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TOTALLY INERT SUBGROUPS OF THE RANK TWO ABELIAN GROUP CONSTRUCTED BY ZASSENHAUS

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A subgroup H of an abelian group G is totally inert if, for every non-zero endomorphism ϕ of G, H is commensurable with $\phi(H)$, that is, $H \cap \phi(H)$ has finite index in H and in $\phi(H)$. In this paper we provide necessary and sufficient conditions for the existence of rank two subgroups which fail to be totally inert of a particular torsion-free group of rank two G such that its endomorphism ring $\mathrm{End}(G)$ equals $\mathbb{Z}[i]$, the ring of Gaussian integers, obtained by a classical construction of Zassenhaus. The results obtained here partially solve a problem raised in a recent paper by Brendan Goldsmith and the author, where totally inert subgroups of general abelian groups are investigated.

1. Introduction

All groups considered in this paper are abelian, so with the word "group" we always mean "abelian group". For unexplaind notions and terminology we refer to the recent monograph by Fuchs [3].

Two subgroups H and K of a group G are commensurable, if $H \cap K$ has finite index in both H and K, equivalently, $(H+K)/H$ and $(H+K)/K$ are both finite. The following notion was introduced in [6].

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Definition 1.1. A subgroup H of a group G is totally inert if H is commensurable with $\phi(H)$ for all non-zero endomorphisms ϕ of G, that is, if $H \cap \phi(H)$ has finite index both in H and in $\phi(H)$.

The restriction in the above definition to non-zero endomorphisms is unavoidable, since, requiring that $(H + \phi(H))/\phi(H)$ is finite also for $\phi = 0$, would force H to be finite, so giving rise to a trivial notion. This restriction does not affect the condition that $(H + \phi(H))/H$ is finite, since for $\phi = 0$ this condition always trivially holds.

The notion of totally inert subgroup strengthens that of fully inert subgroup H of a group G, which only requires that $(H + \phi H)/H$ is finite for all endomorphisms ϕ of G. The notion of fully inert subgroups gave rise to a large number of papers investigating their structure and their applications. For instance, fully inert subgroups of divisible and free groups, and fully inert submodules of free and complete torsion-free J_p-modules, have been characterized in [1], [2] and [8], respectively. Furthermore, in the three papers [7], [4] and [9], concerning abelian p-groups, it was proved, respectively, that direct sums of cyclic p-groups, torsion-complete p-groups and countably totally projective p-groups share the property that all fully inert subgroups are commensurable with fully invariant subgroups. The groups satisfying this property are called *groups with minimal full inertia* and are studied in [5].

The investigation of totally inert subgroups of abelian groups started more recently in the joint paper [4] with Goldsmith, where we proved, *inter alia*, that all subgroups of torsion-free groups of rank one are totally inert, and, more generally, this property is shared by all torsion-free groups whose endomorphism ring is a rational group. Furthermore, we studied the totally inert subgroups of a torsion-free group G of rank two such that its endomorphism ring $\mathrm{End}(G)$ equals $\mathbb{Z}[i]$, the ring of Gaussian integers. The group G is obtained as a particular case of a construction performed almost fifty years ago in [10] by Zassenhaus, who expressed his appreciation to A.L.S. Corner for rectifying and simplifying the proof that was deep and very technical.

We proved in [4] that there are plenty of subgroups of rank two of G which are totally inert and that all the subgroups of rank one are not totally inert. The following problem remained open: does G contain subgroups of rank two which fail to be totally inert?

This paper is devoted to investigate this problem.

In Section 2 we resume part of the Zassenhaus construction of a torsion-free group G of finite rank equal to the rank of the additive group of an assigned ring R, which is assumed to be free, such that the additive group of $\mathrm{End}(G)$ is isomorphic to R. We adapt this construction to the case $R = \mathbb{Z}[i]$, the ring of the Gaussian integers. In this particular case, the group G is the sum of infinitely many subgroups G_k ($k \in \mathbb{N}$) of $\mathbb{Q}[i]$ containing $\mathbb{Z}[i]$ of the form $G_k = \mathbb{Z}[i] + \frac{s}{n} \cdot \mathbb{Z}[i]$, for some $s \in \mathbb{Z}[i] \setminus \mathbb{Z}$ and $n \in \mathbb{Z}$ depending on k, such that the quotient groups $G_k/\mathbb{Z}[i]$ are finite of orders relatively prime in pairs.

