LE MATEMATICHE
Vol. LV (2000) – Fasc. I, pp. 133–159

THE GLOBAL RING OF A SMOOTH PROJECTIVE SURFACE

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Let X be a smooth projective variety over an algebraically closed field k. We associate to X, in a functorial way, a multigraded k-algebra $G(X)$, called the global ring of X. We find some results when $dim\, X = 2$, and we explicitly determine the global ring in two particular cases.

Introduction.

In some papers ([6], [7], [8]) about zero-dimensional subschemes of a quadric surface $\mathbf{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$, it turned out very useful to use the bigraded ring

$$\bigoplus_{a,b\geq 0} H^0(\mathbf{Q}, \mathcal{O}_{\mathbf{Q}}(a, b))$$

Working with this ring we saw that all the geometric properties of the quadric and of its subschemes can be read in it. In this paper we generalize that construction: to any smooth projective variety X we associate the multi-graded ring

$$G(X) = \bigoplus_{[D]} H^0(X, \mathcal{O}_X(D))$$

which we call the global ring of X. In particular we shall consider the case when X is a smooth projective surface. In the first section we give some general results on the global ring of such surfaces. The main result is that two smooth

Entrato in Redazione il 21 dicembre 2000.

projective surfaces are isomorphic if and only if their global rings are. The following sections are devoted to determine the global rings of some particular surfaces. In section two we consider the surface **S** obtained by blowing up the projective plane in six generic points. Recalling of the usual embedding $\mathbf{S} \hookrightarrow \mathbb{P}^3$ we call **S** a "smooth cubic surface". $G(\mathbf{S})$ is a 7-graded ring, and in the generic case, i.e. when **S** has no Eckardt points, it is the quotient of a polynomial ring having 27 variables with an ideal generated by 81 elements. In the last section we consider a particular Mori quartic **M**, namely a Mori quartic whose Picard group is generated by the classes of two curves, H and L. Here H is a very ample divisor which embeds **M** in $\mathbb{P}^3$ as a smooth quartic surface, and L is an isolated curve such that $L \cdot H = 1$. We show that $G(\mathbf{M})$ is a bigraded ring isomorphic to the quotient of a polynomial ring $k[x, y_1, y_2, z_1, z_2, t]$ by an ideal generated by two elements.

1. Definitions and first results.

Let X be a smooth projective variety over an algebraically closed field k. We can identify the three groups (see [9], II, 6):

$Cl(X)$ = group $Div\, X$ of all Weil divisors, modulo the subgroup of principal divisors;

$CaCl(X)$ = group of all Cartier divisors, modulo the subgroup of principal divisors;

$Pic(X)$ = group of isomorphism classes of the invertible sheaves on X.

For any element $[D] \in Pic(X)$ we can consider the k-vector space $H^0(X, \mathcal{O}_X(D))$ which is different from zero only when D is an effective divisor; so we can define the following ring, which we call the *global ring* of X:

$$G(X) = \bigoplus_{[D]\in Pic(X)} H^0(X, \mathcal{O}_X(D))$$

where we have fixed one divisor for each linear equivalence class.

The ring product is given by the product of sections, i.e. by the sum of divisors; hence if $s \in H^0(X, \mathcal{O}_X(D))$, $s' \in H^0(X, \mathcal{O}_X(D'))$ are two elements, their product is an element in $H^0(X, \mathcal{O}_X(D + D'))$. In this sense we say that $G(X)$ is a $Pic(X)$-graded algebra (or a multi-graded algebra). Note that $G(X)$ is a domain since $s \neq 0, s' \neq 0$ implies that D and D' are effective divisors, so $D + D'$ is too. If X, Y are two smooth varieties, a multigraded ring homomorphism

$$\Phi : G(X) \to G(Y)$$

is a couple (φ, f) where

$$\varphi : Pic(X) \to Pic(Y)$$

is a group homomorphism, $f = \{f_{[D]}\}$ is a family of k-linear maps

$$f_{[D]} : H^0(X, \mathcal{O}_X(D)) \to H^0(Y, \mathcal{O}_Y(D'))$$

where for any divisor $D \subset X$, $\varphi([D]) = [D']$, $D' \subset Y$. The family $f = \{f_{[D]}\}$ satisfies the following condition: for any two divisors $D, E \subset X$, calling $\varphi([D]) = [D']$, $\varphi([E]) = [E']$, the following square is commutative

$$\begin{tikzcd} H^0(X, \mathcal{O}_X(D)) \otimes H^0(X, \mathcal{O}_X(E)) \arrow[r] \arrow[d, "f_{[D]}\otimes f_{[E]}"] & H^0(X, \mathcal{O}_X(D+E)) \arrow[d, "f_{[D+E]}"] \\ H^0(Y, \mathcal{O}_Y(D')) \otimes H^0(Y, \mathcal{O}_Y(E')) \arrow[r] & H^0(Y, \mathcal{O}_Y(D'+E')) \end{tikzcd}$$

So far we have defined a subcategory of the category of multi-graded algebras over abelian groups. Indeed we have a contravariant functor from the category of smooth projective varieties X and their morphisms to the category of multi-graded algebras: to any morphism $g : X \to Y$ it corresponds a group homomorphism $\gamma : Pic(Y) \to Pic(X)$ given by $\gamma(\mathcal{L}) = g^*\mathcal{L}$, where g^* is the inverse image functor, and $\mathcal{L}$ is any invertible sheaf on Y. Moreover, the natural morphism $\mathcal{L} \mapsto g_*g^*\mathcal{L}$ induces the morphism:

$$Hom_Y(\mathcal{O}_Y, \mathcal{L}) \to Hom_Y(\mathcal{O}_Y, g_*g^*\mathcal{L}) \cong$$

$$Hom_X(g^*\mathcal{O}_Y, g^*\mathcal{L}) \cong Hom_X(\mathcal{O}_X, g^*\mathcal{L})$$

where we used standard properties of g_* and g^*. Taking the direct sums of these maps we obtain the algebra homomorphism $\Gamma : G(Y) \to G(X)$. So, starting from $g : X \to Y$, we have constructed the couple (γ, Γ) which is a multigraded algebra homomorphism.

Let now $H \subset X$ be a very ample divisor, and let $h^0(X, \mathcal{O}_X(H)) = n + 1$; thus we have a closed embedding $i : X \to \mathbb{P}^n$ such that $\mathcal{O}_X(H) \cong i^*\mathcal{O}_{\mathbb{P}}(1)$. In this case we can consider the subring of $G(X)$:

$$G_H(X) = \bigoplus_n H^0(X, \mathcal{O}_X(nH))$$

this is a $\mathbb{Z}$-graded ring which completely defines X in the sense that $X \cong Proj\ G_H(X)$ (see [9], II, n. 7).

Denote by $Eff\ X \subset Div\, X$ the semigroup of effective divisors. When $Eff\ X$ is finitely generated then $G(X)$ is isomorphic to a quotient of a polynomial ring with a finite number of variables. In fact, assuming as variables the generators of $Eff\ X$, say $x_1, x_2, \dots, x_n$, taking each variable with its own multi-degree, we have a surjection $k[x_1, x_2, \dots, x_n] \to G(X)$, hence

$$G(X) \cong k[x_1, x_2, \dots, x_n]/I$$

for some homogeneous prime ideal I. Of course, here $k[x_1, x_2, \dots, x_n]$ has to be considered as a $Pic(X)$-graded ring. If D is any effective divisor we shall write $G(X)_{[D]}$, $k[x_1, x_2, \dots, x_n]_{[D]}$ and $I_{[D]}$ for the homogeneous component of the $Pic(X)$-grade of D in $G(X)$, $k[x_1, x_2, \dots, x_n]$ and I respectively.

Notice that even if $Pic(X)$ is finitely generated, we may as well have that $Eff\ X$ is not. This happens, for instance, when X is the blow up of the projective plane in nine (or more) suitable points: in this case $Pic(X) \cong \mathbb{Z}^{10}$ but $Eff\ X$ is not finitely generated because X contains infinitely many isolated divisors (see [11]).

Example 1.1. Let $X = \mathbb{P}^n$. Since $Pic(\mathbb{P}^n) \cong \mathbb{Z}$ and $Eff\ \mathbb{P}^n$ is generated by the elements of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1))$, we obtain for the global ring of $\mathbb{P}^n$ the $\mathbb{Z}$-graded ring

$$G(\mathbb{P}^n) = k[x_0, x_1, \dots, x_n].$$

A similar situation occurs for smooth projective varieties which have $Pic(X) \cong \mathbb{Z}$, generated by a very ample divisor H; in this case $\mathcal{O}_X(H)$ gives a closed immersion $X \hookrightarrow \mathbb{P}^n$ and it turns out that $G(X)$ is isomorphic to the coordinate ring of X as a subvariety of $\mathbb{P}^n$. This happens in particular for surfaces which have only complete intersections as divisors: for instance generic surfaces of $\mathbb{P}^3$ having degree ≥ 4.

