APPROXIMATION PROPERTIES OF CERTAIN MODIFIED SZASZ-MIRAKYAN OPERATORS

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We introduce certain modified Szasz-Mirakyan operators in exponential weighted spaces of functions of one variable. We give theorems on the degree of approximation and the Voronovskaya type theorem.

1. Introduction.

1.1. Let $q > 0$ be a fixed number,

\[ v_q(x) := e^{-q x}, \quad x \in R_0 := [0, +\infty), \]

and let $C_q$ be the space of all real-valued functions $f$ continuous on $R_0$ for which $v_q f$ is uniformly continuous and bounded on $R_0$ and the norm is defined by the formula

\[ \| f \|_q \equiv \| f(\cdot) \|_q := \sup_{x \in R_0} v_q(x) |f(x)|. \]

$C_q$ is called exponential weighted space ([1]).

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In the paper [1] were examined approximation properties of the Szasz-
Mirakyan operators

\[(3) \quad S_n(f; x) := \sum_{k=0}^{\infty} \varphi_k(nx)f(k/n), \quad x \in R_0, \quad n \in N := \{1, 2, \ldots\},\]

for functions \( f \in C_q \), where

\[(4) \quad \varphi_k(t) := e^{-t^k/k!}, \quad t \in R_0, \quad k \in N_0 := N \cup \{0\}.\]

In [1] was proved that \( S_n \) is a positive linear operator from the space \( C_q \) into \( C_r \)

provided that \( r > q > 0 \) and \( n > n_0 \), where \( n_0 \) is a fixed natural number such

that \( n_0 > q/\ln(r/q) \). For example: the function \( 1/v_q \), \( q > 0 \), belongs to \( C_q \), but

\( S_n(1/v_q; \cdot) \notin C_q \) for \( n \in N \).

1.2. Denote by \( B_q, q > 0 \), the space of all real-valued functions \( f \) defined on

\( R_0 \) for which \( v_q f \) is bounded function on \( R_0 \) and the norm is given by (2).

Hence

\[(5) \quad C_q \subset B_q \subset C_r, \quad \text{for} \ r > q > 0.\]

In this paper we modify the operator \( S_n \) given in (3). We introduce the operator

\( S_n[f; q], q > 0, n \in N, \) which (see Lemma 1 and (5)) is a positive linear

operator from the space \( C_q \) into \( C_q \).

**Definition.** Let \( q > 0 \) be a fixed number. For functions \( f \in B_q \) and \( n \in N \) we define operators

\[(6) \quad S_n[f; q](x) \equiv S_n(f; q; x) := \sum_{k=0}^{\infty} \varphi_k(nx)f(k/(n + q)), \quad x \in R_0,\]

where \( \varphi_k(\cdot) \) is given in (4).

By elementary calculations we get from (6):

\[(7) \quad S_n(1; q; x) = 1,\]

\[(8) \quad S_n(t - x; q; x) = -q x/(n + q).\]

\[(9) \quad S_n[(t - x)^2; q; x] = \left(q^2 x^2 + n x\right)/(n + q)^2.\]
(10) \( S_n((t-x)^4; q; x) = \left(q^4 x^4 + 6q^2 nx^3 + (3n - 4q)nx^2 + nx\right)/(n+q)^4, \)

for all \( x \in R_0, n \in N \) and for every fixed \( q > 0 \). Moreover we have

(11) \( S_n\left(e^{qt}; q; x\right) = e^{qtx}, \)

\( S_n\left(te^{qt}; q; x\right) = \frac{n}{n+q} e^{qtx} e^{qtx}, \)

\( S_n\left(t^2 e^{qt}; q; x\right) = \frac{n}{(n+q)^2} e^{qtx} \left\{ nx e^{q(n+q)} + 1 \right\} e^{qtx}, \)

for \( x \in R_0 \) and \( n \in N \), where

(12) \( q_n := n \left(e^{q(n+q)} - 1\right). \)

Next properties of \( S_n[f; q] \) we shall give in Section 2. Main theorems will be given in Section 3.

2. Lemmas.

Applying (11)–(12) and (2), we shall prove two main lemmas.