In Section 3 we furnish our main result, Theorem 3.2, with the lemmas needed for his proof. This theorem states, *inter alia*, that there exists a rank two subgroup of G which fails to be totally inert if and only if there are infinitely many $k \in \mathbb{N}$ such that the quotient groups $G_k/\mathbb{Z}[i]$ are not cyclic. It follows, in particular, that this happens if there are infinitely many k such that the subgroups $G_k = \mathbb{Z}[i] + \frac{s}{n} \cdot \mathbb{Z}[i]$ have the corresponding elements s and n which are coprime in $\mathbb{Z}[i]$ (see Remark 3.3). However, the recognition whether the required conditions are satisfied in the Zassenhaus construction seems to be a challenging task.

2. The torsion-free rank two group G with $\mathrm{End}(G) = \mathbb{Z}[i]$ obtained in the Zassenhaus construction

In order to find subgroups of rank two of G which fail to be totally inert, we must enter deeply into the Zassenhaus construction of G. From now on, we consider the construction adapted to the case $R = \mathbb{Z}[i]$, so G will always denote the rank two group constructed by Zassenhaus, according to the following theorem.

Theorem 2.1. (Zassenhaus) There exists a $\mathbb{Z}[i]$-submodule G of $\mathbb{Q}[i]$ satisfying the following properties:

(1) $\mathbb{Z}[i] \leq G$

(2) $\mathbb{Z}[i]G = G$

(3) the only endomorphisms of G are the left multiplications by elements of $\mathbb{Z}[i]$.

Items (2) and (3) of Theorem 2.1 imply that the endomorphism ring $\mathrm{End}(G)$ of G is equal to $\mathbb{Z}[i]$. Moreover, we have the proper inclusions $\mathbb{Z}[i] < G < \mathbb{Q}[i]$ ($\mathbb{Q}$ denotes the field of rational numbers), because the group isomorphism $\mathbb{Z}[i] \cong \mathbb{Z} \oplus \mathbb{Z}$ implies that $\mathrm{End}(\mathbb{Z}[i]) \cong M_2(\mathbb{Z})$, the 2×2 matrix ring over $\mathbb{Z}$, and the group isomorphism $\mathbb{Q}[i] \cong \mathbb{Q} \oplus \mathbb{Q}$ implies that $\mathrm{End}(\mathbb{Q}[i]) \cong M_2(\mathbb{Q})$.

The group G is generated by countably many subgroups G_k of $\mathbb{Q}[i]$ ($k \in \mathbb{N}$) satisfying, for each $k \in \mathbb{N}$, the following conditions:

(i) $\mathbb{Z}[i] < G_k$

(ii) the orders of the quotient groups $G_k/\mathbb{Z}[i]$ are positive integers n_k relatively prime in pairs

(iii) $\mathbb{Z}[i]G_k = G_k$

(iv) there is a group endomorphism σ_k of $\mathbb{Q}[i]$ such that $\sigma_k(\mathbb{Z}[i]) \leq \mathbb{Z}[i]$, $\sigma_k(1) = 0$ and $\sigma_k(G_k)$ is not contained in G_k.

The subgroups G_k, in the notation of Zassenhaus, are of the form $\mathbb{Z}[i] + \frac{s}{n} \cdot \mathbb{Z}[i]$, for some $s \in \mathbb{Z}[i] \setminus \mathbb{Z}$ and $n \in \mathbb{Z}$ depending on k.

The Zassenhaus proof is developed in two steps. First it is shown that, assuming the construction of the subgroups G_k to be accomplished, then the group $G = \sum_{k\in\mathbb{N}} G_k$ satisfies the condition of Theorem 2.1. Then the complicated technical part of the proof is the recursive construction of the subgroups G_k, with the positive integers n_k not well identified.

Let m_k be the minimal positive integer such that $m_k G_k \leq \mathbb{Z}[i]$, i.e., $\mathrm{Ann}_{\mathbb{Z}}(G_k/\mathbb{Z}[i]) = m_k\mathbb{Z}$. As each m_k divides n_k, also the integers m_k ($k \in \mathbb{N}$) are relatively prime in pairs. So $G/\mathbb{Z}[i] = \bigoplus_k G_k/\mathbb{Z}[i]$ is a torsion group which is the direct sum of countably many finite groups of orders relatively prime in pairs.

More information on the subgroups G_k are contained in the following proposition.