We point out that the global ring of a smooth variety X is a ring intrinsically linked to the variety, and we think that it should be possible to study some geometric properties of the variety using this ring. In particular, to every effective divisor on X it corresponds a principal homogeneous ideal of $G(X)$, i.e. to every subscheme of codimension one we can associate its polynomial equation.

In the sequel we shall consider the case when X is a smooth projective surface, and we shall study the case when X is a "smooth cubic surface" **S** (i.e. a surface isomorphic to the blow up of the projective plane at six generic points), and a "Mori quartic" **M** of type (1,0) (i.e. a $K3$ surface **M** such that $Pic(\mathbf{M}) \cong \mathbb{Z}^2$ and $Pic(\mathbf{M})$ is generated by the classes of H and of L, where

H is very ample, such that $h^0(\mathbf{M}, \mathcal{O}_{\mathbf{M}}(H)) = 4$, $h^0(\mathbf{M}, \mathcal{O}_{\mathbf{M}}(L)) = 1$, $H^2 = 4$, and $L^2 = -2$, $H \cdot L = 1$).

Before going on we fix our terminology and give some general results for surfaces. Let X be a smooth projective surface over an algebraically closed field k. For any divisor $D \subset X$, when no confusion can arise, we set for $i = 0, 1, 2$

$$H^i(D) = H^i(X, \mathcal{O}_X(D)) \quad \text{and} \quad h^i(D) = dim_k H^i(D)$$

We denote by K a canonical divisor on X. A *curve* $C \subset X$ is an effective divisor. We say that C is *integral* when it is linearly equivalent to an integral curve. For any divisor $D \subset X$ we call *virtual dimension* of D the number

$$vdim(D) = \frac{1}{2} D \cdot (D - K) + 1 + p$$

where p is the arithmetic genus of X. Notice that $vdim(D) = h^0(D)$ when $h^1(D) = h^2(D) = 0$, as one sees applying the Riemann-Roch Theorem (R-R for short). It is a simple computation to show that for any two divisors $D, D' \subset X$ we have

$$vdim(D + D') = vdim(D) + vdim(D') + D \cdot D' - 1 - p.$$

We say that a curve $C \subset X$ is *isolated* if $h^0(C) = 1$. To any curve $C \subset X$ we can associate its Zariski decomposition (see [12], Theorem 7.7; see also [3], n. 2):

$$C = N(C) + \overline{C}$$

where $N(C)$ is the negative component of C: it is the sum of integral curves, each with its own multiplicity, having negative self-intersection. Thus $N(C)$ is sum of isolated curves ([3], Proposition 2.2). We denote by $Fix(C)$ the fixed component of the linear system corresponding to $H^0(C)$, and by abuse of notation we call it the *fixed part* of C. Since it is not known if every isolated integral curve has negative self-intersection we can only say $N(C) \subset Fix(C)$, and in the cases we shall consider we have $N(C) = Fix(C)$. More precisely, on $\mathbf{S}$ and on $\mathbf{M}$ the isolated curves are "lines". Now, since any line $L \subset \mathbf{S}$ has self-intersection $L^2 = -1$, and any rational integral curve $C \subset \mathbf{M}$ has self-intersection $C^2 = -2$, the assertion follows easily (see [2] n. 2 for curves on $\mathbf{S}$, and [3] n. 4 for curves on Mori quartics).

It is well known that there exists a unique pairing $Pic(X) \times Pic(X) \to \mathbb{Z}$ giving the intersection multiplicity between divisors on X (see [9], V, Theorem 1.1). Now we want to show that this intersection multiplicity can be found, once fixed a very ample divisor on X, starting from the global ring of X, just using the dimensions of certain vector spaces which represent the homogeneous components of this ring.

Lemma 1.2. *Let X be a smooth projective surface and let $E \subset X$ be a very ample divisor. Let $G(X)$ be the global ring of X. Then the intersection multiplicity on X is determined by $G(X)$ and by E.*

Proof. Let $C, D \subset X$ be any two divisors. Denote by $H \subset X$ a multiple of E such that for any $n \geq 1$ and for $i = 1, 2$ we have ([9], III, Theorem 5.3)

$$H^i(nH) = 0, \quad H^i(C + nH) = 0, \quad H^i(D + nH) = 0, \quad H^i(C + D + nH) = 0$$

Then we have:

$$vdim(H + H) = 2vdim(H) + H^2 - 1 - p$$

hence

$$H^2 = h^0(2H) - 2h^0(H) + 1 + p$$

$$vdim(H + (C + H)) = vdim(H) + vdim(C + H) + H^2 + C \cdot H - 1 - p$$

hence

$$\begin{aligned} C \cdot H &= h^0(C + 2H) - h^0(H) - h^0(C + H) - H^2 + 1 + p = \\ &= h^0(C + 2H) - h^0(C + H) - h^0(2H) + h^0(H) \end{aligned}$$

and in the same way

$$D \cdot H = h^0(D + 2H) - h^0(D + H) - h^0(2H) + h^0(H)$$

Finally we can compute

$$vdim((C + H) + (D + H)) = vdim(C + H) + vdim(D + H)$$

$$+C \cdot D + C \cdot H + D \cdot H + H^2 - 1 - p$$

from which by a simple substitution of the previously found values of $C \cdot H$, $D \cdot H$ and H^2 we obtain

$$C \cdot D = h^0(C + D + 2H) - h^0(C + 2H) - h^0(D + 2H) + h^0(2H).$$

□

Now we prove the main result of this section.

Theorem 1.3. *Let X, Y be two smooth projective surfaces and let $G(X)$, $G(Y)$ be their global rings. Then $X \cong Y$ if and only if there exists a multigraded isomorphism $G(X) \cong G(Y)$ between the two global rings.*

Proof. If $X \cong Y$ then $G(X) \cong G(Y)$ by general properties of functors.

Conversely, let $G(X) \cong G(Y)$ via the isomorphism $\Phi = (\varphi, f)$, let $D \subset X$ be a very ample divisor and $\varphi([D]) = [D']$. To the homogeneous component $H^0(D) \subset G(X)$ it corresponds a homogeneous component of $G(Y)$ which is determined by an effective divisor $D' \subset Y$.

Claim: D' is ample. By the Nakai-Moishezon criterion (see [9], V, Theorem 1.10) we need to show that $D' \cdot F' > 0$ for any integral curve $F' \subset Y$. First observe that D' is integral, since the section $s_{D'}$ in $H^0(Y, {}_Y(D'))$ is equal to $f_{[D]}(s_D)$, and D is integral. Moreover, $D'^2 > 0$. Otherwise, if $D'^2 < 0$ then D' should be an isolated curve, so that $h^0(D') = 1$, and D too should be isolated; if $D'^2 = 0$ then a point of Y cannot belong to two curves of $H^0(D')$, so $h^0(D') \leq 2$, i.e. at best D' might move in a pencil, hence $h^0(D) \leq 2$, a contradiction because D is very ample.

Let now $F' \subset Y, F' \neq D'$, be an integral curve, and let $F \subset X$ be the corresponding curve. Suppose that $D' \cdot F' = 0$. Then F' imposes one condition to $h^0(D')$ because any curve linearly equivalent to D' and passing through a point of F' must contain F'. This means that $h^0(D' - F') = h^0(D') - 1$; but then $h^0(D - F) = h^0(D) - 1$, and this is impossible because D is very ample, and a curve cannot impose only one condition to $H^0(D)$ since $H^0(D)$ separates the points of X. Hence the claim is proved. Now, choosing an integer r such that $rD' \subset Y$ is very ample, we have the closed embeddings

$$X \hookrightarrow \mathbb{P}^n, \quad Y \hookrightarrow \mathbb{P}^n$$

where $n = h^0(rD) - 1 = h^0(rD') - 1$. The rings $G_{rD}(X) = \oplus_{n \geq 0} H^0(nrD)$, $G_{rD'}(Y) = \oplus_{n \geq 0} H^0(nrD')$ are two isomorphic $\mathbb{Z}$-graded subrings of $G(X)$ and $G(Y)$ respectively. Since $X \cong Proj\ G_{rD}(X)$ and $Y \cong Proj\ G_{rD'}(Y)$, we have $X \cong Y$. $\square$

The simplest non-trivial surface to which we can apply the above results is the smooth quadric surface $\mathbf{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$. The global ring $G(\mathbf{Q})$ has been used in [6], [7] and [8] to study the zero-dimensional subschemes of $\mathbf{Q}$. In a recent paper [5] this ring has been essential in order to describe the structure of the Rao module of certain curves of $\mathbf{Q}$. There a curve $C \subset \mathbf{Q}$ is associated to a bigraded polynomial which has been used to describe geometric properties of C.

2. The smooth cubic surface S.

Let $\mathbf{S}$ be a surface isomorphic (as a variety) to the blow up of $\mathbb{P}^2$ in six points, no three of them collinear and not all on a conic. It is well known that $Pic(\mathbf{S}) \cong \mathbb{Z}^7$ and that one can choose, as usual, the free base $[L], [E_i] (i = 1, 2, \ldots, 6)$, with intersection multiplicity given by $L^2 = 1$; $L \cdot E_i = 0$; $E_i \cdot E_j = -\delta_{ij}$ $(i, j = 1, 2, \ldots, 6)$. The reader can find in [2] details and references for curves on such surfaces. In the sequel we shall refer to this paper for notation and standard facts.

