Definition 2.16. [5] Let $1 \leq p < +\infty$ and $g : \mathbf{R} \times T \longrightarrow Y$ be a function such that $g(\cdot, z) \in L^p_{loc}(\mathbf{R}, Y)$ for each $z \in T$. Then, $g \in AAS^pU(\mathbf{R} \times T, Y)$ if the following hold:

- **(i)** For each $z \in T$, $g(\cdot, z) \in AAS^p(\mathbf{R}, Y)$.
- **(ii)** g is S^p-uniformly continuous with respect to the second argument on each compact subset K in T, namely: for all $\varepsilon > 0$ there exists $\delta_{K,\varepsilon}$ such that for all $z_1, z_2 \in K$, we have

$$\|z_1 - z_2\| \leq \delta_{K,\varepsilon} \Longrightarrow \left(\int_h^{h+1} \|g(s,z_1) - g(s,z_2)\|_Y^p ds \right)^{\frac{1}{p}} \leq \varepsilon \quad \text{for all } h \in \mathbf{R}. \tag{8}$$

Definition 2.17. Let $\mu \in \mathcal{M}$. A function $g : \mathbf{R} \times T \longrightarrow Y$ such that $g(\cdot, z) \in BS^p(\mathbf{R}, Y)$ for each $z \in T$ is said to be μ-S^p-ergodic in h with respect to z in T if the following hold:

- **(i)** For all $z \in T$, $g(\cdot, z) \in \mathcal{E}^p(\mathbf{R}, Y, \mu)$.
- **(ii)** f is S^p-uniformly continuous with respect to the second argument on each compact subset K in T.

Denote by $\mathcal{E}^pU(\mathbf{R} \times T, Y, \mu)$ the set of all such functions.

Definition 2.18. [5] Let $\mu \in \mathcal{M}$ and $g : \mathbf{R} \times T \longrightarrow Y$ be such that $g(\cdot, z) \in BS^p(\mathbf{R}, Y)$ for each $z \in T$. The function g is μ-S^p- pseudo-almost automorphic if g is written as:

$$g = z + \varphi,$$

where $z \in AAS^pU(\mathbf{R} \times T, Y)$, and $\varphi \in \mathcal{E}^pU(\mathbf{R} \times T, Y, \mu)$.
The space of all such functions is denoted $PAAS^pU(\mathbf{R}, T, \mu)$.

Theorem 2.19. *[5] Let $\mu \in \mathcal{M}$ and $f : \mathbf{R} \times T \to Y$. Assume that:*

(i) $g = z + \varphi \in PAAS^pU(\mathbf{R} \times T, Y, \mu)$ *with* $z \in AAS^pU(\mathbf{R} \times T, Y)$ *and* $\varphi \in \mathcal{E}^pU(\mathbf{R} \times T, Y, \mu)$.

(ii) $z = z_1 + z_2 \in PAA(\mathbf{R}, T, \mu)$*, where* $z_1 \in AA(\mathbf{R}, T)$ *and* $z_2 \in \mathcal{E}^p(\mathbf{R}, T, \mu)$.

(iii) For every bounded subset $B \subset T$ the set $\wedge := \{g(\cdot, z) : z \in B\}$ is bounded in $BS^p(\mathbf{R}, T)$.
Then, $g(\cdot, z(\cdot)) \in PAAS^p(\mathbf{R}, Y, \mu)$.

3. μ-pseudo-almost automorphic solutions of equation 1

In this section, we prove the existence and uniqueness of μ-pseudo-almost automorphic mild solutions to equation 1.

Definition 3.1. A mild solution for equation 1 is the continuous function $z : \mathbf{R} \longrightarrow T$ that satisfies the following variation of constants formula:

$$z(h) = U(h, s)z(s) + \int_s^h U(h, r)g(r, z(r))dr \text{ for all } h \geq s,\ h, s \in \mathbf{R}. \quad (9)$$

Where the function g is Lipschitzian in bounded sets with respect to the second argument.

In the sequel, we assume that:

(H1) The family $A(h)$, $h \in \mathbf{R}$ generates a strongly continuous evolution family $(U(h, s))_{h \geq s}$.

(H2) The evolution family $(U(h, s))_{h \geq s}$ has an exponential dichotomy on $\mathbf{R}$, with constants $M \geq 0$, $\delta > 0$ and Green's function Γ.

(H3) For each $z \in T$, $\Gamma(h, s)z$ for $h, s \in \mathbf{R}$ is bi-almost automorphic.

(H4) The function g is Lipschitzian in bounded sets with respect to the second argument i.e., for all $\sigma > 0$ there exists a nonegative scalar function $L_\sigma(\cdot)$ such that

$$\|g(h,z) - g(h,y)\| \leq L_\sigma(h)\|z-y\|, \quad z,y \in B(0,\sigma),\, h \in \mathbf{R}.$$

Remark 3.2. An explicit example of a strongly bi-almost automorphic Green function i.e., hypothesis **(H3)**, is given in Section 4. Sufficient conditions insuring hypothesis **(H3)** in the case where $A(h) = \delta(h)A + \alpha(h)h \in \mathbf{R}$ and A a generator of a strongly continuous semigroup, provided that $\delta, \alpha \in AAS^1(\mathbf{R})$ with $\inf_{h\in\mathbf{R}} \delta(h) > 0$, which is a weak condition, see Section 4.

In the interest of establishing our problem, we first study the following linear inhomogeneous evolution equation associated to equation 1 :

$$z'(h) = A(h)z(h) + u(h) \text{ for all } h \in \mathbf{R}. \tag{10}$$

where $u : \mathbf{R} \to T$ is locally integrable. We recall that a mild solution to equation 10 is a continuous function $z : \mathbf{R} \to T$ that is given by the following variation of constant formula:

$$z(h) = U(h,s)z(s) + \int_s^h U(h,r)u(r)dr \text{ for all } h \geq s : \tag{11}$$

The following Lemma is needed.

Lemma 3.3. *Let $u \in BS^p(\mathbf{R},T)$ for $1 \leq p < \infty$. Assume that **(H1)-(H2)** hold. Then equation 10 has a unique bounded mild solution given by :*

$$z(h) = \int_{\mathbf{R}} \Upsilon(h,s)u(s)ds, \quad h \in \mathbf{R}. \tag{12}$$

Proof. Let us show first that the integral given in formula 12 is defined and bounded on $\mathbf{R}$. We know from the exponential dichotomy of $(U(h,s))_{h\geq s}$ that

$$\int_{\mathbf{R}} \Upsilon(h,s)u(s)ds = \int_{-\infty}^h U(h,\sigma)P(\sigma)u(\sigma)d\sigma - \int_h^\infty \tilde{U}(h,\sigma)Q(\sigma)u(\sigma)d\sigma$$

for $h \in \mathbf{R}$. Let $p > 1$. Using Hölder's inequality, we have:

$$\begin{aligned}
\Big\| \int_{\mathbb{R}} \Upsilon(h,s)u(s)ds \Big\| &\leq \int_{-\infty}^{h} \|U(h,s)P(s)u(s)\|ds + \int_{h}^{\infty} \|\tilde{U}(h,s)Q(s)u(s)\|ds \\
&\leq \int_{-\infty}^{h} Me^{-\delta(h-s)}\|u(s)\|ds + \int_{h}^{\infty} Me^{-\delta(h-s)}\|u(s)\|ds \\
&\leq \sum_{n\geq 1} \int_{h-n}^{h-n+1} Me^{-\delta(h-s)}\|u(s)\|ds + \sum_{n\geq 1} \int_{h+n-1}^{h+n} Me^{-\delta(h-s)}\|u(s)\|ds \\
&\leq M \sum_{n\geq 1} \left(\int_{h-n}^{h-n+1} e^{-q\delta(h-s)}ds \right)^{\frac{1}{q}} \left(\int_{h-n}^{h-n+1} \|u(s)\|^p ds \right)^{\frac{1}{p}} \\
&+ M \sum_{n\geq 1} \left(\int_{h+n-1}^{h+n} e^{-q\delta(h-s)}ds \right)^{\frac{1}{q}} \left(\int_{h+n-1}^{h+n} \|u(s)\|^p ds \right)^{\frac{1}{p}} \\
&\leq 2M \sum_{n\geq 1} e^{-\delta n} \left(\frac{e^{\delta q}-1}{\delta q} \right)^{\frac{1}{q}} \|u\|_{BS^p} \\
&= 2M\|u\|_{BS^p} \left(\frac{e^{\delta q}-1}{\delta q} \right)^{\frac{1}{q}} \frac{1}{e^{\delta}-1} < \infty.
\end{aligned}$$