Proposition 2.2. (1) Each subgroup G_k in the Zassenhaus construction satisfies the equality $G_k = \frac{z_k}{m_k} \cdot \mathbb{Z}[i]$ for an element $z_k = a_k + ib_k \in \mathbb{Z}[i]$ and an integer $m_k > 1$ which is minimal such that $m_k G_k \leq \mathbb{Z}[i]$.

(2) If $a_k = 0$ or $b_k = 0$, then $G_k = \frac{1}{m_k} \cdot \mathbb{Z}[i]$.

(3) If $a_k \neq 0 \neq b_k$, the integer $a_k^2 + b_k^2$ divides $m_k a_k$ and $m_k b_k$, it properly divides m_k^2, and a_k and b_k are coprime.

Proof. (1) As, by construction, each G_k is a $\mathbb{Z}[i]$-submodule of $\mathbb{Q}[i]$ containing $\mathbb{Z}[i]$, also $m_k G_k$ is a $\mathbb{Z}[i]$-submodule of $\mathbb{Z}[i]$, hence an ideal of $\mathbb{Z}[i]$. Being $\mathbb{Z}[i]$ a PID, we get that $m_k G_k = z_k \mathbb{Z}[i]$ for some $z_k = a_k + ib_k \in \mathbb{Z}[i]$, hence $G_k = \frac{z_k}{m_k} \cdot \mathbb{Z}[i]$ with $\frac{z_k}{m_k} \in \mathbb{Q}[i]$.

(2) We may assume $b_k = 0$, since $a_k = 0$ implies that $G_k = \frac{ib_k}{m_k} \cdot \mathbb{Z}[i] = \frac{b_k}{m_k} \cdot \mathbb{Z}[i]$, so we can deal this case as when $b_k = 0$. From $G_k = \frac{a_k}{m_k} \cdot \mathbb{Z}[i] > \mathbb{Z}[i]$ we get that $1 = \frac{a_k}{m_k} \cdot w$ for some $w \in \mathbb{Z}[i]$. Clearly $w = \frac{m_k}{a_k} \in \mathbb{Z}[i] \cap \mathbb{Q} = \mathbb{Z}$ implies $a_k = 1$, since $(a_k, m_k) = 1$.

(3) The inclusion $\mathbb{Z}[i] \leq G_k$ is equivalent to the equality $1 = \frac{z_k}{m_k} \cdot w$ for some $w \in \mathbb{Z}[i]$. Since w is the inverse of $\frac{z_k}{m_k}$, $w = m_k \frac{a_k - ib_k}{a_k^2 + b_k^2} \in \mathbb{Z}[i]$ amounts to say that $a_k^2 + b_k^2$ divides $m_k a_k$ and $m_k b_k$. From $(a_k^2 + b_k^2) r = m_k a_k$ and $(a_k^2 + b_k^2) s = m_k b_k$ ($r, s \in \mathbb{Z}$ non-zero) it follows that $(a_k^2 + b_k^2)^2 (r^2 + s^2) = m_k^2 (a_k^2 + b_k^2)$, therefore $(a_k^2 + b_k^2)(r^2 + s^2) = m_k^2$, where $r^2 + s^2 > 1$. Finally, assume that there exists a prime p such that p^h is its maximal power dividing both a_k and b_k, for an exponent $h \geq 1$. Then $p^h a' = a_k$ and $p^h b' = b_k$, therefore $p^{2h}(a_k'^2 + b_k'^2)$ divides $m_k p^h a_k'$; furthermore, since p cannot divide m_k, p^h divides a_k', and similarly p^h divides b_k', absurd. Consequently no prime p divides both a_k and b_k, which are coprime. □

We provide a particular example of a subgroup G_k containing $\mathbb{Z}[i]$, which needs the following result, to be used also in the next section.