The surface $\mathbf{S}$ can be embedded in $\mathbb{P}^3$ as a smooth cubic surface by means of the divisor $H = 3L - \sum_{i=1}^{6} E_i$ which is of type (3; 1, 1, 1, 1, 1, 1) in $Pic(\mathbf{S})$, so we simply call $\mathbf{S}$ a smooth cubic surface. Observe that two surfaces $\mathbf{S}$, $\mathbf{S'}$ raising in this way are isomorphic if and only if there exist six skew lines $L_1, L_2, \ldots, L_6 \subset \mathbf{S}$ and six skew lines $L'_1, L'_2, \ldots, L'_6 \subset \mathbf{S'}$ such that, denoting by π: $\mathbf{S} \to \mathbb{P}^2$ and π': $\mathbf{S'} \to \mathbb{P}^2$ the morphisms for which $L_1, L_2, \ldots, L_6$ and $L'_1, L'_2, \ldots, L'_6$ are the exceptional lines respectively, and $P_i = \pi(L_i)$, $P'_i = \pi'(L'_i)$ for $i = 1, 2, \ldots, 6$, an isomorphism $\alpha : \mathbb{P}^2 \to \mathbb{P}^2$ exists such that $\alpha(Pi) = P'_i$.

If $D \subset \mathbf{S}$ is a divisor of type $(a; b_1, b_2, b_3, b_4, b_5, b_6)$, we shall call degree of D the number $d = deg\, D = H \cdot D = 3a - \sum_{i=1}^{6} b_i$; the genus of D is the number $g = \binom{a-1}{2} - \sum_{i=1}^{6} \binom{b_i}{2}$. A simple computation shows that the virtual dimension of D is the number $vdim(D) = d + g$.

Eff $\mathbf{S}$ is finitely generated (cf. [11], Theorem 1), hence the global ring of $\mathbf{S}$ is generated as a k-algebra by the 27 curves of degree one (ibidem); we shall call these generators ℓ_{ij}, e_r, g_s, with $i, j, r, s = 1, 2, \ldots, 6, i < j$: following the notation used in [9] (see for instance V, Theorem 4.9) the 15 variables ℓ_{ij} correspond to the lines L_{ij}, the variables e_r correspond to the lines E_r, and the variables g_s correspond to the lines G_s. Observe that if three of the 27 lines pass through a same point P, i.e. if an "Eckardt point" there exists on $\mathbf{S}$ (see [1] and [2], Appendix 1), then the corresponding variables are linearly dependent. We will consider the general case, in which the surface $\mathbf{S}$ has no Eckardt points and the 27 generators are linearly independent. Anyway the global ring of $\mathbf{S}$ can be seen as a quotient of the ring $k[\ell_{ij}, e_r, g_s]$ with $i, j, r, s = 1, 2, \ldots, 6, i < j$, modulo a homogeneous ideal J. Note that by the definition of degree of divisors on $\mathbf{S}$, the ring $G(\mathbf{S})$ has also a $\mathbb{Z}$-grading in which all the indeterminates have degree one. We will see (Theorem 2.5) that the ideal J is generated by elements of degree two in this grading.

Recall that the conics are the only integral curves on $\mathbf{S}$ which move in a pencil; to each of the 27 lines there corresponds a pencil of conics on $\mathbf{S}$. Look, for instance, at the conics of type (1; 1, 0, 0, 0, 0, 0). There are 5 monomials of this type, namely $\ell_{1j}e_j$, $j = 2, 3, \ldots, 6$; but of course only two of them are linearly independent in the k-vector space they generate. So there are three relations among them; one can see this explicitly, by taking the blow up morphism $\pi\colon \mathbf{S} \to \mathbb{P}^2$ such that $\pi(E_i) = P_i$, $i = 1, 2, \ldots, 6$. Denoting (with some abuse of notation) by L_{ij} the plane line passing through P_i and P_j, clearly $L_{14} = aL_{12} + bL_{13}$ for some $a, b \in k$, and looking at the total inverse image of this line one gets: $e_1\ell_{14}e_4 = ae_1\ell_{12}e_2 + be_1\ell_{13}e_3$; taking away the common factor e_1 one obtains one of the three relations. The same thing happens for all the 27 pencils of conics of $\mathbf{S}$: this can be seen using the Weyl group "E6" on the configuration of the 27 lines ([9], V, Ex. 4.11). So we get 81 elements of degree two in the 27 variables we have chosen, which vanish on $\mathbf{S}$: call I the ideal they generate in the polynomial ring $k[\ell_{ij}, e_r, g_s]$. We shall see that I defines the k-algebra $G(\mathbf{S})$, i.e. that $I = J$. Note that the vector subspaces of $k[\ell_{ij}, e_r, g_s]$ and of $G(\mathbf{S})$ having $\mathbb{Z}$-degree 1 are isomorphic, so that we can identify $k\ell_{ij}$ and $H^0(L_{ij})$, ke_r and $H^0(E_r)$, kg_s and $H^0(G_s)$.

Recall that a curve on $\mathbf{S}$, say D of type $(a; b_1, b_2, b_3, b_4, b_5, b_6)$ with $b_1 \geq b_2 \geq \ldots \geq b_6$, is said to be of type $(*)$ if it is integral (i.e. linearly equivalent to an integral curve): this happens exactly when D is zero, a line, a conic, or, when $deg\, D > 2$, the following conditions hold:

$$1)\, a > 0, \quad 2)\, b_6 \geq 0, \quad 3)\, a - b_1 - b_2 \geq 0,$$

$$4)\, 2a - b_1 - b_2 - \ldots - b_5 \geq 0, \quad 5)\, D^2 = a^2 - \sum_{i=1}^{6} b_i^2 > 0$$

Moreover a divisor D free from fixed components is composed with a pencil of conics if and only if it satisfies conditions 1), 2), 3), 4) above, and $D^2 = 0$.

In order to prove that $I = J$ it will be enough to consider classes of curves free from fixed components, i.e. composed with a pencil or of type $(*)$.

Proposition 2.1. *Let $D \subset \mathbf{S}$ be an integral cubic curve. Suppose that in $k[\ell_{ij}, e_r, g_s]$ there exist n monomials belonging to $H^0(D)$ and $n > h^0(D) = m$. Then the $n - m$ relations among these monomials belong to I (the ideal generated by the 81 conics).*

Proof. First of all consider the curves D of type $(1; 0, 0, 0, 0, 0, 0)$; in $H^0(D)$ there are 15 monomials, $\ell_{ij}e_ie_j$, $i, j = 1, 2, \ldots, 6$, $i < j$. Since $h^0(D) = 3$ we have 12 relations among them; we have to show that they are in I.

Calling $\Gamma_1, \ldots, \Gamma_6$ the conics of type $(1; 1, 0, 0, 0, 0, 0), \ldots, (1; 0, 0, 0, 0, 0, 1)$, we can choose a base for each vector space $H^0(\Gamma_i)$, $i = 1, \ldots, 6$, $H^0(D)$ using respectively monomials in $k[\ell_{ij}, e_r, g_s]_{[\Gamma_i]}$, $k[\ell_{ij}, e_r, g_s]_{[D]}$ and call them $\mathcal{B}_i$ $(i = 1, \dots, 6)$, $\mathcal{B}$. Now we can identify $H^0(\Gamma_i) =< \mathcal{B}_i >$, $H^0(D) =< \mathcal{B} >$ so we can write

$$k[\ell_{ij}, e_r, g_s]_{[\Gamma_i]} =< \mathcal{B}_i > \oplus K_i, \quad k[\ell_{ij}, e_r, g_s]_{[D]} =< \mathcal{B} > \oplus K$$

Consider the following commutative diagram in which φ', φ are the multiplication maps and the vertical maps are the projections onto the first component:

$$\begin{tikzcd}
\bigoplus_{i=1}^{6} \big(< \mathcal{B}_i > \oplus K_i \big) \otimes k e_i \arrow[r, "\varphi'"] \arrow[d, "(\oplus p_{1i})\otimes 1"] & < \mathcal{B} > \oplus K \arrow[r] \arrow[d, "p_1"] & 0 \\
\bigoplus_{i=1}^{6} H^0(\Gamma_i) \otimes H^0(E_i) \arrow[r, "\varphi"] & H^0(D) \arrow[r] & 0
\end{tikzcd}$$

An element $\sum (M_i \otimes a_i E_i) \in ker\ \varphi$ is image of $\sum (m_i, 0) \otimes a_i e_i \in ker\ \varphi'$ by the above identification, so there is a surjection $ker\ \varphi' \to ker\ \varphi$. By the snake lemma we get a surjection $ker\ (\oplus p_{1i}) \otimes 1 \to ker\ p_1$ so that any element in $ker\ p_1$ is generated by elements in $ker\ (\oplus p_{1i}) \subset I$.

The above proof holds for all the other curves of degree 3 and genus 0 again by the action of the Weyl group.