On the other hand, for $p = 1$, it follows that

$$\begin{aligned}
\Big\| \int_{\mathbb{R}} \Upsilon(h,s)u(s)ds \Big\| &\leq \int_{-\infty}^{h} \|U(h,s)P(s)u(s)\|ds + \int_{h}^{\infty} \|\tilde{U}(h,s)Q(s)u(s)\|ds \\
&\leq \sum_{n\geq 1} \int_{h-n}^{h-n+1} Me^{-\delta(h-s)}\|u(s)\|ds + \sum_{n\geq 1} \int_{h+n-1}^{h+n} Me^{-\delta(h-s)}\|u(s)\|ds \\
&\leq 2M \sum_{n\geq 1} e^{-\delta n} \|u\|_{BS^1} \\
&= 2M \frac{1}{e^{\delta}-1} \|u\|_{BS^1} < \infty.
\end{aligned}$$

Hence, 12 is well defined. Now, the fact that the mild solution of equation 10 is given by 12 can proved as in [5, Theorem 4.2-(i)]. □

Theorem 3.4. *Let* $1 \leq p < \infty$ *and* $u \in AAS^p(\mathbf{R},T)$. *Assume that* ***(H1)-(H3)*** *are satisfied, then equation 10 has a unique mild solution* $z \in AA(\mathbf{R},T)$ *given by 12, it means that.*

$$z(h) = \int_{\mathbf{R}} \Upsilon(h,s)u(s)ds, \quad h \in \mathbf{R}.$$

Proof. Let $1 \leq p < \infty$ and $u \in AAS^p(\mathbf{R},T)$. From Lemma 3.3 we infer that z is the unique mild solution to equation 10 given by 12. Now, we show that $z \in AA(\mathbf{R},T)$. Let $k \in \mathbf{N}$. Then, for $p > 1$, we have

$$\begin{aligned}
\|z_k(h)\| &\leq \int_{h-k}^{h-k+1} \|U(h,s)P(s)u(s)\|ds + \int_{h+k-1}^{h+k} \|\tilde{U}(h,s)Q(s)u(s)\|ds \\
&\leq \int_{h-k}^{h-k+1} Me^{-\delta(h-s)}\|u(s)\|ds + \int_{h+k-1}^{h+k} Me^{\delta(h-s)}\|u(s)\|ds \\
&\leq M\left(\int_{h-k}^{h-k+1} e^{-q\delta(h-s)}ds\right)^{\frac{1}{q}} \left(\int_{h-k}^{h-k+1} \|u(s)\|^p ds\right)^{\frac{1}{p}} \\
&+ M\left(\int_{h+k-1}^{h+k} e^{-q\delta(h-s)}ds\right)^{\frac{1}{q}} \left(\int_{h+k-1}^{h+k} \|u(s)\|^p ds\right)^{\frac{1}{p}} \\
&\leq 2M\|u\|_{BS^p}\left(\frac{e^{\delta q}-1}{\delta q}\right)^{\frac{1}{q}} e^{-\delta k} \text{ for all } h \in \mathbf{R}.
\end{aligned}$$

By the same way, for $p = 1$, we have

$$\begin{aligned}
\|z_k(h)\| &\leq \int_{h-k}^{h-k+1} \|U(h,s)P(s)u(s)\|ds + \int_{t+k-1}^{h+k} \|\tilde{U}(h,s)Q(s)u(s)\|ds \\
&\leq 2M\|u\|_{BS^1}\left(\frac{e^{\delta q}-1}{\delta q}\right)^{\frac{1}{q}} e^{-\delta k} \text{ for all } h \in \mathbf{R}.
\end{aligned}$$

Since $\sum_{k\geq 1} e^{-\delta k} = \frac{e^{-\delta}}{1-e^{-\delta}} < \infty$, it follows from Weierstrass theorem that the serie $\sum_{k\geq 1} z_k(h)$ is uniformly convergent on $\mathbf{R}$. Then, we define

$$z(h) = \sum_{k\geq 1} z_k(h) \text{ for all } h \in \mathbf{R}.$$

In fact, let $n \in \mathbf{N}$. Then, for $p > 1$, we have

$$\begin{aligned}
\|z(h) - \sum_{k=1}^{n} z_k(h)\| &= \Big\| \int_{\mathbb{R}} \Upsilon(h,s)u(s)ds - \sum_{k=1}^{n} \int_{h-k}^{h-k+1} U(h,s)P(s)u(s)ds \\
&+ \sum_{k=1}^{n} \int_{h+k-1}^{h+k} \tilde{U}(h,s)Q(s)u(s)ds \Big\| \\
&\leq \| \sum_{k\geq n+1} \int_{h-k}^{h-k+1} U(h,s)P(s)u(s)ds\| + \| \sum_{k\geq n+1} \int_{h+k-1}^{h+k} \tilde{U}(h,s)Q(s)u(s)ds\| \\
&\leq \sum_{k\geq n+1} \int_{h-k}^{h-k+1} \|U(h,s)P(s)u(s)\|ds + \sum_{k\geq n+1} \int_{h+k-1}^{h+k} \|\tilde{U}(h,s)Q(s)u(s)\|ds \\
&\leq \sum_{k\geq n+1} \int_{h-k}^{h-k+1} Me^{-\delta(h-s)}\|u(s)\|ds + \sum_{k\geq n+1} \int_{h+k-1}^{h+k} Me^{\delta(h-s)}\|u(s)\|ds \\
&\leq M \sum_{k\geq n+1} \left(\int_{h-k}^{h-k+1} e^{-q\delta(h-s)}ds \right)^{\frac{1}{q}} \left(\int_{h-k}^{h-k+1} \|u(s)\|^p ds \right)^{\frac{1}{p}} \\
&+ M \sum_{k\geq n+1} \left(\int_{h+k-1}^{h+k} e^{-q\delta(h-s)}ds \right)^{\frac{1}{q}} \left(\int_{h+k-1}^{h+k} \|u(s)\|^p ds \right)^{\frac{1}{p}} \\
&\leq 2M \left(\frac{e^{\delta q} - 1}{\delta q} \right)^{\frac{1}{q}} \|u\|_{BS^p} \sum_{k\geq n+1} e^{-\delta k} \to 0 \ \ as \ \ n \to \infty
\end{aligned}$$

uniformly in $h \in \mathbf{R}$.
In otherwise, for $p = 1$, we obtain that

$$\begin{aligned}
\|z(h) - \sum_{k=1}^{n} u_k(h)\| &= \Big\| \int_{\mathbb{R}} \Upsilon(h,s)u(s)ds - \sum_{k=1}^{n} \int_{h-k}^{h-k+1} U(h,s)P(s)u(s)ds \\
&\qquad + \sum_{k=1}^{n} \int_{h+k-1}^{h+k} \tilde{U}(h,s)Q(s)u(s)ds \Big\| \\
&\qquad \leq 2M\|u\|_{BS^1} \sum_{k\geq n+1} e^{-\delta k} \to 0 \ \ as \ \ n \to \infty
\end{aligned}$$

uniformly in $h \in \mathbf{R}$.
To conclude, it suffices to prove that for all $k \in \mathbf{N}$, z_k belongs to $AA(\mathbf{R}, T)$. Let (s'_n) be a sequence of real numbers, $u \in AAS^p(\mathbf{R}, T)$ and Υ is bi-almost automorphic, then there exist a subsequence $(s_n) \subset (s'_n)$ a measurable functions $\tilde{u}$ and $\tilde{\Upsilon}$ such that for all $h,\ s \in \mathbf{R}$,

$$\lim_{n\to\infty} \left(\int_{h}^{h+1} \|u(s+s_n) - \tilde{u}(s)\|^p ds \right)^{\frac{1}{p}} = 0$$

and

$$\lim_{n\to\infty}\left(\int_h^{h+1}\|\tilde{u}(s-s_n)-u(s)\|^p ds\right)^{\frac{1}{p}}=0.$$