Lemma 2.1. *Let $w = \frac{z}{m}$, with $z = a + ib \in \mathbb{Z}[i]$, $\mathrm{Ann}_{\mathbb{Z}}(w + \mathbb{Z}[i]) = m\mathbb{Z}$, and $w\mathbb{Z}[i] \geq \mathbb{Z}[i]$. Then the quotient group $w\mathbb{Z}[i]/\mathbb{Z}[i]$ has cardinality $\frac{m^2}{a^2+b^2}$.*

Proof. We have the isomorphism

$$w\mathbb{Z}[i]/\mathbb{Z}[i] \cong (w\mathbb{Z}[i]/mw\mathbb{Z}[i])/(\mathbb{Z}[i]/mw\mathbb{Z}[i]). \qquad (*)$$

Since $w\mathbb{Z}[i]/mw\mathbb{Z}[i] \cong \mathbb{Z}[i]/m\mathbb{Z}[i]$, the numerator of the right term in (*) has cardinality m^2. So we must prove that the cardinality of $\mathbb{Z}[i]/(a+ib)\mathbb{Z}[i]$ is (a^2+b^2). In the exact sequence

$$0 \to (a+ib)\mathbb{Z}[i]/(a^2+b^2)\mathbb{Z}[i] \to \mathbb{Z}[i]/(a^2+b^2)\mathbb{Z}[i] \to \mathbb{Z}[i]/(a+ib)\mathbb{Z}[i] \to 0$$

the first non-zero term is isomorphic to the last non-zero term, because $a^2+b^2 = (a+ib)(a-ib)$ and $\mathbb{Z}[i]/(a-ib)\mathbb{Z}[i] \cong \mathbb{Z}[i]/(a+ib)\mathbb{Z}[i]$ via the isomorphism induced by the conjugation. Thus the cardinality of the central term, which equals $(a^2+b^2)^2$, equals also the square of the cardinality of $\mathbb{Z}[i]/(a+ib)\mathbb{Z}[i]$. Consequently the denominator of the right term in (*) equals (a^2+b^2), as desired. $\square$

Example 2.3. Let $G_k = \frac{z_k}{m_k} \cdot \mathbb{Z}[i]$ be a subgroup of $\mathbb{Q}[i]$ containing $\mathbb{Z}[i]$, where $z_k = a_k + ib_k \in \mathbb{Z}[i]$, $a_k \neq 0 \neq b_k$ and m_k is a positive integer minimal such that $m_k G_k \leq \mathbb{Z}[i]$. Assume that $m_k = p_k$ is a prime integer; then the following facts hold:

(1) $p_k = a_k^2 + b_k^2$, so p_k is congruent to 1 modulo 4;

(2) $G_k/\mathbb{Z}[i]$ is isomorphic to the simple cyclic group $\mathbb{Z}(p_k)$, so every subgroup $H_k \leq G_k$ containing $\mathbb{Z}[i]$ is equal either to G_k or to $\mathbb{Z}[i]$.

To check (1), since $(a_k^2+b_k^2)r = p_k a_k$ and $(a_k^2+b_k^2)s = p_k b_k$, p_k must divide $a_k^2+b_k^2$, otherwise $r = p_k r'$ and $s = p_k s'$ would imply that $(a_k^2+b_k^2)r' = a_k$ and $(a_k^2+b_k^2)s' = b_k$, that are evidently absurd. Henceforth, $a_k^2+b_k^2 = p_k h$ for an $h \in \mathbb{N}$. Thus $rh = a_k$ and $sh = b_k$, and since a_k and b_k are coprime, $h = 1$, proving fact (1).

To check (2), observe that the order of $G_k/\mathbb{Z}[i]$, by Lemma 2.1, is equal to $\frac{p_k^2}{p_k} = p_k$, thus $G_k/\mathbb{Z}[i] \cong \mathbb{Z}(p_k)$.

3. Rank two subgroups of G which fail to be totally inert

Our main result, next Theorem 3.2, provides two equivalent conditions for the existence of a rank two subgroup not totally inert of the rank two group G such that $\mathrm{End}(G) = \mathbb{Z}[i]$, obtained with the Zassenhaus construction. The two conditions concern the elements generating the subgroups $G_k/\mathbb{Z}[i]$ and the structure of these subgroups, that cannot be cyclic. The next example provides a subgroup G_k of $\mathbb{Q}[i]$ such that the factor group $G_k/\mathbb{Z}[i]$ is not cyclic, differently from Example 2.3.