Let now D be a curve of degree 3 and genus 1; they are all of type (3;1,1,1,1,1,1) and there are 45 monomials in $H^0(D)$, fifteen given by $\ell_{ij}\ell_{pq}\ell_{rs}$ with $\{i, j, p, q, r, s\} = \{1, 2, 3, 4, 5, 6\}$ and the remaining by $\ell_{ij} e_i g_j$, $\ell_{ij} e_j g_i$.

Call $\Gamma_{ij} = H - L_{ij}$ the conics which are complementary with the 15 lines ℓ_{ij} and choose a base for each vector space $H^0(\Gamma_{ij})$ $(i = 1, \dots, 6, i < j)$, $H^0(D)$ using respectively monomials in $k[\ell_{ij}, e_r, g_s]_{[\Gamma_{ij}]}$, $k[\ell_{ij}, e_r, g_s]_{[D]}$; now, using the same arguments as for rational cubics, one can construct a similar diagram and conclude in the same way. $\square$

Now we want to prove that relations of any degree behave exactly as those of degree three, i.e. they are defined by polynomials in I; we shall use induction on the degree, but we need first some technical results.

Lemma 2.2. *Let $D \subset \mathbf{S}$ be a curve; if there exists a conic $\Gamma \subset \mathbf{S}$ such that $H^1(D - \Gamma) = 0$ then the cup product morphism $H^0(D) \otimes H^0(\Gamma) \to H^0(D + \Gamma)$ is surjective.*

Proof. Applying R-R to $D - \Gamma$ one gets:

$$h^0(D-\Gamma) = vdim(D-\Gamma) + h^1(D-\Gamma) - h^0(-D+\Gamma-H) = d + g - D\cdot\Gamma - 1$$

where d, g are degree and genus of D . Here we used the hypothesis $h^1(D - \Gamma) = 0$, the formula for $vdim(D - \Gamma)$ and the fact that $-D - (H - \Gamma)$ is not effective. Similarly, for the divisor D one has $h^0(D) = d + g + h^1(D)$, and since the conic Γ can give at most $D \cdot \Gamma + 1$ conditions, one gets $h^0(D) - h^0(D - \Gamma) = h^1(D) + D \cdot \Gamma + 1 \leq D \cdot \Gamma + 1$ so that $h^1(D) = 0$. The same argument gives $h^0(D + \Gamma) = d + g + D \cdot \Gamma + 1$.

Now choose a base $\{D_1, \dots, D_r\}$ of $H^0(D - \Gamma)$, a base $\{\Gamma_1, \Gamma_2\}$ of $H^0(\Gamma)$, and extend $\{D_1\Gamma_1, \dots, D_r\Gamma_1\}$ to a base for $H^0(D)$: $\{D_1\Gamma_1, \dots, D_r\Gamma_1, D_{r+1}, \dots, D_{d+g}\}$; now we can consider the following elements:

$$D_1\Gamma_1^2, D_2\Gamma_1^2, \dots, D_r\Gamma_1^2, D_{r+1}\Gamma_1, \dots, D_{d+g}\Gamma_1$$

$$D_{r+1}\Gamma_2, \dots, D_{d+g}\Gamma_2$$

One sees that they are independent elements of $H^0(D + \Gamma)$, in number of $d + g + D \cdot \Gamma + 1$ and this is exactly the dimension of $H^0(D + \Gamma)$. $\square$

Lemma 2.3. *If $D \subset \mathbf{S}$ is a curve of type $(*)$ with $d = degD > 2$ then $D \cdot \Gamma > 0$ for any conic $\Gamma \subset \mathbf{S}$.*

Proof. The assertion is trivial if D is a plane curve, i.e. if D is linearly equivalent to H. Let $\Gamma \subset \mathbf{S}$ be any conic, and let $L \subset \mathbf{S}$ be the complementary line, i.e. the line such that $\Gamma + L$ is linearly equivalent to H. Assuming that $D \cdot \Gamma \leq 0$ we have

$$d = D \cdot H = D \cdot (\Gamma + L) = D \cdot \Gamma + D \cdot L \leq D \cdot L$$

a contradiction, because D is not a plane curve so it cannot have a d-secant line. $\square$

Proposition 2.4. *Let $D \subset \mathbf{S}$ be a curve free from fixed components having degree $d \geq 5$. Then there exists a conic Γ such that $vdim(D - 2\Gamma) \geq 0$ (i.e. $d + g \geq 2D \cdot \Gamma + 2$).*

Proof. If D is composed with a pencil, $D = r\Gamma$ $(r > 2)$ the assertion is trivially true. Hence we assume that D is of type $(*)$. Notice that $vdim(D - 2\Gamma) = vdim(D) - 2D \cdot \Gamma - 2 = g + d - 2D \cdot \Gamma - 2$, where g is the genus of D. We use induction on the degree d of D. For $d = 5$ the lemma can be directly proved by considering all the curves of type $(*)$ on $\mathbf{S}$ of degree 5.

By a base change in $Pic(\mathbf{S})$ we can assume that the conic Γ of type $(1; 1, 0, 0, 0, 0, 0)$ is a minimal secant for D, and that $(a; b_1, b_2, b_3, b_4, b_5, b_6)$, with $b_1 \geq b_2 \geq \dots \geq b_6 \geq 0$, is the type of D after this base change (it is enough to choose the line G_1 as a maximal secant and $G_2, \dots, G_6$ such that

$D \cdot G_1 \geq D \cdot G_2 \geq \ldots \geq D \cdot G_6$). Hence, by the minimality of $D \cdot \Gamma$, a direct computation with the other two conics which might be minimal secants for D gives:

$$a - b_1 \leq 2a - b_1 - b_2 - b_3 - b_4$$

$$a - b_1 \leq 3a - 2b_1 - b_2 - b_3 - b_4 - b_5 - b_6$$

The case when $b_1 = 0$ gives no problem: $d = 3a, D \cdot \Gamma = a, g = (1/2)(a-1)(a-2)$, hence (recall that $a \geq 2$):

$$vdim(D - 2\Gamma) = \frac{1}{2}(a-1)(a-2) + a - 2 \geq 0$$

When $b_1 > 0$ we consider three cases.
1^{st} *case*. Suppose that

$$\begin{cases} a > b_1 + b_2 \\ 2a > b_1 + b_2 + b_3 + b_4 + b_5 \end{cases}$$

Consider the curve D' of type $(a; b_1 + 1, b_2, b_3, b_4, b_5, b_6)$ having degree $d' = d - 1$ and genus $g' = g - b_1$. This curve is still free from fixed components, hence by the inductive hypothesis, denoting by Γ' a conic minimal secant for D', we have

$$vdim(D' - 2\Gamma') = g' + d' - 2D' \cdot \Gamma' - 2 \geq 0$$

Now, since $D' = D - E_1$, we have $D' \cdot \Gamma' \leq D' \cdot \Gamma = D \cdot \Gamma - E_1 \cdot \Gamma = D \cdot \Gamma - 1$ and $D' \cdot \Gamma' = D \cdot \Gamma' \geq D \cdot \Gamma - E_1 \cdot \Gamma'$, hence either $D' \cdot \Gamma' = D \cdot \Gamma - 1$ or $D' \cdot \Gamma' = D \cdot \Gamma - 2$. In the first case a simple substitution shows that $vdim(D - 2\Gamma) \geq 0$ because $b_1 > 0$; the second case can take place only when $2a = b_1 + b_2 + b_3 + b_4 + b_5 + b_6$. In this situation, recalling that $d > 5$, we can have $b_1 < 3$ only if D is of type: $(6; 2, 2, 2, 2, 2, 2)$, and for this curve $vdim(D - 2\Gamma) = 0$. If $b_1 \geq 3$ a simple substitution gives the result.

2^{nd} *case*. Suppose that

$$\begin{cases} a = b_1 + b_2 \\ 2a > b_1 + b_2 + b_3 + b_4 + b_5 \end{cases}$$

Notice that we only need to consider the case $a > b_1 + b_5$. In fact, if $a = b_1 + b_5$ then D is of type $(a; b_1, a - b_1, a - b_1, a - b_1, a - b_1, b_6)$; since we know that

$a - b_1 \leq 3a - 2b_1 - 4(a - b_1) - b_6$, i.e. that $2a \leq 3b_1 - b_6$, we can directly compute $vdim(D - 2\Gamma)$:

$$vdim(D - 2\Gamma) = g - 3a + 5b_1 - b_6 - 2 \geq g + 2b_1 - a - 2 \geq 0$$

because $2b_1 > a + 1$, as one can check.