**Lemma 1.** Let \( q > 0 \) be a fixed number. Then

(13) \( \|S_n\left[1/v_q; q\right]\|_q \leq 1, \quad n \in N. \)

Moreover

(14) \( \|S_n[f; q]\|_q \leq \|f\|_q, \)

for every \( f \in B_q \) and \( n \in N \).

The formulas (1)–(6) and the inequality (14) show that \( S_n[f; q], n \in N, \) is a positive linear from the space \( B_q \) into \( C_q \).
Proof. For \( n \in \mathbb{N} \) and \( q > 0 \) we have
\[
0 < e^{q/(n+q)} - 1 < \sum_{k=1}^{\infty} \left( \frac{q}{n+q} \right)^k = \frac{q}{n},
\]
which by (12) implies \( 0 < q_n < q \) for all \( n \in \mathbb{N} \). From this and by (11) and (1) we get
\[
v_q(x) S_n\left(1/v_q(t); q; x\right) = e^{q/(n+q)x} \leq 1 \quad \text{for} \quad x \in R_0, \ n \in \mathbb{N},
\]
and by (2) follows (13).

Thus the proof is completed. \( \square \)

Lemma 2. For every fixed \( q > 0 \) and for all \( x \in R_0 \) and \( n \in \mathbb{N} \) we have
\[
v_q(x) S_n\left(\frac{(t-x)^2}{v_q(t)}; q; x\right) \leq \frac{4e^2q^2x^2}{(n+q)^2} + \frac{3x}{n+q}.
\]

Proof. From (11)–(12) it follows that
\[
S_n\left(\frac{(t-x)^2}{v_q(t)}; q; x\right) = \left\{ x^2 \left( \frac{n}{n+q} e^{q/(n+q)} - 1 \right)^2 + \frac{nx}{(n+q)^2} e^{q/(n+q)} \right\} e^{q/(n+q)},
\]
for \( x \in R_0 \) and \( n \in \mathbb{N} \). By the inequality \( e^t - 1 \leq t e^t \) for \( t \geq 0 \), we get
\[
\left( \frac{n}{n+q} e^{q/(n+q)} - 1 \right)^2 \leq 2 \left( \frac{n}{n+q} - 1 \right)^2 e^{2q/(n+q)} + \left( e^{q/(n+q)} - 1 \right)^2 \leq \frac{4q^2}{(n+q)^2} e^{2q/(n+q)} < \frac{4e^2q^2}{(n+q)^2} \quad \text{for} \quad n \in \mathbb{N}.
\]
From the above and by (13) we easily obtain (15). \( \square \)

From (8)–(10) we obtain

Lemma 3. Assuming that \( q > 0 \) is a fixed number, we have
\[
\lim_{n \to \infty} n S_n(t-x; q; x) = -qx, \quad \lim_{n \to \infty} n S_n((t-x)^2; q; x) = x,
\]
\[
\lim_{n \to \infty} n^2 S_n((t-x)^3; q; x) = 3x^2,
\]
for every \( x \in R_0 \).
3. Theorems.

3.1. First we shall give two theorems on point-convergence of the sequence \( \{S_n(f; q; \cdot)\}_{n=1}^{\infty} \).

**Theorem 1.** Suppose that \( f \in B_q \) with a fixed \( q > 0 \) and let \( x_0 \in R_0 \) be a point of continuity of \( f \). Then

\[
\lim_{n \to \infty} S_n(f; q; x_0) = f(x_0).
\]

**Proof.** From (6) we get

\[
S_n(f; q; 0) = f(0), \quad n \in N.
\]

If \( x_0 > 0 \), then by (6) and (7) we have

\[
S_n(f; q; x_0) - f(x_0) = \sum_{k=0}^{\infty} \varphi_k(nx_0)(f(k/(n + q)) - f(x_0)), \quad n \in N.
\]

Choose \( \varepsilon > 0 \). By our assumptions there exists \( \delta = \delta(\varepsilon; x_0) > 0 \) such that

\[
|f(k/(n + q)) - f(x_0)| < \varepsilon/2 \quad \text{if} \quad |k/(n + q) - x_0| < \delta.
\]