Example 3.1. *Let $G_k = \frac{1}{m_k} \cdot \mathbb{Z}[i]$ ($1 < m_k \in \mathbb{Z}$) be a subgroup of $\mathbb{Q}[i]$. Then $G_k/\mathbb{Z}[i] \cong \mathbb{Z}[i]/m_k\mathbb{Z}[i] \cong \mathbb{Z}(m_k) \oplus \mathbb{Z}(m_k)$ ($\mathbb{Z}(m_k)$ denotes the cyclic group of order m_k). The above isomorphism is crucial to obtain a non totally inert subgroup of rank two of G, as proved in the next Theorem 3.2.*

The proof of Theorem 3.2 is based on several lemmas, which use the following notation. Let $G = \sum_{k\in\mathbb{N}} G_k$ be the group obtained by the Zassenhaus construction applied to $\mathbb{Z}[i]$, where $G_k = \frac{a_k+ib_k}{m_k} \cdot \mathbb{Z}[i]$ for all $k \in \mathbb{N}$, with $a_k, b_k, m_k \in \mathbb{Z}$ ($m_k \neq 0$) without a common divisor $\neq 1$.

The first lemma proves the equivalence of (2) and (3) in Theorem 3.2. It makes use of the well known fact that a finite group whose order is the minimal positive integer annihilating it is cyclic.

Lemma 3.1. *The following conditions are equivalent:*

(i) $\frac{a_k^2+b_k^2}{m_k} \in \mathbb{Z}$;

(ii) $a_k^2 + b_k^2 = m_k$;

(iii) $G_k/\mathbb{Z}[i]$ *is isomorphic to* $\mathbb{Z}(m_k)$, *the cyclic group of order* m_k.

Proof. (i) $\Rightarrow$ (ii) From Proposition 2.2 we know that $(a_k^2 + b_k^2)r = m_k a_k$ and $(a_k^2 + b_k^2)s = m_k b_k$ for some $r, s \in \mathbb{Z}$, hence the equality $a_k^2 + b_k^2 = m_k t$ for an element $t \in \mathbb{Z}$ is equivalent to the two equalities $a_k = tr$ and $b_k = ts$. But $(a_k, b_k) = 1$, therefore $t = 1$ and $a_k^2 + b_k^2 = m_k$.

(ii) $\Rightarrow$ (i) is obvious.

(ii) $\Rightarrow$ (iii) From Lemma 2.1 we know that the order of $G_k/\mathbb{Z}[i]$ is equal to $\frac{m_k^2}{a_k^2+b_k^2}$, hence from $(a_k^2 + b_k^2) = m_k$ it follows that this order equals m_k. As $m_k\mathbb{Z}$ is the annihilator ideal of the group $G_k/\mathbb{Z}[i]$, it follows that this group is cyclic, isomorphic to $\mathbb{Z}(m_k)$.

(iii) $\Rightarrow$ (ii) If $G_k/\mathbb{Z}[i] \cong \mathbb{Z}(m_k)$, then $\frac{m_k^2}{a_k^2+b_k^2} = m_k$, hence $(a_k^2 + b_k^2) = m_k$. □

The next lemma is the main tool in proving (1) $\Rightarrow$ (2) of Theorem 3.2.

Lemma 3.2. *Let H be a subgroup of G containing $\mathbb{Z}[i]$ such that $H/\mathbb{Z}[i] = \bigoplus_{k\in\mathbb{N}} H_k/\mathbb{Z}[i]$, where $\mathbb{Z}[i] \leq H_k \leq G_k$ for all k. If $iH_k \leq H_k$ for almost all $k \in \mathbb{N}$, then H is commensurable with a fully invariant subgroup of G, hence H is totally inert.*

Proof. Let us note that $iH_k \leq H_k$ is equivalent to $iH_k = H_k$ and that $H = \sum_{k\in\mathbb{N}} H_k$. Let $iH_k \leq H_k$ for $k \in E_1 \subseteq \mathbb{N}$, where $E_2 = \mathbb{N} \setminus E_1$ is finite. Then

$$H = \sum_{k\in E_1} H_k + \sum_{k\in E_2} H_k$$

therefore

$$H/\sum_{k\in E_1} H_k \cong (H/\mathbb{Z}[i])/(\sum_{k\in E_1} H_k/\mathbb{Z}[i])$$

is isomorphic to a quotient of the finite group $\sum_{k\in E_2} H_k/\mathbb{Z}[i]$, hence H is commensurable with $\sum_{k\in E_1} H_k$, which is fully invariant, because

$$i\sum_{k\in E_1} H_k = \sum_{k\in E_1} iH_k \leq \sum_{k\in E_1} H_k$$

and by [4, Lemma 4.2], so we are done in view of [4, Proposition 2.1]. □

The proof of (2) ⇒ (1) of Theorem 3.2 needs the following three lemmas.