And for each $z\in T$,

$$\lim_{n\to\infty}\|\Upsilon(h+s_n,s+s_n)z-\tilde{\Upsilon}(h,s)z\|=0$$

and

$$\lim_{n\to\infty}\|\tilde{\Upsilon}(h-s_n,s-s_n)z-\Upsilon(h,s)z\|=0.$$

Let $z_k(h)=\Phi_k(h)-\Psi_k(h)$, where

$$\Phi_k(h)=\int_{h-k}^{h-k+1}\Upsilon(h,s)u(s)ds \text{ and } \Phi_k(h)=\int_{h+k-1}^{h+k}\Upsilon(h,s)u(s)ds.$$

Thus, we define the measurable function by

$$\begin{aligned}\tilde{z}_k(h) &= \int_{h-k}^{h-k+1}\tilde{\Upsilon}(h,s)\tilde{u}(s)ds-\int_{h+k-1}^{h+k}\tilde{\Upsilon}(h,s)\tilde{u}(s)ds\\ &= \tilde{\Phi}_k(h)-\tilde{\Psi}_k(h),\end{aligned}$$

where

$$\tilde{\Phi}_k(h):=\int_{h-k}^{h-k+1}\tilde{\Upsilon}(h,s)\tilde{u}(s)ds \quad\text{and}\quad \tilde{\Psi}_k(h):=\int_{h+k-1}^{h+k}\tilde{\Upsilon}(h,s)\tilde{u}(s)ds, h\in\mathbf{R}.$$

Therefore, for $p>1$, we have

$$
\begin{aligned}
&\|\Phi_k(h+s_n)-\tilde{\Phi}_k(h)\|\\
&\leq \|\int_{h+s_n-k}^{h+s_n-k+1}\Upsilon(h+s_n,s)u(s)ds-\int_{h-k}^{h-k+1}\tilde{\Upsilon}(h,s)\tilde{u}(s)ds\|\\
&\leq \|\int_{k-1}^{k}\Upsilon(h+s_n,h+s_n-s)u(h+s_n-s)ds-\int_{k-1}^{k}\tilde{\Upsilon}(h,h-s)\tilde{u}(h-s)ds\|\\
&\leq \int_{k-1}^{k}\|\Upsilon(h+s_n,h+s_n-s)u(h+s_n-s)-\tilde{\Upsilon}(h,h-s)\tilde{u}(h-s)\|ds\\
&\leq \int_{k-1}^{k}\|\Upsilon(h+s_n,h+s_n-s)u(h+s_n-s)-\Upsilon(h+s_n,h+s_n-s)\tilde{u}(h-s)\|ds\\
&+\int_{k-1}^{k}\|\Upsilon(h+s_n,h+s_n-s)\tilde{u}(h-s)-\tilde{\Upsilon}(h,h-s)\tilde{u}(h-s)\|ds\\
&\leq \int_{k-1}^{k}\|\Upsilon(h+s_n,h+s_n-s)\left[u(h+s_n-s)-\tilde{u}(h-s)\right]\|ds\\
&+\int_{k-1}^{k}\|\Upsilon(h+s_n,h+s_n-s)\tilde{u}(h-s)-\tilde{\Upsilon}(h,h-s)\tilde{u}(h-s)\|ds\\
&\leq M\left(\int_{k-1}^{k}e^{-q\delta s}ds\right)^{\frac{1}{q}}\left(\int_{k-1}^{k}\|u(h+s_n-s)-\tilde{u}(h-s)\|^p ds\right)^{\frac{1}{p}}\\
&+\int_{k-1}^{k}\|\Upsilon(h+s_n,h+s_n-s)\tilde{u}(h-s)-\tilde{\Upsilon}(h,h-s)\tilde{u}(h-s)\|ds\\
&=I_1+I_2,
\end{aligned}
$$

where

$$
I_1 := M\left(\int_{k-1}^{k}e^{-q\delta s}ds\right)^{\frac{1}{q}}\left(\int_{k-1}^{k}\|u(h+s_n-s)-\tilde{u}(h-s)\|^p ds\right)^{\frac{1}{p}}
$$

and

$$
I_2 := \int_{k-1}^{k}\|\Upsilon(h+s_n,h+s_n-s)\tilde{u}(h-s)-\tilde{\Upsilon}(h,h-s)\tilde{u}(h-s)\|ds.
$$

As $u\in AAS^p(\mathbf{R},T)$, $I_1\to 0$, as $n\to\infty$ for all $h\in\mathbf{R}$. From **(H3)** and since

$$
\begin{aligned}
\|\Upsilon(h+s_n,h+s_n-s)\tilde{u}(h-s)-\tilde{\Upsilon}(h,h-s)\tilde{u}(h-s)\| &\leq Me^{-\delta s}\|\tilde{u}(h-s)\|\\
&+\|\tilde{\Upsilon}(h,h-s)\tilde{u}(h-s)\|,
\end{aligned}
$$

it follows in view of the dominated convergence Theorem, that $I_2\to 0$ as $n\to\infty$ for all $h\in\mathbf{R}$. Hence

$$
\lim_{n\to\infty}\|\Phi_k(h+s_n)-\tilde{\Phi}_k(h)\|=0 \text{ for all } \ t\in\mathbf{R}.
$$

We can show in a similar way that

$$\lim_{n\to\infty} \|\tilde{\Phi}_k(h-s_n) - \Phi_k(h)\| = 0 \quad \text{for all} \quad h \in \mathbf{R}.$$

Moreover, by the same way, for $p = 1$, we obtain that

$$\begin{aligned}
\|\Phi_k(h+s_n) - &\tilde{\Phi}_k(h)\| \\
&\leq \left\| \int_{h+s_n-k}^{h+s_n-k+1} \Upsilon(h+s_n,s)u(s)ds - \int_{h-k}^{h-k+1} \tilde{\Upsilon}(h,s)\tilde{u}(s)ds \right\| \\
&\leq M \int_{k-1}^{k} \|u(h+s_n-s) - \tilde{u}(h-s)\| ds \\
&+ \int_{k-1}^{k} \|\Upsilon(h+s_n,h+s_n-s)\tilde{u}(h-s) - \tilde{\Upsilon}(h,h-s)\tilde{u}(h-s)\| ds \\
&= J_1 + I_2,
\end{aligned}$$

where

$$J_1 := M \int_{k-1}^{k} \|u(h+s_n-s) - \tilde{u}(h-s)\| ds$$

Then, the result follows from the fact that $u \in AAS^1(\mathbf{R},T)$. This proves that $\Phi_k \in AA(\mathbf{R},T)$ for each $k \in \mathbf{R}$. By the same way, we prove the result for Ψ_k. We recall that the serie $\sum_{k\geq 1} u_k(h)$ is uniformly convergent on $\mathbf{R}$, which implies that $z \in AA(\mathbf{R},T)$. □