Denoting by \( Z_1 = \{k \in N_0 : |k/(n + q) - x_0| < \delta\} \) and \( Z_2 = \{k \in N_0 : |k/(n + q) - x_0| \geq \delta\} \) we can write

\[
v_q(x_0)|S_n(f; q; x_0) - f(x_0)| \leq \left( \sum_{k \in Z_1} + \sum_{k \in Z_2} \right) v_q(x_0) \varphi_k(nx_0)|f(k/(n + q)) - f(x_0)|
\]

and

\[
\sum_{k \in Z_1} \leq \frac{\varepsilon}{2} \sum_{k=0}^{\infty} \varphi_k(nx_0) = \frac{\varepsilon}{2}, \quad n \in N.
\]

If \( k \in Z_2 \), then \( 1 \leq \delta^{-2}(k/(n + q) - x_0)^2 \). Moreover for \( f \in B_q \) we have

\[
|f(k/(n + q)) - f(x_0)| \leq \|f\|_q \left( e^{qk/(n+q)} + e^{qnx_0} \right), \quad k \in N_0, n \in N.
\]
Hence
\[
\sum_2 \leq \|f\|_q \delta^{-2} \nu_q(x_0) \sum_{k \in \mathbb{Z}_2} \varphi_k(n x_0 \left( e^{\theta k/(n+q)} + e^{\theta x_0} \right) (k/(n + q) - x_0)^2 \leq
\]
\[
\leq \|f\|_q \delta^{-2} \left\{ e^{\theta x_0} S_n \left( e^{\theta (t - x_0)^2}; q; x_0 \right) + S_n \left( (t - x_0)^2; q; x_0 \right) \right\}.
\]
Applying (9) and (15), we get
\[
\sum_2 \leq \|f\|_q \delta^{-2} \left( \frac{(4e^2 + 1)q^2 x_0^2}{(n + q)^2} + \frac{4x_0}{n + q} \right), \quad n \in \mathbb{N}.
\]
It is obvious that for fixed positive \(x_0, \delta\) and \(\|f\|_q\) there exists \(n_0 \in \mathbb{N}\) such that
\[
\sum_2 < \frac{\varepsilon}{2} \quad \text{for all } n > n_0.
\]
Consequently,
\[
(18) \quad e^{-q x_0} |S_n(f; q; x_0) - f(x_0)| < \varepsilon \quad \text{for } n > n_0.
\]
From (17) and (18) follows (16). \(\square\)

Analogously as Theorem 1 we obtain

Theorem 2. Suppose that \(f \in B_{2q}\) with a fixed \(q > 0\) and let \(x_0 \in R_0\) be a point of continuity of \(f\). Then assertion (16) is satisfied.

Theorems 1 and 2 imply

Corollary 1. If \(f \in C_q\) or \(f \in C_{2q}\) with a fixed \(q > 0\), then
\[
(19) \quad \lim_{n \to \infty} S_n(f; q; x) = f(x), \quad x \in R_0.
\]

3.2. Now we shall given two theorems on the degree of approximation. Let \(\omega_1(f; C_q; \cdot)\) and \(\omega_2(f; C_q; \cdot)\) be the modulus of continuity and the modulus of smoothness of \(f \in C_q, q > 0\), i.e.,
\[
\omega_1(f; C_q; t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_q, \quad \omega_2(f; C_q; t) := \sup_{0 \leq h \leq t} \left\|\Delta^2_h f(\cdot)\right\|_q,
\]
for \(t \geq 0\), where
\[
\Delta_h f(x) := f(x + h) - f(x), \quad \Delta^2_h f(x) := f(x) - 2f(x + h) + f(x + 2h)
\]
for \( x, h \in R_0 \). Let for fixed \( m \in N \) and \( q > 0 \)
\[
C^m_q = \left\{ f \in C_q : f^{(k)} \in C_q, k = 1, 2, \ldots, m \right\},
\]
and let
\[
\psi(x) := (1 + x^2)^{-1}, \quad x \in R_0,
\]
(20)
\[
\lambda_{n,q} := \left( \frac{1 + q}{n + q} \right)^{1/2}, \quad n \in N, q > 0.
\]

**Theorem 3.** Suppose that \( f \in C^2_q \) with a fixed \( q > 0 \). Then
\[
\|S_n\{f ; q \} - f\|_q \leq \frac{q}{n + q} \|f'\|_q + \left( 4e^2 + 1 \right) \|f''\|_q \lambda_{n,q}^2
\]
for all \( n \in N \).