Lemma 3.3. *Let* $G_k = \frac{a_k+ib_k}{m_k}\cdot\mathbb{Z}[i]$ *be such that* $q_k = \frac{a_k^2+b_k^2}{m_k} \in \mathbb{Q}\setminus\mathbb{Z}$. *Then* $q_k \in G_k$ *and* $\mathrm{Ann}_{\mathbb{Z}[i]}(q_k+\mathbb{Z}[i]) = m_k\cdot\mathbb{Z}[i]$.

Proof. Our hypotheses ensure that $\mathrm{Ann}_{\mathbb{Z}}(q_k+\mathbb{Z}[i])\cdot\mathbb{Z}[i] = m_k\cdot\mathbb{Z}[i]$. From $q_k = \frac{a_k^2+b_k^2}{m_k} = \frac{a_k+ib_k}{m_k}\cdot(a_k-ib_k)$ it follows that $q_k \in G_k$. Since the inclusion

$$\mathrm{Ann}_{\mathbb{Z}[i]}(q_k+\mathbb{Z}[i]) \geq m_k\cdot\mathbb{Z}[i]$$

always holds, we must verify the opposite inclusion. So, assume that $(x+iy)q_k \in \mathbb{Z}[i]$ $(x,y\in\mathbb{Z})$. Then $xq_k, yq_k \in \mathbb{Z}$ imply that $x,y \in \mathrm{Ann}_{\mathbb{Z}}(q_k+\mathbb{Z}[i])$, hence $x+iy \in m_k\cdot\mathbb{Z}[i]$. □

Lemma 3.4. *Let* $x\in\mathbb{Q}[i]\setminus\mathbb{Z}[i]$ *with* $\mathrm{Ann}_{\mathbb{Z}}(x+\mathbb{Z}[i]) = r\mathbb{Z}$. *If* $\mathrm{Ann}_{\mathbb{Z}[i]}(x+\mathbb{Z}[i]) = r\mathbb{Z}[i]$, *then*

$$(x\mathbb{Z}+\mathbb{Z}[i])/\mathbb{Z}[i] \bigcap (ix\mathbb{Z}+\mathbb{Z}[i])/\mathbb{Z}[i] = 0.$$

Proof. Assume that $ax - bix \in \mathbb{Z}[i]$ for some $a,b\in\mathbb{Z}$. We must prove that $ax\in\mathbb{Z}[i]$. From the above assumption we get that $x(a-ib)\in\mathbb{Z}[i]$, so $a-ib \in \mathrm{Ann}_{\mathbb{Z}[i]}(x+\mathbb{Z}[i])$ and the hypothesis implies that $a-ib\in r\mathbb{Z}[i]$. Therefore $a-ib = rz$ for some $z\in\mathbb{Z}[i]$, hence $a\in r\mathbb{Z}$ and consequently $ax\in\mathbb{Z}[i]$, as desired. □

Lemma 3.5. *If there exist elements* $x_k \in G_k\setminus\mathbb{Z}[i]$ *with* $\mathrm{Ann}_{\mathbb{Z}}(x_k+\mathbb{Z}[i]) = r_k\mathbb{Z}$ *such that* $\mathrm{Ann}_{\mathbb{Z}[i]}(x_k+\mathbb{Z}[i]) = r_k\mathbb{Z}[i]$ *for all k belonging to an infinite subset E of* $\mathbb{N}$, *then the subgroup* $M = \sum_{k\in E} x_k\mathbb{Z}$ *is not totally inert in G.*

Proof. We have the following equalities:

$$(M+iM+\mathbb{Z}[i])/\mathbb{Z}[i] = \bigoplus_{k\in E}[(x_k\mathbb{Z}+\mathbb{Z}[i])/\mathbb{Z}[i]\oplus(ix_k\mathbb{Z}+\mathbb{Z}[i])/\mathbb{Z}[i]]$$

where the sum in the square parenthesis is direct, by Lemma 3.4. Assume, by way of contradiction, that M is totally inert, so $(M+iM)/M$ is finite (or, equivalently, $(M+iM)/iM$ is finite). Then also $(M+iM+Z[i])/(M+\mathbb{Z}[i])$ is finite. But

$$(M+iM+\mathbb{Z}[i])/(M+\mathbb{Z}[i]) \cong [(M+iM+\mathbb{Z}[i])/\mathbb{Z}[i]]/[(M+\mathbb{Z}[i])/\mathbb{Z}[i]] \cong$$

$$\cong \bigoplus_{k\in E}(ix_k\mathbb{Z}+\mathbb{Z}[i])/\mathbb{Z}[i])$$

and the last direct sum is an infinite group. So we reach the desired contradiction. □

We have now all the tools to prove our main result.