Consider the curve D' obtained from D increasing by 1 the first b_i $(i = 3, 4, 5)$ such that $b_1 + b_i < a$. First suppose $i = 3$, so that D' is the curve of type $(a; b_1, b_2, b_3 + 1, b_4, b_5, b_6)$ of degree $d' = d - 1$ and genus $g' = g - b_3$. If $b_3 > 0$ we can repeat the same argument as in the first case because either $D' \cdot \Gamma' = D \cdot \Gamma$ or $D' \cdot \Gamma' = D \cdot \Gamma - 1$; if $b_3 = 0$ we can directly work on the curve D. In fact, assuming that $b_3 = 0$, D is of type $(a; b_1, a - b_1, 0, 0, 0, 0)$ so that

$$vdim(D - 2\Gamma) = g + 2a - 2a + 2b_1 - 2 = g + 2b_1 - 2 \geq 0$$

If $i = 4$ and $b_4 \neq 0$ we repeat the same argument, while when $b_4 = 0$ again we give a direct computation: in this case D is of type $(a; b_1, a - b_1, a - b_1, 0, 0, 0)$ with $b_1 > 1$ (otherwise $a - b_1 = 1$ and $a = 2$: this leads to $d = 3$). Now a direct computation gives

$$vdim(D - 2\Gamma) = g + a + b_1 - 2a + 2b_1 - 2 = g + 3b_1 - a - 2 \geq 0$$

Finally, if $i = 5$ we are reduced to the case $b_5 = 0$, so D is of type $(a; b_1, a - b_1, a - b_1, a - b_1, 0, 0)$; again a direct computation gives

$$vdim(D - 2\Gamma) = g + 2(2b_1 - a - 1)$$

Now, since $a - b_1 \leq 2a - b_1 - 3(a - b_1)$ (Γ is a minimal secant of D), we have $b_1 \geq 2a/3$, hence

$$2b_1 - a - 1 \geq \frac{a}{3} - 1 \geq 0$$

because $a \geq 3$ (if $a \leq 2$ then $d < 5$).

3^{rd} *case*. Suppose that $2a = b_1 + b_2 + b_3 + b_4 + b_5$. This can happen only when $b_6 = 0$. In fact, since $a - b_1 \leq 3a - 2b_1 - b_2 - b_3 - b_4 - b_5 - b_6$, we have $2a \geq b_1 + b_2 + b_3 + b_4 + b_5 + b_6$.

Now, from the inequality $a - b_1 \leq 2a - b_1 - b_2 - b_3 - b_4 = b_5$, we see that $a = b_1 + b_5$, so D is of type $(a; b_1, a - b_1, a - b_1, a - b_1, a - b_1, 0)$. Hence $b_1 = 2a/3$, with $a = 3n$, $n \geq 2$, and a direct computation gives

$$vdim(D - 2\Gamma) = g + \frac{a}{3} - 2 \geq 0. \qquad \square$$

Theorem 2.5. *Let $C \subset \mathbf{S}$ be a curve of type $(*)$, having degree $d \geq 4$. If C has not one of the following types ($n > 1$):*

(a) $(n; 0, 0, 0, 0, 0, 0)$;
(b) $(2n; n, n, n, 0, 0, 0)$;
(c) $(3n; 2n, n, n, n, n, 0)$;
(d) $(4n; 2n, 2n, 2n, n, n, n)$;
(e) $(5n; 2n, 2n, 2n, 2n, 2n, 2n)$;
(f) $(3; 1, 1, 1, 1, 1, 0)$, $(4; 2, 2, 1, 1, 1, 1)$, $(5; 2, 2, 2, 2, 2, 1)$;
(g) $(6; 2, 2, 2, 2, 2, 2)$

then there exists a conic $\Gamma \subset \mathbf{S}$ such that the cup product morphism

$$H^0(C - \Gamma) \otimes H^0(\Gamma) \to H^0(C)$$

is surjective.

Proof. If $C - 2\Gamma$ is not effective for any conic Γ, then $0 = vdim(C - 2\Gamma) + h^1(C - 2\Gamma)$ and Proposition 2.4 implies $h^1(C - 2\Gamma) = vdim(C - 2\Gamma) = 0$. Moreover $vdim(C - 2\Gamma) = d + g - 2C \cdot \Gamma - 2 = 0$, so $d + g = 2C \cdot \Gamma + 2$ and one has $vdim(C - \Gamma) = d + g - C \cdot \Gamma - 1 = C \cdot \Gamma + 1 > 0$ by Lemma 2.3. Hence $C - \Gamma$ is effective and $H^1(C - 2\Gamma) = 0$: the result follows applying Lemma 2.2.

If $C - 2\Gamma$ is effective for some conic Γ, then $C - \Gamma$ is too; now we want to discard those divisors such that $H^1(C - 2\Gamma) \neq 0$; of course we shall find the types listed in the theorem.

We will choose Γ such that $C \cdot \Gamma$ is minimum; according with the various types of conics one sees that this minimum can be attained by $\Gamma_1 = (1; 1, 0, 0, 0, 0, 0)$, $\Gamma_2 = (2; 1, 1, 1, 1, 0, 0)$, $\Gamma_3 = (3; 2, 1, 1, 1, 1, 1)$, since, as usual, we consider divisors C of type $(a; b_1, b_2, b_3, b_4, b_5, b_6)$ with $b_1 \geq b_2 \geq \ldots \geq b_6$. We have three cases.

I) Γ_1 is a minimal secant, i.e. $C \cdot \Gamma_3 \geq C \cdot \Gamma_1$ and $C \cdot \Gamma_2 \geq C \cdot \Gamma_1$. In this case $C - 2\Gamma_1$ is of type $(a - 2; b_1 - 2, b_2, b_3, b_4, b_5, b_6)$, so the following inequalities must be satisfied:

$$(I*) \qquad \begin{cases} 2a \geq \sum_{i=1}^{6} b_i \\ a \geq b_2 + b_3 + b_4 \end{cases}$$

E_1 is a double fixed line of $C - 2\Gamma_1$ if and only if $b_1 = 0$, i.e. if C has the type of (a). L_{23} is a double fixed line of $C - 2\Gamma_1$ if and only if $a = b_2 + b_3$. In this case by $(I*)$ we must have $b_4 = 0$; moreover $a \geq b_1 + b_2 \geq b_1 + b_3 \geq b_2 + b_3 = a$ implies that C has the type of (b).

G_6 is a double fixed line of $C - 2\Gamma_1$ if and only if $2a = b_1 + b_2 + b_3 + b_4 + b_5 \leq b_1 + b_5 + a$, so $a = b_1 + b_5$ and $b_2 = b_3 = b_4 = b_5$. C must have the type of (c).

G_1 is a double fixed line for $C - 2\Gamma_1$ if and only if $2a - 2 = b_2 + b_3 + \ldots + b_6$; using $(I*)$ one sees that $b_1 \leq 2$ and considering all the possibilities one proves that C must have one of the following types: $(6; 2, 2, 2, 2, 2, 2)$, $(4; 2, 2, 1, 1, 1, 1)$, $(3; 1, 1, 1, 1, 1, 0)$ and these are listed in (f), (g).

II) Γ_3 is a minimal secant, i.e. $C \cdot \Gamma_1 \geq C \cdot \Gamma_3$ and $C \cdot \Gamma_2 \geq C \cdot \Gamma_3$. In this case $C - 2\Gamma_3$ is of type $(a - 6; b_1 - 4, b_2 - 2, b_3 - 2, b_4 - 2, b_5 - 2, b_6 - 2)$ and the following inequalities must hold:

$$(II*) \qquad \begin{cases} 2a \leq \sum_{i=1}^{6} b_i \\ a \leq b_1 + b_5 + b_6 \end{cases}$$

E_1 is a double fixed line for $C - 2\Gamma_3$ if and only if $b_1 = 2$. By $(II*)$ one gets $a \leq 6$ and one can check that the only possibilities are curves of type $(4; 2, 2, 1, 1, 1, 1)$, $(5; 2, 2, 2, 2, 2, 1)$, $(6; 2, 2, 2, 2, 2, 2)$ which are listed in (f) and (g).

E_6 is a double fixed line of $C - 2\Gamma_3$ if and only if $b_6 = 0$; $(II*)$ gives $a = b_1 + b_5$, thus $b_2 = b_3 = b_4 = b_5$ and after some computation one finds again (c). Moreover, one can see that $b_i = 0$ for $i = 2, 3, 4, 5$, implies that C is not of type $(*)$, so that no line E_i, $i = 2, \ldots, 5$, can be a double fixed line for $C - 2\Gamma_3$.

L_{23} is a double fixed line of $C - 2\Gamma_3$ if and only if $a = b_2 + b_3$; again $(II*)$ gives $b_1 = b_2 = b_3 = a/2$; hence $b_5 + b_6 \geq a/2$. Now, we have $2a \geq b_1 + b_2 + b_3 + b_4 + b_5 \geq b_1 + b_2 + b_3 + b_4 + b_6 \geq b_1 + b_2 + b_3 + b_5 + b_6$, and since $b_1 + b_2 + b_3 = 3a/2$, one gets $b_4 = b_5 = b_6 = a/4$; we obtain (d).

G_1 is a double fixed line of $C - 2\Gamma_3$ if and only if $2a = b_2 + b_3 + \ldots + b_6$. Now, $2a = b_2 + b_3 + b_4 + b_5 + b_6 \leq b_1 + b_3 + b_4 + b_5 + b_6 \leq \ldots \leq b_1 + b_2 + b_3 + b_4 + b_5 \leq 2a$, hence we have $b_1 = b_2 = b_3 = b_4 = b_5 = b_6$ and we obtain a curve of type (e).