Theorem 3.5. *Let $\mu \in \mathcal{M}$ satisfy* ***(M)****. Assume that* ***(H1)-(H3)*** *are satisfied and that $u \in PAAS^p(\mathbf{R},T,\mu)$. Then equation 10 has a unique mild solution $z \in PAA(\mathbf{R},T,\mu)$ given by,*

$$z(h) = \int_{\mathbf{R}} \Upsilon(h,s)u(s)ds, \;\; h \in \mathbf{R}.$$

Proof. Let $u = \tilde{u} + \varphi \in PAAS^p(\mathbf{R},T,\mu)$, where $\tilde{u} \in AAS^p(\mathbf{R},T)$ and $\varphi \in \mathcal{E}^p(\mathbf{R},T,\mu)$. Thus, z has a unique decomposition

$$z = z_1 + z_2$$

where, for all $h \in \mathbf{R}$, we have

$$z_1(h) = \int_{\mathbf{R}} \Upsilon(h,s)u(s)ds,$$

and

$$\begin{aligned}
z_2(h) &= \int_{\mathbf{R}} (h,s)\varphi(s)ds \\
&:= z_2^a(h) + z_2^l(h),
\end{aligned}$$

where

$$z_2^a(h) := \int_{-\infty}^{h} U(h,s)P(s)\varphi(s)ds \text{ and } z_2^l(h) := -\int_{h}^{\infty} \tilde{U}(t,s)Q(s)\varphi(s)ds.$$

Using Theorem 3.4, we obtain that $u_1 \in AA(\mathbf{R},T)$. Let us prove that $z_2 \in \mathcal{E}(\mathbf{R},T,\mu)$. It suffices to show that $z_2^a, z_2^l \in \mathcal{E}(\mathbf{R},T,\mu)$. Let $r > 0$ and $p > 1$, then

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{-r}^{r} \|z_2^a(h)\| d\mu(h) \\
&\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{h} \|U(h,s)P(s)\varphi(s)\| ds d\mu(h) \\
&\leq \frac{M}{\mu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{h} e^{-\delta(h-s)}\|\varphi(s)\| ds d\mu(h) \\
&\leq \frac{M}{\mu([-r,r])}\int_{-r}^{r}\left(\int_{-\infty}^{h} e^{\frac{-\delta}{2}q(h-s)}ds\right)^{\frac{1}{q}}\left(\int_{-\infty}^{h} e^{\frac{-\delta}{2}p(h-s)}\|\varphi(s)\|^p ds\right)^{\frac{1}{p}} d\mu(h) \\
&\leq \frac{M}{\mu([-r,r])}\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\int_{-r}^{r}\left(\sum_{k\geq 1}\int_{h}^{h+1} e^{\frac{-\delta}{2}p(h-s+k)}\|\varphi(s-k)\|^p ds\right)^{\frac{1}{p}} d\mu(h) \\
&\leq \left(\frac{M}{\mu([-r,r])}\right)^{\frac{1}{q}+\frac{1}{p}}\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\int_{-r}^{r}\left(\sum_{k\geq 1}\int_{h}^{h+1} e^{\frac{-\delta}{2}p(h-s+k)}\|\varphi(s-k)\|^p ds\right)^{\frac{1}{p}} d\mu(h) \\
&\leq L\left(\sum_{k\geq 1} e^{\frac{-\delta}{2}pk}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{h}^{t+1}\|\varphi(s-k)\|^p ds d\mu(h)\right)^{\frac{1}{p}}
\end{aligned}$$

where $L = \dfrac{M}{\mu([-r,r])^{\frac{1}{q}}}\left(\dfrac{2}{q\delta}\right)^{\frac{1}{q}}$.

As $\mathcal{E}^p(\mathbf{R},T,\mu)$ is invariant by translation and by $\varphi \in \mathcal{E}^p(\mathbf{R},T,\mu)$, we have

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{h}^{h+1}\|\varphi(s-k)\|^p ds d\mu(h) = 0 \text{ for all } k \geq 1.$$

Since,

$$\left(\sum_{k\geq 1} e^{\frac{-\delta}{2}pk}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{t}^{h+1}\|\varphi(s-k)\|^p ds d\mu(h)\right)^{\frac{1}{p}} \leq \sum_{k\geq 1} e^{\frac{-\delta}{2}pk}\|\varphi\|_{BS^p},$$

and by the dominated convergence Theorem, we obtain that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|z_2^a(h)\| d\mu(h) = 0. \tag{13}$$

Now, for $p=1$, it follows by the argument that

$$\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|z_2^a(h)\|d\mu(h) \leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{h}\|U(h,s)P(s)\varphi(s)\|dsd\mu(h)$$
$$\leq \frac{M}{\mu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{h} e^{-\delta(h-s)}\|\varphi(s)\|dsd\mu(h)$$
$$\leq M\sum_{k\geq 1} e^{-\delta pk}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{h}^{h+1}\|\varphi(s-k)\|dsd\mu(h) \to 0 \text{ as } r\to\infty.$$

Arguing as above, we show that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|z_2^l(h)\|d\mu(h)=0. \tag{14}$$

From (13) and (14), we have

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|z_2(h)\|d\mu(h)=0.$$

Hence, $z\in\mathcal{E}(\mathbf{R},T,\mu)$. □

In the sequel we prove the existence and uniqueness of μ-pseudo-almost automorphic solutions to the semilinear equation 1.

Theorem 3.6. *Let $p\geq 1$ and $\mu\in\mathcal{M}$ satisfy **(M)**. Asumme that **(H1)-(H4)** hold and $g\in PAAS^pU(\mathbf{R}\times T,\mu)$ with $L_\sigma\in BS^p(\mathbf{R},T)$. If there exist $\sigma>0$ such that,*

$$\sigma > \min\left\{\left(2M\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\frac{1}{1-e^{-\frac{\delta}{2}}}\right)^{\frac{1}{p}}\right),\left(\frac{2M}{1-e^{-\delta}}\right)\right\}^{-1}\|g(\cdot,0)\|_\infty > 0, \tag{15}$$

and

$$\|L_\rho\|_{BS^p} \leq \min\left\{\left(2M\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\frac{1}{1-e^{-\frac{\delta}{2}}}\right)^{\frac{1}{p}}\right),\left(\frac{2M}{1-e^{-\delta}}\right)\right\}^{-1} - \sigma^{-1}\|g(\cdot,0)\|_\infty. \tag{16}$$

Then, equation 1 has a unique mild solution $u\in PAA(\mathbf{R},T,\mu)$ with $0\leq u(h)\leq\sigma$ for all $h\in\mathbf{R}$.