*Proof.* Let \( x \in R_0 \) be a fixed point. Then for \( f \in C^2_q \) and \( t \in R_0 \) we can write
\[
f(t) = f(x) + f'(x)(t - x) + \int_x^t \int_x^s f''(u) \, du \, ds,
\]
which implies
\[
f(t) = f(x) + f'(x)(t - x) + \int_x^t (t - u) f''(u) \, du.
\]
From this and by (7) we deduce that
\[
S_n(f(t); q; x) = f(x) + f'(x)S_n(t - x; q; x) + S_n\left( \int_x^t (t - u) f''(u) \, du ; q; x \right),
\]
\[
n \in N.
\]
But by (1) and (2),
\[
\left| \int_x^t (t - u) f''(u) \, du \right| \leq \|f''\|_q \left( \frac{1}{v_q(t)} + \frac{1}{v_q(x)} \right)(t - x)^2.
\]
From the above and by (9) and (15) it follows that
\[
|v_q(x)|S_n(f(t); q; x) - f(x)| \leq \|f''\|_q |S_n(t - x; q; x)| + \|f''\|_q \left\{ v_q(x)S_n\left( \frac{(t - x)^2}{v_q(t)} ; q; x \right) + S_n\left( (t - x)^2 ; q; x \right) \right\} \leq \|f''\|_q \frac{qx}{n + q} + \|f''\|_q \left\{ \frac{4e^2 + 1}{n + q} q^2x^2 + \frac{4x}{n + q} \right\}
\]
for \( n \in N \), which by (2) and (20) and (21) yields (22). \( \square \)
Theorem 4. Assume that \( f \in C_q \) with a fixed \( q > 0 \). Then

\[
\| \{ S_n [ f ; q ] - f \} \Psi \|_q \leq \frac{M_1}{\sqrt{n + q}} \omega_1 ( f ; C_q ; \lambda_{n,q}) + \\
+(11 + 36 \epsilon^2) \omega_2 ( f ; C_q ; \lambda_{n,q})
\]

for all \( n \in N \), where

\[
M_1 = \left( \frac{5q}{\sqrt{1 + q}} \right) \exp \frac{\sqrt{q} + q^2}{2}
\]

and \( \lambda_{n,q} \) is defined by (21).

Proof. As in [1] we shall use the Stiegl function \( f_h \) of \( f \in C_q \):

\[
f_h (x) := \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \left[ 2f(x+s+t) - f(x) - f(x + 2(s+t)) \right] dsdt, \quad x \in \mathbb{R}_0, h > 0.
\]

From this we get

\[
f_h (x) = \frac{1}{h^2} \int_0^{h/2} \left[ 8\Delta_h f(x + s) - 2\Delta_h f(x + 2s) \right] ds.
\]

and consequently \( f_h \in C_q^2 \) if \( f \in C_q \). Moreover, for \( h > 0 \), we have

\[
\| f_h - f \|_q \leq \omega_2 ( f ; C_q ; h),
\]

\[
\left\| f_h' \right\|_q \leq 5e^{2h}h^{-1} \omega_1 ( f ; C_q ; h),
\]

\[
\left\| f_h'' \right\|_q \leq 9h^{-2} \omega_2 ( f ; C_q ; h),
\]