III) Γ_2 is a minimal secant, i.e. $C \cdot \Gamma_1 \geq C \cdot \Gamma_2$ and $C \cdot \Gamma_3 \geq C \cdot \Gamma_2$. In this case $C - 2\Gamma_2$ is of type $(a - 4; b_1 - 2, b_2 - 2, b_3 - 2, b_4 - 2, b_5, b_6)$ with the inequalities

$$(III*) \qquad \begin{cases} a \leq b_2 + b_3 + b_4 \\ a \geq b_1 + b_5 + b_6 \end{cases}$$

E_4 is a double fixed line of $C - 2\Gamma_2$ if and only if $b_4 = 0$. By $(III*)$ one gets $a = b_2 + b_3$ and $b_4 = b_5 = b_6$; we again obtain (b). If G_1 would be a double fixed line of $C - 2\Gamma_2$, then $2a = b_2 + \ldots + b_6$, which implies $b_2 = \ldots = b_6$ and this contradicts $(III*)$. Analogous computations show that G_2 is a double fixed line of $C - 2\Gamma_2$ if and only if $C = b(3; 2, 1, 1, 1, 1, 1)$, i.e. C is not of type $(*)$; for G_3 one finds $C = a(1; 1, 1, 0, 0, 0, 0)$, a contradiction. G_4 is a double fixed line of $C - 2\Gamma_2$ if and only if $2a = b_1 + b_2 + b_3 + b_5 + b_6$. Since $2a = b_1+b_2+b_3+b_5+b_6 \leq b_1+b_2+b_3+b_4+b_6 \leq b_1+b_2+b_3+b_4+b_5$ one gets $b_6 = b_5 = b_4 =: b$. By $(III*)$ $b_1 + 2b \leq a \leq b_2 + b_3 + b$ whence $2a = b_1 + b_2 + b_3 + 2b \leq a + b_2 + b_3$, so $a \leq b_2 + b_3$ and $a = b_2 + b_3 \leq b_1 + b_3 \leq b_1 + b_2 \leq a$. Hence $b_1 = b_2 = b_3 =: \overline{b}$; $a = 2\overline{b}$ and $2a = 3\overline{b} + 2b = 4\overline{b}$ so that $2b = \overline{b}$. C must be of type $b(4; 2, 2, 2, 1, 1, 1)$, which is (d).

L_{45} is a double fixed line of $C - 2\Gamma_2$ if and only if $a = b_4 + b_5$ and this implies $b_1 = b_2 = b_3 = b_4 = b_5 =: b$, so that $a = 2b$ and this is a contradiction with C of type $(*)$. Finally L_{56} is a double fixed line of $C - 2\Gamma_2$ if and only if $a = b_5 + b_6 + 2$. $(III*)$ implies $b_1 \leq 2$. By checking all the possibilities one finds that C must be of type (a) or (f) or (g). $\square$

Recall that for any pencil of conics on $\mathbf{S}$ there are 5 degree two monomials of that type in the 27 variables; one can use two of them to express the remaining three. In this way we have found 81 degree two elements in $k[\ell_{ij}, e_r, g_s]$ which vanish in $G(\mathbf{S})$.

Theorem 2.6. *The ideal I of $k[\ell_{ij}, e_r, g_s]$ generated by the 81 above elements defines the ring $G(\mathbf{S})$, i.e. $I = J$, so that $G(\mathbf{S}) \cong k[\ell_{ij}, e_r, g_s]/I$.*

Proof. We must prove that $I = J$. Fix a curve $D \subset \mathbf{S}$ free from fixed components. Suppose that D is composed with a pencil of conics, $D = r\Gamma$. We know that $H^0(\Gamma) =< x, y >$, x, y a base, hence $H^0(r\Gamma)$ has base $x^r, x^{r-1}y, \ldots, y^r$. In $H^0(\Gamma)$ we have five monomials of $k[\ell_{ij}, e_r, g_s]$, so there are three generators of I in $H^0(\Gamma)$. $H^0(r\Gamma)$ has dimension $r + 1$ and contains $\binom{4+r}{4}$ monomials of $k[\ell_{ij}, e_r, g_s]$, so in J there are $\binom{4+r}{4} - (r+1)$ relations. The same number of independent relations can be constructed starting by the three relations of $H^0(\Gamma)$ belonging to I.

If D is of type $(*)$ and $deg\ D = 3$ then the result follows from Proposition 2.1, so suppose that $deg\ D \geq 4$ and that for D Theorem 2.5 holds. Hence there exists a conic Γ such that $H^0(D - \Gamma) \otimes H^0(\Gamma) \to H^0(D)$ is surjective. Choose a base for each vector space $H^0(D - \Gamma)$, $H^0(\Gamma)$, $H^0(D)$ using respectively monomials in $k[\ell_{ij}, e_r, g_s]_{[D-\Gamma]}$, $k[\ell_{ij}, e_r, g_s]_{[\Gamma]}$, $k[\ell_{ij}, e_r, g_s]_{[D]}$, and call them $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$. Now we can identify $H^0(D - \Gamma) =< \mathcal{B}_1 >$, $H^0(\Gamma) =< \mathcal{B}_2 >$,

$H^0(D) =< \mathcal{B}_3 >$ so we can write

$$k[\ell_{ij}, e_r, g_s]_{[D-\Gamma]} =< \mathcal{B}_1 > \oplus K_1,$$
$$k[\ell_{ij}, e_r, g_s]_{[\Gamma]} =< \mathcal{B}_2 > \oplus K_2,$$
$$k[\ell_{ij}, e_r, g_s]_{[D]} =< \mathcal{B}_3 > \oplus K_3$$

Consider the commutative diagram

$$\begin{tikzcd}
(< \mathcal{B}_1 > \oplus K_1) \otimes (< \mathcal{B}_2 > \oplus K_2) \arrow[r, "\varphi'"] \arrow[d, "p\otimes p'"] & < \mathcal{B}_3 > \oplus K_3 \arrow[r] \arrow[d, "p''"] & 0 \\
H^0(D-\Gamma) \otimes H^0(\Gamma) \arrow[r, "\varphi"] & H^0(D) \arrow[r] & 0
\end{tikzcd}$$

where φ, φ' are given by multiplication, the vertical maps are the projections onto the first component. Now an element $\sum M_i \otimes \Gamma_i \in ker\ \varphi$ is image of $\sum(m_i, 0) \otimes (\gamma_i, 0) \in ker\ \varphi'$ by the above identifications. Hence there exists a surjection $ker\ \varphi' \rightarrow ker\ \varphi$. Using the snake lemma we get a surjection $ker\ p \otimes p' \rightarrow ker\ p''$ so that every element in $ker\ p''$ is generated by elements in $ker\ p' \subset I$.

We have to prove now the theorem for the curves of types $(a) - (g)$ listed in Theorem 2.5. As to the multiples of rational cubics (types $(a) - (e)$), we can consider the cubic $\Delta \subset \mathbf{S}$ of type $(1; 0, \ldots, 0)$ and observe that the map $H^0((n-1)\Delta) \otimes H^0(\Delta) \rightarrow H^0(n\Delta)$ is surjective. Since by Proposition 2.1 we know that $I_{[\Delta]} = J_{[\Delta]}$ one can construct as above a commutative diagram which gives the result.

For the quartics of genus 1 listed in (f), consider for instance the quartic Δ of type $(3; 1, 1, 1, 1, 1, 0)$. Denoting by $\Gamma_1, \ldots, \Gamma_5$ the conics of type $(1; 1, \ldots, 0), \ldots, (1; 0, \ldots, 1, 0)$ and by $\Gamma'_1, \ldots, \Gamma'_5$ the conics of type $(2; 0, 1, 1, 1, 1, 0), \ldots, (2; 1, 1, 1, 1, 0, 0)$ one has the surjection

$$\bigoplus_{i=1}^{5} H^0(\Gamma_i) \otimes H^0(\Gamma'_i) \rightarrow H^0(\Delta)$$

Now the same commutative diagram as above can be constructed and the result follows.