Proof. Consider the set $\Xi_\sigma^{PAA} := \{v\in PAA(\mathbf{R},T) : \sup_{h\in\mathbf{R}}\|u(h)\|\leq\sigma\}$ and define the map $Z:\Xi_\sigma^{PAA}\longrightarrow PAA(\mathbf{R},T)$ by

$$Zu(h)=\int_{\mathbf{R}}G(h,s)g(s,u(s))ds, \quad h\in\mathbf{R}.$$

First, we show that $Z\Xi_{\sigma}^{PAA} \subset \Xi_{\sigma}^{PAA}$. Indeed, let $u \in \Xi_{\sigma}^{PAA}$. Then, by assumptions on g, we obtain that

$$
\begin{aligned}
& \|Zu(h)\|_{\alpha} \\
\leq \; & \int_{\mathbf{R}} \|G(h,s)g(s,u(s))\|_{\alpha} ds \\
\leq \; & M\int_{-\infty}^{h} e^{-\delta(h-s)}\|g(s,u(s))-g(s,0)\|ds + M\int_{-\infty}^{h} e^{-\delta(h-s)}\|g(s,0)\|ds \\
+ \; & M\int_{t}^{+\infty} e^{-\delta(s-h)}\|g(s,u(s))-g(s,0)\|ds + M\int_{h}^{+\infty} e^{-\delta(s-h)}\|g(s,0)\|ds \\
\leq \; & \sigma M\int_{-\infty}^{h} e^{-\delta(h-s)}\left[L_{\sigma}(s)+\sigma^{-1}\|g(s,0)\|\right]ds \\
+ \; & \sigma M\int_{h}^{+\infty} e^{-\delta(s-h)}\left[L_{\sigma}(s)+\sigma^{-1}\|g(s,0)\|\right]ds \\
\leq \; & \sigma M\left(\int_{-\infty}^{h} e^{-q\frac{\delta}{2}(h-s)}ds\right)^{\frac{1}{q}}\left(\sum_{k\geq 1}\int_{h-k}^{h-k+1} e^{-p\frac{\delta}{2}(h-s)}\left[|L_{\sigma}(s)|^{p}+\sigma^{-p}\|g(s,0)\|^{p}\right]ds\right)^{\frac{1}{p}} \\
+ \; & \sigma M\left(\int_{h}^{+\infty} e^{-q\frac{\delta}{2}(s-h)}ds\right)^{\frac{1}{q}}\left(\sum_{k\geq 1}\int_{h+k-1}^{h+k} e^{-p\frac{\delta}{2}(s-h)}\left[|L_{\sigma}(s)|^{p}+\sigma^{-p}\|g(s,0)\|^{p}\right]ds\right)^{\frac{1}{p}} \\
\leq \; & \sigma\left[M\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\sum_{k\geq 1}\int_{h-k}^{h-k+1} e^{-\frac{\delta}{2}p(h-s)}ds\right)^{\frac{1}{p}} + M\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\sum_{k\geq 1}\int_{h+k-1}^{h+k} e^{-\frac{\delta}{2}p(s-h)}ds\right)^{\frac{1}{p}}\right] \\
& \times\left[\|L_{\sigma}\|_{BS^{p}}+\rho^{-1}\|g(\cdot,0)\|_{BS^{p}}\right] \\
\leq \; & 2M\|L_{\sigma}\|_{BS^{p}}\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\frac{1}{1-e^{-\frac{p\delta}{2}}}\right)^{\frac{1}{p}}\left[\|L_{\sigma}\|_{BS^{p}}+\sigma^{-1}\|g(\cdot,0)\|_{BS^{p}}\right] \\
\leq \; & \sigma, \quad h\in\mathbf{R}.
\end{aligned}
$$

Hence, $Z\Xi_{\sigma}^{PAA} \subset \Xi_{\sigma}^{PAA}$. Therefore, let $u,\ v \in \Xi_{\sigma}^{PAA}$. Then, one has :

$$
\begin{aligned}
(Zz)(h) &= \int_{-\infty}^{h} U(h,s)P(s)g(s,z(s))ds - \int_{h}^{\infty} \tilde{U}(h,s)Q(s)g(s,z(s))ds \\
&= (Z^{a}z)(h) + (Z^{l}z)(h) \text{ for all } h\in\mathbf{R},
\end{aligned}
$$

where

$$
\begin{aligned}
(Z^{a}z)(h) = \int_{-\infty}^{h} U(h,s)P(s)g(s,z(s))ds \text{ and } (Z^{a}z)(h) \\
= -\int_{h}^{\infty} \tilde{U}(h,s)Q(s)g(s,z(s))ds, h\in\mathbf{R}.
\end{aligned}
$$

By the composition Theorem 2.19, it is clear that $g(\cdot,z(\cdot)) \in PAAS^p(\mathbf{R},Y,\mu)$. Moreover, for $p>1$, we have

$$\|(Z^a z)(h)-(Z^a v)(h)\| \le \int_{-\infty}^{h} \|U(h,s)P(s)g(s,z(s))-U(h,s)P(s)g(s,v(s))\|ds$$

$$\le M\int_{-\infty}^{h} e^{-\delta(h-s)}\|g(s,z(s))-g(s,v(s))\|ds$$

$$\le M\left(\int_{-\infty}^{h} e^{-\frac{\delta}{2}q(h-s)}ds\right)^{\frac{1}{q}}\left(\int_{-\infty}^{h} e^{-\frac{\delta}{2}p(h-s)}\|g(s,z(s))-g(s,v(s))\|^p ds\right)^{\frac{1}{p}}$$

$$\le M\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\sum_{k\ge1}\int_{h-k}^{h-k+1} e^{-\frac{\delta}{2}p(h-s)}L_g^p(s)\|z(s)-v(s)\|^p ds\right)^{\frac{1}{p}}$$

$$\le M\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\sum_{k\ge1}\int_{h-k}^{h-k+1} e^{-\frac{\delta}{2}p(h-s)}L_g^p(s)ds\right)^{\frac{1}{p}}\|z-v\|_\infty$$

$$\le M\|L_g\|_{BS^p}\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\frac{1}{1-e^{-\frac{p\delta}{2}}}\right)^{\frac{1}{p}}\|z-v\|_\infty$$

Arguing as above, we have also

$$\begin{aligned}\|(Z^l z)(h)-(Z^l v)(h)\| &\le \int_{h}^{\infty} \|\tilde{U}(h,s)Q(s)g(s,z(s))-\tilde{U}(h,s)Q(s)g(s,v(s))\|ds\\ &\le M\int_{h}^{\infty} e^{-\delta(h-s)}\|g(s,z(s))-g(s,v(s))\|ds\\ &\le M\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\sum_{k\ge1}\int_{h+k-1}^{h+k} e^{-\frac{\delta}{2}p(s-h)}L_g^p(s)ds\right)^{\frac{1}{p}}\|z-v\|_\infty\\ &\le M\|L_\sigma\|_{BS^p}\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\frac{1}{1-e^{-\frac{p\delta}{2}}}\right)^{\frac{1}{p}}\|z-v\|_\infty.\end{aligned}$$

Now, for $p=1$, we obtain that

$$\begin{aligned}\|(Z^a z)(h)-(Z^a v)(h)\| &\le \int_{-\infty}^{h} \|U(h,s)P(s)g(s,z(s))-U(h,s)P(s)g(s,v(s))\|ds\\ &\le M\int_{-\infty}^{h} e^{-\delta(h-s)}\|g(s,z(s))-g(s,v(s))\|ds\\ &\le M\sum_{k\ge1} e^{-\delta k}\int_{h-k}^{h-k+1} L_g(s)ds\|z-v\|_\infty\\ &\le M\|L_\sigma\|_{BS^1}\left(\frac{1}{1-e^{-\delta}}\right)\|z-v\|_\infty\end{aligned}$$

and that

$$\begin{aligned}\|(Z^l z)(h)-(Z^l v)(h)\| &\leq \int_h^{\infty} \|\tilde{U}(h,s)Q(s)g(s,z(s))-\tilde{U}(h,s)Q(s)g(s,v(s))\|ds \\ &\leq M\|L_\sigma\|_{BS^1}\left(\frac{1}{1-e^{-\delta}}\right)\|z-v\|_\infty .\end{aligned}$$

Consequently, we have

$$\|(Zz)(h)-(Zv)(h)\| \leq C\|z-v\|_\infty$$

where $C=\|L_\sigma\|_{BS^p}\min\left[\left(2M\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\left(\frac{1}{1-e^{-\frac{p\delta}{2}}}\right)^{\frac{1}{p}}\right),\left(\frac{2M}{1-e^{-\delta}}\right)\right]$.
Therefore, by Banach fixed point Theorem, Z has a unique fixed point $z\in\Xi_\sigma^{PAA}$ such that $Zz=z$. This proves the result. □

4. Application

For illustration, we propose to study the following dynamics of a two-species competition Lotka-Volterra type model with diffusion which is a combinaison of the models in [19].