Hence we can write

\[
\| \{ S_n [ f ; q ] - f \} \Psi \|_q \leq \| \{ S_n [ f - f_h ; q ] - f \} \Psi \|_q + \\
+ \| \{ S_n [ f_h ; q ] - f_h \} \Psi \|_q + \| \{ f_h - f \} \Psi \|_q
\]

for \( n \in N, h > 0 \). By (20), (14) and (24), we get

\[
\| \{ S_n [ f - f_h ; q ] \} \Psi \|_q \leq \| f - f_h \|_q \leq \omega_2 ( f ; C_q ; h),
\]
for \( n \in N \) and \( h > 0 \). Applying Theorem 3 and (25) and (26), we get

\[
\|S_n[f; q] - f_h\|_q \leq \frac{q}{n+q} \|f_h\|_q + (4e^2 + 1) \|f_h\|_q \lambda_{n,q}^2 \\
\leq \frac{5qe^{\eta h}}{h(n+q)} \omega_1(f; C_qh) + 9(4e^2 + 1)h^{-2}\lambda_{n,q}^2 \omega_2(f; C_qh)
\]

for \( n \in N \) and \( h > 0 \). Collecting (24), (28) and (29), we obtain from (27)

\[
\|S_n[f; q] - f\|_q \leq \frac{5qe^{\eta h}}{h(n+q)} \omega_1(f; C_q; h) + 2 + 9(4e^2 + 1)h^{-2}\lambda_{n,q}^2 \omega_2(f; C_q; h)
\]

for all \( n \in N \) and \( h > 0 \). Now setting \( h = \lambda_{n,q} \), for fixed \( n \in N \) and \( q > 0 \), we obtain (23). \( \square \)

Theorem 4 implies the following

**Corollary 2.** If \( f \in C_q, q > 0, \) then

\[
\lim_{n \to \infty} [S_n(f; q; x) - f(x)] = 0
\]

uniformly on every interval \([x_1, x_2], x_2 > x_1 \geq 0\).

**Corollary 3.** If \( f \in C_q \) with a fixed \( q > 0 \) and if \( \omega_2(f; C_q; t) = O(t^{\alpha}) \) for a fixed \( 0 < \alpha \leq 2 \), then there exists positive constant \( M_2(q, \alpha) \), depending only on \( q \) and \( \alpha \), such that

\[
\|S_n[f; q] - f\|_q \leq M_2(q, \alpha) \cdot (n + q)^{-\alpha/2}
\]

for all \( n \in N \).

**3.3.** Applying Lemma 3 and Theorem 2, we shall prove the Voronovskaya type theorem.

**Theorem 5.** Let \( f \in C^2_q \) with a fixed \( q > 0 \). Then

\[
\lim_{n \to \infty} n[S_n(f; q; x) - f(x)] = -qx f'(x) + \frac{x}{2} f''(x)
\]

for every \( x \in R_0 \).
Proof. By (17) follows (30) for $x = 0$. Choosing $x > 0$, we have by the Taylor formula for $f \in C_q^2$:

$$f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + \varepsilon_1(t; x)(t - x)^2, t \in R_0,$$

where $\varepsilon_1(t) \equiv \varepsilon_1(t; x)$ is a function belonging to $C_q$ and $\varepsilon_1(x) = 0$. From this and by (7) we get

$$S_n(f; q; x) = f(x) + f'(x)S_n(t - x; q; x) + \frac{1}{2} f''(x)S_n((t - x)^2; q; x) + S_n(\varepsilon_1(t)(t - x)^2; q; x),$$

which by Lemma 3 yields

$$\lim_{n \to \infty} n[S_n(f; q; x) - f(x)] = -qx f'(x) + \frac{x}{2} f''(x) + \lim_{n \to \infty} nS_n(\varepsilon_1(t)(t - x)^2; q; x).$$

By the Hölder inequality we have

$$\left| S_n(\varepsilon_1(t)(t - x)^2; q; x) \right| \leq \left| S_n(\xi^2(t); q; x) \right|^{1/2} \left| S_n((t - x)^4; q; x) \right|^{1/2},$$

for $n \in N$. Since $\xi^2 \in C_{2q}$, we get by Theorem 2

$$\lim_{n \to \infty} S_n(\varepsilon_1(t); q; x) = \varepsilon_1(x) = 0.$$  

From this and by Lemma 3 we deduce that

$$\lim_{n \to \infty} nS_n(\varepsilon_1(t)(t - x)^2; q; x) = 0$$

and from (31) follows (30). \qed

3.4. Now we shall give some properties of derivatives $(S_n[f; q])^{(r)}$, $r \in N$.

**Theorem 6.** Suppose that $f \in B_q$ with a fixed $q > 0$. Then $S_n[f; q] \in C_q^\infty$ for every fixed $r \in N$ and $n \in N$