Theorem 3.2. Let $G=\sum_{k\in\mathbb{N}} G_k$ be the group obtained by the Zassenhaus construction applied to $\mathbb{Z}[i]$, where $G_k = \frac{a_k+ib_k}{m_k}\cdot\mathbb{Z}[i]$ for all $k\in\mathbb{N}$, with $a_k, b_k, m_k \in \mathbb{Z}$ $(m_k\neq 0)$ without a common divisor $\neq 1$. The following conditions are equivalent:

(1) there exists a subgroup M of rank two of G which fails to be totally inert;

(2) there are infinitely many $k\in\mathbb{N}$ such that $(a_k^2+b_k^2)\neq m_k$;

(3) there are infinitely many $k\in\mathbb{N}$ such that the quotient groups $G_k/\mathbb{Z}[i]$ are not cyclic;

Proof. (1) $\Rightarrow$ (2). Assume, by way of contradiction, that $(a_k^2+b_k^2)=m_k$ for almost all $k\in\mathbb{N}$. By Lemma 3.1, $G_k/\mathbb{Z}[i]\cong\mathbb{Z}(m_k)$ for almost all $k\in\mathbb{N}$, therefore, setting $w_k=\frac{a_k+ib_k}{m_k}$, $G_k/\mathbb{Z}[i]=(w_k\mathbb{Z}+\mathbb{Z}[i])/\mathbb{Z}[i]$. Then $iw_k\in w_k\mathbb{Z}+\mathbb{Z}[i]$. This implies that for every $x\in G_k\setminus\mathbb{Z}[i]$, since $x+\mathbb{Z}[i]=nw_k+\mathbb{Z}[i]$ for some integer n, $ix+\mathbb{Z}[i]=niw_k+\mathbb{Z}[i]\leq nw_k+\mathbb{Z}[i]=x+\mathbb{Z}[i]$. Therefore, for every subgroup H of G containing $\mathbb{Z}[i]$ such that $H/\mathbb{Z}[i]=\bigoplus_{k\in\mathbb{N}}H_k/\mathbb{Z}[i]$, $iH_k\leq H_k$ for almost all $k\in\mathbb{N}$, Thus the hypothesis of Lemma 3.2 is satisfied, so every subgroup H of G is totally inert, This gives the desired contradiction.

(2) $\Rightarrow$ (1). The hypotheses of Lemma 3.5 with $x_k=q_k=\frac{a_k^2+b_k^2}{m_k}$ are satisfied, in view of Lemma 3.3 and Lemma 3.1, for infinitely many $k\in\mathbb{N}$. Hence the subgroup $M=\sum_{k\in E} q_k\mathbb{Z}$ is not totally inert in G.

(2) $\Leftrightarrow$ (3) follows immediately from Lemma 3.1. □

Remark 3.3. (1) If $a_k=0$ or $b_k=0$, then, in view of Proposition 2.2 (2), we may assume $a_k=1$ and $b_k=0$, thus certainly $(a_k^2+b_k^2)\neq m_k$. In fact, in this case $G_k/\mathbb{Z}[i]\cong\mathbb{Z}(m_k)\oplus\mathbb{Z}(m_k)$ is not cyclic.

(2) If a subgroups G_k of $\mathbb{Q}[i]$ is of the form $G_k=\mathbb{Z}[i]+\frac{s}{n}\cdot\mathbb{Z}[i]$ $(s\in\mathbb{Z}[i]\setminus\mathbb{Z}$, $n\in\mathbb{Z})$ and s is coprime with n in $\mathbb{Z}[i]$, then $n\mathbb{Z}[i]+s\mathbb{Z}[i]=\mathbb{Z}[i]$ is equivalent to $G_k=\frac{1}{n}\cdot\mathbb{Z}[i]$, so $G_k/\mathbb{Z}[i]$ is not cyclic, as required by Theorem 3.2.

The conclusion is that the following question remains open: does the group G constructed by Zassenhaus such that $\mathrm{End}(G)=\mathbb{Z}[i]$ satisfy the equivalent conditions of Theorem 3.2?

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