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,\xi)=d_1(t)\Delta u(t,\xi)+a(t)u(t,\xi)-c_1(t)\dfrac{v(t,\xi)u(t,\xi)}{1+v(t,\xi)}+k_1(t,\xi),\\ \qquad\qquad t\in\mathbf{R},\xi\in\Omega,\\ \dfrac{\partial}{\partial t}v(t,\xi)=d_2(t)\Delta v(t,\xi)-b(t)v(t,\xi)+c_2(t)\dfrac{u(t,\xi)v(t,\xi)}{1+|u(t,\xi)|}+k_2(t,\xi),\\ \qquad\qquad t\in\mathbf{R},\xi\in\Omega,\\ u(t,\xi)|_{\partial\Omega}=0;\ v(t,\xi)|_{\partial\Omega}=0,\ t\in\mathbf{R},\xi\in\partial\Omega\end{cases} \tag{17}$$

where,
• $u(t,\xi)$ and $v(t,\xi)$ are respectively the local densities of the preys and the predators at time t and at location ξ.

- $\Omega\subset\mathbf{R}^N$ $(N\geq 1)$ is an open bounded domain with Lipschitz type boundary $\partial\Omega$.
- $\Delta:=\sum_{k=1}^{n}\dfrac{\partial^2}{\partial\xi_i^2}$ is the Laplace operator on Ω, $d_i\in C^\mu(\mathbf{R},\mathbf{R}^+)$, $0<\mu\leq 1$ $(\mu=1)$, $i=1,2$, are eventually the diffusion terms preys and the predators respectively, such that $0<d_i^0:=\inf_{t\in\mathbf{R}}(d_i)\leq d_i^1:=|d_i|_\infty<\infty$.

- $a,\ b \in L^1_{loc}(\mathbf{R},\mathbf{R}^+)$ correspond to the growth and the death terms $u(t,\xi)$ and terms $v(t,\xi)$ respectively in the absence of interaction, of the populations.

- $g_i : \mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^N \longrightarrow \mathbf{R}$ are the nonlinear terms defined by

$$g_1(t,u(t,\xi),v(t,\xi),\nabla v(t,\xi)) = a(t)u(t,\xi) - c_1(t)\frac{v(t,\xi)u(t,\xi)}{1+v(t,\xi)}$$

and

$$g_2(t,u(t,\xi),v(t,\xi),\nabla u(t,\xi)) = c_2(t)\frac{u(t,\xi)v(t,\xi)}{1+|\nabla u(t,\xi)|}$$

where $c_i \in L^1_{loc}(\mathbf{R},\mathbf{R}^+)$ for $i=1,2$ are the growth and the death terms, due to interactions, of the preys and the predators respectively.

4.1. Concerning the abstract formulation

We consider the following Banach space $X := C_0(\overline{\Omega})\times C_0(\overline{\Omega})$, equipped with the given norm : $\| \begin{pmatrix}\varphi\\ \psi\end{pmatrix} \| = \|\varphi\|_\infty + \|\psi\|_\infty$, where $C_0(\overline{\Omega})$ is the space of continuous functions $\varphi : \overline{\Omega} \longleftrightarrow \mathbf{R}$ such that $\varphi|_{\partial\Omega} = 0$. We define the closed linear operators $(A(t),D(A(t)))$, $t\in\mathbf{R}$, by

$$\begin{cases} A(t) := \begin{pmatrix} d_1(t)\Delta & 0\\ 0 & d_2(t)\Delta - b(t)\end{pmatrix}, \\ D(A(t)) = C_0^2(\overline{\Omega})\times C_0^2(\overline{\Omega}) := D \end{cases} \tag{18}$$

where $C_0^2(\overline{\Omega}) := \left\{\varphi\in C_0(\overline{\Omega})\cap H_0^1(\Omega) :\ \Delta\varphi\in C_0(\overline{\Omega})\right\}$. See [19].

Therefore, the nonlinear term $f : \mathbf{R}\times X \longrightarrow X$ is defined by

$$f(t,\begin{pmatrix}\varphi\\ \psi\end{pmatrix})(\xi) = \begin{pmatrix} g_1(t,\varphi(\xi),\psi(\xi),\nabla\psi(\xi))\\ g_2(t,\varphi(\xi),\psi(\xi),\varphi(\xi))\end{pmatrix}.$$

Hence, 17 takes the eventually abstract form equation 1:

$$u'(t) = A(t)u(t) + f(t,u(t)), \quad t\in\mathbf{R}.$$

4.2. Main results

For prove the existence and uniqueness of almost automorphic solutions to 1, we consider the main hypotheses **(H1)-(H4)**. We also prove that $(A(t),D(A(t))), t \in \mathbf{R}$ satisfy hypotheses **(H1)-(H3)**. The operators $A_1(t) = d_1(t)\Delta$ and $A_2(t) = d_2(t)\Delta$ are defined. The operator $A_0 := -\Delta$ on $C_0(\overline{\Omega})$ is defined in [12], is sectorial with constant $M \geq 1$ and angle of sectoriality $\theta \in (\frac{\pi}{2}, \pi)$ such that

$$\|\lambda R(-\lambda, A_0)\|_{\mathcal{L}(X)} \leq M \quad \text{for all } \lambda \in \Sigma_{0,\theta}. \tag{19}$$

Then, using 19 and by assumptions on d_i , we claim that

$$\|\lambda R(-\lambda, A_i(t))\| = \|\frac{\lambda}{d_i(t)} R(-\frac{\lambda}{d_i(t)}, A_0)\| \leq M \quad \text{for all } t \in \mathbf{R}.$$

Then, for each $t \in \mathbf{R}$, $A_i(t)$ generates a bounded analytic semigroup $(T_t^i(\tau))_{\tau \geq 0}$ (with uniform bound M with respect to t and the same angle θ) on $C_0(\overline{\Omega})$ such that

$$\| T_t^i(\tau) \| \leq M e^{-d_i^0 \lambda_1 \tau} \qquad \text{for } \tau \geq 0, \tag{20}$$

where $\lambda_1 := \min\{\lambda : \lambda \in \sigma(A_0)\} > 0$ and $\sigma(A_0)$ is the spectrum of $-\Delta$ in $H_0^1(\Omega)$ and $M = e^{\lambda_1 |\Omega|^{2/N} (4\pi)^{-1}}$, see [10] for more details. Moreover,

$$\sup_{t,s \in \mathbf{R}} \|A_i(t) A_i(s)^{-1}\| = \sup_{t,s \in \mathbf{R}} \frac{d_i(t)}{d_i(s)} < \infty.$$

Furthermore, by

$$\|A_i(t)A_i(s)^{-1} - I_X\| = d_i(s)^{-1} |d_i(t) - d_i(s)| \leq (d_i^0)^{-1} |d_i(t) - d_i(s)| \leq L_i |t-s|^{\mu}$$

where $L_i := L_{0,i}(d_i^0)^{-1}$ with $L_{0,i}$ is the Hölder constants of d_i. Hence, by 5 one has, for each $i = 1,2$, $(A_i(t))_{t \in \mathbf{R}}$ generates an evolution family $(U_i(t,s))_{t \geq s}$ on $C_0(\overline{\Omega})$. We have, by 20, the semigroups $(T_t^i(\tau))_{\tau \geq 0}$ are hyperbolic with projections $P(t) = I_X$ and $Q(t) = 0$, $t \in \mathbf{R}$ with

$$\|\tau A_i(t) T_t^i(\tau) x\| \leq M e^{-d_i^0 \lambda_1 \tau} \quad \text{for } \tau > 0.$$