$$\left\| (S_n[f; q])^{(r)} \right\|_q \leq n^r \left\| \Delta_{1/(n+q)}^r f(\cdot) \right\|_q,$$

where

$$\Delta_h^r f(x) := \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} f(x + kh).$$
Proof. From (6) we derive the formula
\[ \frac{d}{dx} S_n(f(t); q; x) = -nS_n(f(t); q; x) + nS_n(f(t + 1/(n + q)); q; x) = \]
\[ = nS_n(\Delta_{1/(n+q)} f(t); q; x) \]
and next for every \( r \in N \) we have
\[ (34) \quad \frac{d^r}{dx^r} S_n(f(t); q; x) = n^r S_n \left( \Delta_{1/(n+q)}^r f(t); q; x \right), \quad x \in R_0, \ n \in N, \]
where \( \Delta_{1/(n+q)} f(\cdot) \) is defined by (33). Applying Lemma 1, we immediately obtain (32) from (34).

Corollary 4. Assuming as in Theorem 6, we obtain from (32) and (33) and (2)
\[ \left\| (S_n[f; q])^{(r)} \right\|_q \leq \left( 1 + e^{q/(n+q)} \right)^r \| f \|_q, \]
for \( n, r \in N \).

Formulas (7) and (34) imply the following

Corollary 5. Let \( f \in C_q, \ q > 0 \). Then
(a) if \( f \) is a increasing (decreasing) function on \( R_0 \), then \( S_n(f; q; \cdot), n \in N, \) is also increasing (decreasing) on \( R_0 \);
(b) if \( f \) is a convex (concave) function on \( R_0 \), then \( S_n(f; q; \cdot), n \in N, \) is also convex (concave) on \( R_0 \).

Finally we shall prove analogy of Theorem 1 for first derivate.

Theorem 7. Suppose that \( f \in B_q, \ q > 0, \) and for given \( x_0 > 0 \) there exists \( f'(x_0) \). Then
\[ (35) \quad \lim_{n \to \infty} (S_n[f; q])'(x_0) = f'(x_0). \]

Proof. By assumptions on \( f \) we have
\[ (36) \quad f(t) = f(x_0) + f'(x_0)(t - x_0) + \varepsilon_2(t; x_0)(t - x_0), \]
for \( t \in R_0 \), where \( \varepsilon_2 \) is function continuous at \( x_0 \) and \( \varepsilon_2 \in B_q \).
From (6) it follows that

\[(37) \quad (S_n[f; q])'(x) = -n S_n(f(t); q; x) + \frac{n + q}{x} S_n(t f(t); q; x) =\]
\[= q S_n(f(t); q; x) + \frac{n + q}{x} S_n((t - x) f(t); q; x),\]
for \(x > 0\) and \(n \in \mathbb{N}\). By (36) and (37) we get

\[(38) \quad (S_n[f; q])'(x_0) = f(x_0) \left\{ \frac{n + q}{x_0} S_n(t - x_0; q; x_0) + q \right\} +\]
\[+ f'(x_0) \left\{ \frac{n + q}{x_0} S_n((t - x_0)^2; q; x_0) + q S_n(t - x_0; q; x_0) \right\} +\]
\[+ q S_n(\varepsilon_2(t)(t - x_0); q; x_0) + \frac{n + q}{x_0} S_n(\varepsilon_2(t)(t - x_0)^2; q; x_0).\]

Properties of \(\varepsilon_2\) and Theorem 2 imply that

\[(39) \quad \lim_{n \to \infty} S_n(\varepsilon_2(t)(t - x_0); q; x_0) = 0.\]

Arguing analogously as in the proof of Theorem 5, we get

\[(40) \quad \lim_{n \to \infty} n S_n(\varepsilon_2(t)(t - x_0)^2; q; x_0) = 0.\]

Applying (8), (9), (39) and (40), we obtain (35) from (38).  \(\Box\)

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