Finally, for the curves listed in (g), the surjection $H^0(H) \otimes H^0(H) \rightarrow H^0(2H)$ allows to conclude as above. $\square$

3. The Mori quartic.

In this section we assume $char\, k = 0$. Let $\mathbf{M}$ be a $K3$ surface, i.e. a smooth surface having trivial canonical sheaf and zero irregularity. We assume that $\mathbf{M}$ has a very ample divisor, H, such that $h^0(H) = 4$ and $H^2 = 4$. Moreover we assume that $Pic(\mathbf{M}) \cong \mathbb{Z}^2$ is generated by the classes of H and L, where L is an integral curve such that $H \cdot L = 1$ and $L^2 = -2$. If for any divisor $D \subset \mathbf{M}$ we call degree of D the number $deg\, D = H \cdot D$, then we can say that L is a line since it has degree one and genus zero. All these properties can be summarized by saying that $\mathbf{M}$ is a *Mori quartic of type (1,0)* (see [10] for the Mori's construction, [3] and [4] for more properties of such surfaces). In fact, $\mathbf{M}$ can be embedded in $\mathbb{P}^3$ as a quartic surface and in this embedding the generators of $Pic(\mathbf{M})$ are the class of a plane section H and the class of the line L, i.e. the class of an integral curve of degree 1 and genus 0. Here we want to describe the global ring $G(\mathbf{M})$ of $\mathbf{M}$. We start by giving a description of the curves (i.e. of the effective divisors) on $\mathbf{M}$. Recalling that L is the only isolated curve of $\mathbf{M}$ (see [4], Proposition 4.3), any divisor $D \subset \mathbf{M}$ is linearly equivalent to $aH + bL$, with $a, b \in \mathbb{Z}$. In this case we say that D is of type (a, b), so D has degree $deg\, D = 4a + b$, and genus $g = 2a^2 + ab - b^2 + 1$. Now, applying the algorithm explained in [2], Remark 3.3, it is easy to see that the curves of $\mathbf{M}$ are the following:

(1) curves of type $(a, -b)$ with $a \geq b \geq 0$;
(2) curves of type (a, b) with $a, b \geq 0$.

The curves (1) always are integral, except for $a = b$: in this case they are composed with a pencil. Namely $bH - bL$ is the union of b curves, each linearly equivalent to $H - L$ hence of degree three and genus one, belonging to the same pencil.

The curves (2) are integral only when $a \geq 2b$. If $a < 2b$ then the Zariski decomposition of these curves is

$$aH + bL = (b - \lfloor \frac{a}{2} \rfloor)L + (aH + \lfloor \frac{a}{2} \rfloor L)$$

where $\lfloor - \rfloor$ means the integer part. In this case $(b - \lfloor a/2 \rfloor)L$ is the *fixed part* of $aH + bL$, and this means that any curve linearly equivalent to $aH + bL$ must contain the line L with multiplicity $b - \lfloor a/2 \rfloor$. $aH + \lfloor a/2 \rfloor L$ is the *moving part* of this curve. If $C \subset \mathbf{M}$ is a curve, we can easily compute the dimensions $h^i(C)$ $(i = 0, 1, 2)$ in the following way (see [3], n. 3 and n. 4). For curves (1) we have

$$h^0(aH - bL) = \begin{cases} 2a^2 - b^2 - ab + 2 = vdim(aH - bL) & \text{if } b < a \\ b + 1 & \text{if } b = a \end{cases}$$

For curves (2) we consider their Zariski decomposition: $C = xL + (yH + zL)$ with $x, y, z \geq 0$; then $h^0(C) = h^0(yH + zL) = 2y^2 - z^2 + yz + 2 = vdim(yH + zL)$. Moreover, a simple application of R-R shows that for curves (2) we have

$$h^1(xL) = x^2 - 1; \quad h^1(xL + (yH + zL)) = x^2 - xL \cdot (yL + zH) = x^2 - xy + 2xz$$

For curves (1) we have $H^1(aH - bL) \neq 0$ only when $a = b$, i.e. when the curve is composed with a pencil, and $h^1(bH - bL) = b - 1$. Of course, for any non-zero curve $C \subset \mathbf{M}$ we have $H^2(C) = 0$. Recall that $Eff\ \mathbf{M}$ denotes the semigroup of effective divisors of $\mathbf{M}$. We want to show that $Eff\ \mathbf{M}$ is finitely generated.

Theorem 3.1. *Let* $\mathbf{M}$ *be a Mori quartic of type (1,0). Then* $Eff\ \mathbf{M}$ *is generated by the following elements: 1 section of* $H^0(L)$*, 2 sections of* $H^0(H - L)$*, 2 sections of* $H^0(H)$*, 1 section of* $H^0(2H + L)$*. In particular,* $Eff\ \mathbf{M}$ *is generated in degree* ≤ 9.

Proof. Let us choose the generators of $Eff\ \mathbf{M}$ among the curves of $\mathbf{M}$ of low degree.

$d = 1$ We have only one curve, L, of type $(0, 1)$. Let $x \in H^0(L)$.

$d = 2$ We have only 2 times L, and the corresponding section is x^2. In the sequel we shall omit such curves.

$d = 3$ We have only curves of type $(1, -1)$. Let $y_1, y_2 \in H^0(H - L)$.

$d = 4$ We have only curves of type $(1, 0)$. Let $z_1, z_2 \in H^0(H)$ such that xy_1, xy_2, z_1, z_2 are a base of $H^0(H)$.

$d = 5$ We have only curves of type $(1, 1)$. All these curves have L as fixed part, hence every section of $H^0(H + L)$ is obtained multiplying by x the sections of $H^0(H)$.

$d = 6$ We have curves of types $(1, 2)$ and $(2, -2)$. For the first curves we repeat the same argument as for $(1, 1)$; for the second observe that any section of $H^0(2H - 2L)$ is given by a quadratic form of y_1, y_2.

$d = 7$ Disregarding curves of type $(1, 3)$ (for which we can repeat the same argument as before), we have only curves of type $(2, -1)$. For these curves observe that the cup-product morphism

$$H^0(H - L) \otimes H^0(H) \to H^0(2H - L)$$

is surjective (see [4], Theorem 2.2) because $H^1(-L) \cong H^1(L) = 0$ and $H^1(H - L) = 0$. Hence the sections $xy_1y_2, xy_1^2, xy_2^2, y_1z_1, y_2z_1, y_1z_2, y_2z_2$ give a base of $H^0(2H - L)$.

$d = 8$ Disregarding curves of type (1,4), we have only curves of type (2,0). Here again Theorem 2.2 of [4] shows that there is the surjection

$$H^0(H) \otimes H^0(H) \to H^0(2H)$$

$d = 9$ Essentially we have the curves of type (2,1). $h^0(2H + L) = 11$, but one can check that no cup-product surjects on $H^0(2H + L)$. Using the previous generators we can construct only 10 sections of $H^0(2H + L)$: the sections of $h^0(2H)$ each multiplied by x; so, in order to have a base of $h^0(2H + L)$, we need a new section $t \in H^0(2H + L)$ corresponding to a curve not containing L as a component.

Now the result follows by the following claim.

Claim: $Eff\ \mathbf{M}$ is generated by x, y_1, y_2, z_1, z_2, t.

By induction on the degree. We only need to consider a curve C of degree ≥ 10 free from fixed components. First we consider curves of type $(a, -b)$, with $a > 0, 0 \leq b \leq a$. If $b = 0$ then $a \geq 3$: in this case, again by Theorem 2.2 of [4], we have the surjection

$$H^0((a-1)H) \otimes H^0(H) \to H^0(aH)$$

If $a = b$ then any curve of type $(b, -b)$ is the union of b curves of the same pencil, so the conclusion follows trivially. Consider now $0 < b < a$. We want to show that the cup product

$$\varphi : H^0((a-1)H - (b-1)L) \otimes H^0(H - L) \to H^0(aH - bL)$$

is surjective. Here we have:

$$h^0(aH - bL) = 2a^2 - b^2 - ab + 2, \qquad h^0((a-1)H - (b-1)L) =$$

$$= 2a^2 - b^2 - ab + 2 - 3a + 3b = s$$

Consider $H^0((a-2)H - (b-2)L)$: this space has dimension $r = 2a^2 - b^2 - ab + 2 - 6a + 6b$ (even in the "degenerate case" $a = 3, b = 1$, as a simple computation shows). Let $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be a base of $H^0((a-2)H - (b-2)L)$, and let $\{y_1, y_2\}$ be a base of $H^0(H - L)$. Consider now the following base of $H^0((a-1)H - (b-1)L)$:

$$\alpha_1 y_1, \alpha_2 y_1, \dots, \alpha_r y_1, \beta_{r+1}, \dots, \beta_s$$

and notice that no element of the vector space spanned by $\beta_{r+1}, \dots, \beta_s$ can contain y_1 as a component. Now it is easy to see that the following $s+(s-r) = h^0(aH - bL)$ elements of $Im\,\varphi$ are independent:

$$\alpha_1 y_1^2, \alpha_2 y_1^2, \dots, \alpha_r y_1^2, \beta_{r+1} y_1, \dots, \beta_s y_1,$$

$$\beta_{r+1} y_2, \dots, \beta_s y_2$$

Now consider curves of type (a, b), with $a \geq 2b \geq 0$. The case $b = 0$ has already been considered, so we assume $b > 0$. First we consider curves for which $a > 2b$. In this case, again by Theorem 2.2 of [4], we have the surjection

$$H^0((a-1)H + bL) \otimes H^0(H) \to H^0(aH + bL)$$

because $H^1((a-1)H+bL) = 0$, and $H^1((a-2)H+bL) = 0$: the first equality is trivial, and the second is not trivial only when $a - 1 = 2b$. In this case the Zariski decomposition of $(2b-1)H + bL$ is

$$(2b-1)H + bL = L + ((2b-1)H + (b-1)L)$$

and one can check that the assertion is true because $L\cdot((2b-1)H+(b-1)L) = 1$ (apply the formula for h^1). Finally, suppose $a = 2b$, i.e. consider curves of type $(2b, b)$, with $b > 1$. We have just seen that the cup-product morphism

$$\psi : H^0((2b-1)H + (b-1)L) \otimes H^0(H) \to H^0(2bH + (b-1)L)$$

is surjective. In this way, multiplying by x a base of $Im\,\psi$, we obtain $h^0(2bH + (b-1)L) = 9b^2 + 1$ independent elements of $H^0(2bH + bL)$. Since $h^0(2bH + bL) = 9b^2 + 2$, we only need one more section which is independent from the previous ones. This section is given by t^b. $\square$

A consequence of the above theorem is that the global ring of $\mathbf{M}$ is the bigraded ring

$$G(\mathbf{M}) = k[x, y_1, y_2, z_1, z_2, t]/I$$

where I is a bihomogeneous ideal. Now we look for the relations among the variables x, y_1, y_2, z_1, z_2, t, in $G(\mathbf{M})$, i.e. for the generators of I. Notice that, as in the previous section, the definition of degree of a divisor on $\mathbf{M}$ gives a $\mathbb{Z}$-grading to $G(\mathbf{M})$; in this grading our variables have degree 1, 3, 3, 4, 4, 9 respectively.