So by taking $\phi_i(\sigma) := M e^{-d_i^0 \lambda_1 \sigma} \chi_{\sigma > 0}$ we obtain that $L_i \|\phi_i\|_1 := L_i M (d_i^0 \lambda_1)^{-1}$. Then, the condition $L_i M (d_i^0 \lambda_1)^{-1} < 1$ yields that any evolution family $U_i(t,s)$ for $t \geq s$, is hyperbolic with the same projection I_0 and exponent δ_i satisfying

$$0 < \delta_i < (d_i^0 \lambda_1 - L_i M)/2M,$$

and the associated Green functions $G_i(t,s) = U_i(t,s)$, $t \geq s$. Thus, the family of matrix-valued operators $\begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix}_{t\in\mathbf{R}}$ generates the hyperbolic evolution family

$\left(V(t,s) = \begin{pmatrix} U_1(t,s) & 0 \\ 0 & U_2(t,s) \end{pmatrix}\right)_{t\geq s}$ with projections $\left(P(t) = \begin{pmatrix} I_0 & 0 \\ 0 & I_0 \end{pmatrix}\right)$ for $t \in \mathbf{R}$ and exponent $\delta = \min\{\delta_1;\delta_2\}$. Moreover, by rescaling, we obtain that the following hyperbolic evolution family generates by $(A(t))_{t\in\mathbf{R}}$

$$\left(U(t,s) = \begin{pmatrix} U_1(t,s) & 0 \\ 0 & e^{-\int_s^t b(\sigma)d\sigma}U_2(t,s) \end{pmatrix}\right)_{t\geq s}$$

with projections $\left(P(t) = \begin{pmatrix} I_0 & 0 \\ 0 & I_0 \end{pmatrix}\right)_{t\in\mathbf{R}}$ and exponent

$$\delta_0 = \min\Big(ig\delta_1;\delta_2 + |b|^{BS^1}\Big).$$

Hence, hypotheses **(H1)** and **(H2)** hold with Green function $G(t,s) := U(t,s)$, $t \geq s$ provided that $L_iM(d_i^0\lambda_1^{-2}) < 1$. We need the following preliminary result to check hypothesis **(H3)**.

Lemma 4.1. $\forall\, i = 1,2$, *if* $A_i(\cdot) \in AA(\mathbf{R},\mathcal{L}(C_0^2(\overline{\Omega}),C_0(\overline{\Omega})))$. *Then, the associated evolution family* $U_i(\cdot,\cdot)$ *is bi-almost automorphic.*

Proof. Assume that $\forall i = 1,2$; $A_i = \in AA(\mathbf{R},\mathbf{R})$. Then, for every sequence $(\alpha_k)_{k\geq 0}$ of real numbers, there exists a subsequence $(s_k)_{k\geq 0} \subset (\alpha_k)_{k\geq 0}$ such that

$$\begin{cases} \lim\limits_{k\to+\infty} \|A_i(t+s_k) - A_i(t)\|_{\mathcal{L}(C_0^2(\overline{\Omega}),C_0(\overline{\Omega}))} = 0 \\ \lim\limits_{k\to+\infty} \|A_i(t-s_k) - A_i(t)\|_{\mathcal{L}(C_0^2(\overline{\Omega}),C_0(\overline{\Omega}))} = 0, \end{cases} \tag{21}$$

for all $t \in \mathbf{R}$. For each $i = 1,2$ fixed and $t,\tau \in \mathbf{R}$, one has:

$$A_i(t)^{-1} - A_i(t+\tau)^{-1} = A_i(t+\tau)^{-1}(A_i(t+\tau) - A_i(t))A_i(t)^{-1}. \tag{22}$$

It suffices to show that $A_i^{-1}(\cdot) \in AA(\mathbf{R},\mathcal{L}(C_0(\overline{\Omega})))$. Let $\varphi \in C_0(\overline{\Omega})$, one has:

$$\begin{aligned} & \|A_i(t+s_k)^{-1}\varphi - A_i(t)^{-1}\varphi\| \\ = \;& \|A_i(t+s_k)^{-1}(A_i(t+s_k) - A_i(t))A_i(t)^{-1}\varphi\| \\ \leq \;& \|A_i(t+s_k)^{-1}\|_{\mathcal{L}(C_0(\overline{\Omega}))}\|A_i(t+s_k) - A_i(t)\|_{\mathcal{L}(C_0^2(\overline{\Omega}),C_0(\overline{\Omega}))}\|A_i(t)^{-1}\varphi\|_{C_0^2(\overline{\Omega})}. \end{aligned}$$

By 21 hence

$$\lim_{k\to+\infty} \|A_i(t+s_k)^{-1}\varphi - A_i(t)^{-1}\varphi\| = 0$$

Arguing as above, we obtain that

$$\lim_{k\to+\infty} \|A_i(t-s_k)^{-1}\varphi - A_i(t)^{-1}\varphi\| = 0$$

□

Therefore, we have the following main result.

Proposition 4.2. *Let $d_i \in AA(\mathbf{R})$, $i = 1,2$ and $b \in AAS^1(\mathbf{R},\mathbf{R}^+)$. Then, for each $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in X$, the Green function $G(\cdot,\cdot)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ is bi-almost automorphic. Hence, hypothesis **(H3)** is satisfied.*

Proof. Since $d_i \in AA(\mathbf{R},\mathbf{R}^+)$, it follows that $A_i(\cdot) \in AA(\mathbf{R},\mathcal{L}(C_0^2(\overline{\Omega}),C_0(\overline{\Omega})))$ for $i = 1,2$. Then, by Lemma 4.1, we obtain that $U_i(\cdot,\cdot)$ are bi-almost automorphic. Using the definitions of almost automorphic solutions we show that $e^{-\int_\cdot^\cdot b(\sigma)d\sigma}U_2(\cdot,\cdot)$ is bi-almost automorphic. Let $(\sigma_n)_n$ be any sequence of real numbers, since $b \in AAS^1(\mathbf{R})$, we can find a subsequence $(\tau_n)_n \subset (\sigma_n)_n$ and functions $\tilde{b}$ and $\tilde{U}_2(\cdot,\cdot)$ by definitions of almost automorphic solutions, for b and $U_2(\cdot,\cdot)$ respectively. Define the function $e^{-\int_\cdot^\cdot \tilde{b}(\sigma)d\sigma}\tilde{U}_2(\cdot,\cdot)$. Then, we obtain that

$$\begin{aligned}
&\|e^{-\int_{s+\tau_n}^{t+\tau_n} b(\sigma)d\sigma}U_2(t+\tau_n,s+\tau_n) - e^{-\int_s^t \tilde{b}(\sigma)d\sigma}\tilde{U}_2(t,s)\| \\
\leq\ & e^{-\int_s^t \tilde{b}(\sigma)d\sigma}\|U_2(t+\tau_n,s+\tau_n)\| \left| e^{-\int_s^t \left[b(\sigma+\tau_n)-\tilde{b}(\sigma)\right]d\sigma} - 1\right| \\
+\ & e^{-\int_s^t \tilde{b}(\sigma)d\sigma}\|U_2(t+\tau_n,s+\tau_n) - \tilde{U}_2(t,s)\| \\
\leq\ & Me^{|\tilde{b}|_{BS^1}(t-s)} \left| e^{-\int_s^t \left[b(\sigma+\tau_n)-\tilde{b}(\sigma)\right]d\sigma} - 1\right| \\
+\ & e^{|\tilde{b}|_{BS^1}(t-s)}\|U_2(t+\tau_n,s+\tau_n) - \tilde{U}_2(t,s)\|.
\end{aligned}$$

Thus, one has $\|U_2(t+\tau_n, s+\tau_n) - \tilde{U}_2(t,s)\| \to 0$ as $n \to \infty$ and

$$e^{-\int_s^t \left[b(\sigma+\tau_n) - \tilde{b}(\sigma)\right] d\sigma} \leq e^{\sum_{k=[s]}^{[t]} \int_k^{k+1} |b(\sigma+\tau_n) - \tilde{b}(\sigma)| d\sigma}$$
$$\leq C e^{\sup_k \int_k^{k+1} |b(\sigma+\tau_n) - \tilde{b}(\sigma)| d\sigma (t-s)} \to 1 \text{ as } n \to \infty,$$

uniformly in $t, s \in \mathbf{R}$, $t \geq s$. Thus,

$$\left\| e^{-\int_{s+\tau_n}^{t+\tau_n} b(\sigma) d\sigma} U_2(t+\tau_n, s+\tau_n) - e^{-\int_s^t \tilde{b}(\sigma) d\sigma} \tilde{U}_2(t,s) \right\| \to 0 \text{ as } n \to \infty,$$

uniformly in $t, s \in \mathbf{R}$, $t \geq s$. Hence $e^{-\int_\cdot^\cdot b(\sigma) d\sigma} U_2(\cdot,\cdot)$ is bi-almost automorphic. Consequently, if we consider $\tilde{G}(\cdot,\cdot) = \begin{pmatrix} \tilde{U}_1(\cdot,\cdot) & 0 \\ 0 & e^{-\int_\cdot^\cdot \tilde{b}(\sigma) d\sigma} \tilde{U}_2(\cdot,\cdot) \end{pmatrix}$. Hence, for $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in X$, we obtain

$$\begin{aligned} \|G(t+\tau_n, s+\tau_n) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \tilde{G}(t,s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \| &\leq \|U_1(t+\tau_n, s+\tau_n) - \tilde{U}_1(t,s)\| \|\varphi\|_\infty \\ + \quad \|e^{-\int_{s+\tau_n}^{t+\tau_n} b(\sigma) d\sigma} U_2(t+\tau_n, s+\tau_n) - e^{-\int_s^t \tilde{b}(\sigma) d\sigma} & \tilde{U}_2(t,s)\| \|\psi\|_\infty \to 0 \text{ as } n \to \infty, \end{aligned}$$

uniformly in $t, s \in \mathbf{R}$, $t \geq s$. The result is proved. $\square$

Proposition 4.3. *The function f satisfies **(H4)** with*

$$L_\rho(t) = a(t) + c_1(t) + (c_1(t) + c_2(t))\rho, \quad t \in \mathbf{R}.$$

Proof. Let $\begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} \in X$ and $\rho > 0$ be such that $\left\| \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} \right\|, \left\| \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} \right\| \leq \rho$. Then, for g_2 we have:

$$
\begin{aligned}
&|g_2(t,\varphi_1(\xi),\psi_1(\xi))-g_2(t,\varphi_2(\xi),\psi_2(\xi)|\\
&=c_2(t)\Big|\frac{\varphi_1(\xi)\psi_1(\xi)}{1+|\varphi_1(\xi)|}-\frac{\varphi_2(\xi)\psi_2(\xi)}{1+|\varphi_2(\xi)|}\Big|\\
&=c_2(t)\Big|\Big[\frac{\varphi_1(\xi)\psi_1(\xi)-\varphi_2(\xi)\psi_2(\xi)}{(1+|\varphi_2(\xi)|)(1+|\varphi_1(\xi)|)}\Big]\\
&+\Big[\frac{\varphi_1(\xi)\psi_1(\xi)|\varphi_2(\xi)|\varphi_2(\xi)\psi_2(\xi)|\varphi_1(\xi)|}{(1+|\varphi_1(\xi)|)(1+|\varphi_2(\xi)|)}\Big]\Big|\\
&=c_2(t)\Big|\frac{\psi_2(\xi)}{(1+|\varphi_2(\xi)|)(1+|\varphi_1(\xi)|)}(\varphi_1(\xi)-\varphi_2(\xi))\\
&+\frac{\varphi_1(\xi)}{(1+|\varphi_2(\xi)|)((1+|\varphi_1(\xi)|))}(\psi_1(\xi)-\psi_2(\xi))\\
&+\frac{\varphi_1(\xi)\psi_1(\xi)}{(1+|\varphi_1(\xi)|)(1+|\varphi_2(\xi)|)}(|\varphi_2(\xi)|-|\varphi_1(\xi)|)\\
&+\frac{|\varphi_1(\xi)|(\varphi_1(\xi)\psi_1(\xi)-\varphi_2(\xi)\psi_2(\xi))}{(1+|\varphi_1(\xi)|)(1+|\varphi_2(\xi)|)}\Big|\\
&=c_2(t)\Big|\frac{\psi_2(\xi)}{(1+|\varphi_2(\xi)|)(1+|\varphi_1(\xi)|)}(\varphi_1(\xi)-\varphi_2(\xi))\\
&+\frac{\varphi_1(\xi)}{(1+|\varphi_2(\xi)|)(1+|\varphi_1(\xi)|)}(\psi_1(\xi)-\psi_2(\xi))\\
&+\frac{\varphi_1(\xi)\psi_1(\xi)}{(1+|\varphi_1(\xi)|)(1+|\varphi_2(\xi)|)}(|\varphi_2(\xi)|-|\varphi_1(\xi)|)\Big|\\
&=c_2(t)\Big|\frac{\psi_2(\xi)}{(1+|\varphi_2(\xi)|)(1+|\varphi_1(\xi)|)}(\varphi_1(\xi)-\varphi_2(\xi))\\
&+\frac{\varphi_1(\xi)\psi_1(\xi)}{(1+|\varphi_2(\xi)|)(1+|\varphi_1(\xi)|)}(|\varphi_1(\xi)-\varphi_2(\xi)|)\\
&+\frac{\varphi_1(\xi)}{(1+|\varphi_1(\xi)|)(1+|\varphi_2(\xi)|)}(\psi_1(\xi)-\psi_2(\xi))\Big|\\
&\le c_2(t)(\rho^2+\rho)(|\varphi_1-\varphi_2|)+c_2(t)\rho(|\psi_1-\psi_2|)\\
&\le c_2(t)(\rho^2+\rho)(|\varphi_1-\varphi_2|+|\psi_1-\psi_2|)\\
&\le c_2(t)(\rho^2+\rho)\Big\|\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix}-\begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\Big\|,\quad \xi\in\overline{\Omega},\ t\in\mathbf{R}.
\end{aligned}
$$

Arguing as above, we have that

$$
\begin{aligned}
&|g_1(t,\varphi_1(\xi),\psi_1(\xi),\nabla\varphi_1(\xi))-g_1(t,\varphi_2(\xi),\psi_2(\xi),\nabla\varphi_2(\xi))|\\
\le\quad &(a(t)+c_1(t)(\rho+1))\|\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix}-\begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\|,
\end{aligned}
$$

for $\xi \in \overline{\Omega}$, $t \in \mathbf{R}$. Then, we have that

$$\left\| f(t, \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix}) - f(t, \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix}) \right\| \leq \big(a(t) + c_1(t) + (c_1(t) + c_2(t))\rho + c_2(t)\rho^2\big) \times \left\| \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} - \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} \right\|$$

for every $t \in \mathbf{R}$. Therefore, f satisfies **(H4)** with $L_\rho(t) = a(t) + c_1(t) + (c_1(t) + c_2(t))\rho + c_2(t)\rho^2$. □

By Theorem 3.6, we deduce the following result.

Theorem 4.4. *Under the above assumptions, if L_ρ is small enough, then equation 17 has a unique μ-pseudo almost automorphic solution.*

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