Going on in writing explicitly the monomials for curves in $\mathbf{M}$ of low degree, as done in the proof of Theorem 3.1, we find two relations in degree

12, among the monomials of type (3,0). In fact $h^0(3H) = 20$, while we find the following 22 monomials

$$x^3y_1^3,\, x^3y_1^2y_2,\, x^3y_1y_2^2,\, x^3y_2^3;$$

$$x^2y_1^2z_1,\, x^2y_1y_2z_1,\, x^2y_2^2z_1,\, x^2y_1^2z_2,\, x^2y_1y_2z_2,\, x^2y_2^2z_2;$$

$$xy_1z_1^2,\, xy_2z_1^2,\, xy_1z_1z_2,\, xy_2z_1z_2,\, xy_1z_2^2,\, xy_2z_2^2;$$

$$z_1^3,\, z_1^2z_2,\, z_1z_2^2,\, z_2^3;$$

$$y_1t,\, y_2t.$$

Now, the 20 monomials of the first four rows are independent because they are the only 20 different monomials in the image of the cup-product

$$H^0(2H) \otimes H^0(H) \to H^0(3H)$$

which is surjective. This means that both y_1t and y_2t must be linear combination of the previous monomials. So we obtain two generators of I:

$$\sigma_1 = y_1t - F_1 \ \text{ where } \ F_1 = F_1(x, y_1, y_2, z_1, z_2),$$

$$\sigma_2 = y_2t - F_2 \ \text{ where } \ F_2 = F_2(x, y_1, y_2, z_1, z_2).$$

A direct computation shows that there are no new relations among monomials of degree 13 and 14. The same happens in degree 15, but we want to analyse this case in detail because we meet an interesting situation in bidegree $(4, -1)$: we have $h^0(4H - L) = 29$, while we find the following 30 monomials:

$$x^3y_1^4,\, x^3y_1^3y_2,\, x^3y_1^2y_2^2,\, x^3y_1y_2^3,\, x^3y_2^4;$$

$$x^2y_1^3z_1,\, x^2y_1^2y_2z_1,\, x^2y_1y_2^2z_1,\, x^2y_2^3z_1,\, x^2y_1^3z_2,\, x^2y_1^2y_2z_2,\, x^2y_1y_2^2z_2,\, x^2y_2^3z_2;$$

$$xy_1^2z_1^2,\, xy_1y_2z_1^2,\, xy_2^2z_1^2,\, xy_1^2z_1z_2,\, xy_1y_2z_1z_2,\, xy_2^2z_1z_2,\, xy_1^2z_2^2,\, xy_1y_2z_2^2,\, xy_2^2z_2^2;$$

$$y_1z_1^3,\, y_2z_1^3,\, y_1z_1^2z_2,\, y_2z_1^2z_2,\, y_1z_1z_2^2,\, y_2z_1z_2^2,\, y_1z_2^3,\, y_2z_2^3.$$

Here we excluded monomials containing the variable t because of the relations σ_1 and σ_2. Moreover we have in the ideal (σ_1, σ_2) the element $\sigma = y_2\sigma_1 - y_1\sigma_2 = y_1F_2 - y_2F_1$. This is an element of type $(4, -1)$ not containing t, and it is non-zero since otherwise y_i would divide F_i $(i = 1, 2)$, hence the variable t would be expressed by means of the other variables. In particular, we have no new relation in degree 15. Notice that any element of the ideal (σ_1, σ_2) not containing t must be a multiple of σ: in fact σ is the resultant of σ_1, σ_2 with respect to the variable t.

We want to show that the two elements σ_1 and σ_2 generate I. To this end we shall prove that

$$R = k[x, y_1, y_2, z_1, z_2, t]/(\sigma_1, \sigma_2) \cong G(\mathbf{M})$$

Thus, denoting by $R_{(a,b)}$ the bihomogeneous components of type (a, b) of R, we need to show that for any couple (a, b) we have

$$h^0(a, b) = dim_k R_{(a,b)}$$

Of course we are interested only in couples (a, b) such that $aH + bL$ is effective. Moreover, we can only consider couples such that $(a - 3)H + bL$ is effective (recall that σ_1 and σ_2 are of type $(3, 0)$): in the other cases a direct check gives the result.

We need a technical lemma.

Lemma 3.2. *For any integer $n > 0$ we have*

$$n + (n - 1)2 + \ldots + 2(n - 1) + n = \binom{n+2}{3}$$

Proof. By induction on n. It is a simple exercise. □

Theorem 3.3. *Let* $\mathbf{M}$ *be a Mori quartic of type* $(1, 0)$. *Then the global ring of* $\mathbf{M}$ *is the bigraded ring*

$$G(\mathbf{M}) = k[x, y_1, y_2, z_1, z_2, t]/(\sigma_1, \sigma_2)$$

where σ_1, σ_2 are the elements determined above.

Proof. Following the previously exposed argument, we need to determine the dimensions of $R_{(a,b)}$ for any couple (a, b) such that $(a - 3)H + bL$ is effective. Of course it is enough to consider only couples (a, b) such that $aH + bL$ is free from fixed components. For any such bidegree we want to count the number of monomials of R in the variables x, y_1, y_2, z_1, z_2, t having bidegree (a, b). Since $R_{(a,b)}$ is a quotient of $k[x, y_1, y_2, z_1, z_2, t]_{(a,b)}$, we have to take account of the relations in $(\sigma_1, \sigma_2)_{(a,b)}$.

Notice that we have the isomorphism of k-vector spaces

$$R \cong S \oplus \bigoplus_{n>0} S' t^n$$

where $S = k[x, y_1, y_2, z_1, z_2]/(\sigma)$ and $S' = k[x, z_1, z_2]$. In fact any monomial of R containing $y_1^{r_1} y_2^{r_2} t^s$ can be decomposed in a sum of elements of S (and nothing else if $r_1 + r_2 \geq s$) plus elements of $\bigoplus_{n>0} S't^n$. Moreover, $S'/(\sigma) = S'$.

We consider separately the curves of type (1) and (2) listed at the beginning of this section.

1^{st} *case*. Curves of bidegree $(a, -b)$, with $a - 3 \geq b > 0$. Notice that all such elements are in $S_{(a,-b)}$: this immediately follows recalling the bidegrees of the variables. Hence we need to determine $dim_k S_{(a,-b)}$.

In the following tables we list on the left side all the possible types of the factors of the monomials of type $(a, -b)$ and on the right their number.

	type of monomials		# of monomials
$(0, a-b)$	$(a, -a)$		$a+1$
$(0, a-b-1)$	$(a-1, -a+1)$	$(1, 0)$	$a \cdot 2$
$\ldots$	$\ldots$	$\ldots$	$\ldots$
$(0, 1)$	$(b+1, -b-1)$	$(a-b-1, 0)$	$(b+2) \cdot (a-b)$
	$(b, -b)$	$(a-b, 0)$	$(b+1) \cdot (a-b+1)$

Notice that in counting the monomials of type $(r, 0)$ we have used only the variables z_1, z_2, so that we have $r + 1$ monomials. Summing all we obtain

$$(b+1)\big[(a-b+1)+(a-b)+\ldots+2+1\big]+\big[(a-b)+2(a-b-1)+\ldots +(a-b-1)2+(a-b)\big]=$$

$$=(b+1)\binom{a-b+2}{2}+\binom{a-b+2}{3}=\frac{(a-b+2)(a-b+1)(a+2b+3)}{6}$$

where we have used Lemma 3.2.

Now, in order to compute $dim_k S_{(a,-b)}$ we need to compute the dimension of $(\sigma)_{(a,-b)}$, i.e. the number of monomials in $k[x, y_1, y_2, z_1, z_2]$ of type $(a-4, -b+1)$ because σ is of type (4,-1).

	type of monomials		# of monomials
$(0, a-b-3)$	$(a-4, -a+4)$		$a-3$
$(0, a-b-4)$	$(a-5, -a+5)$	$(1, 0)$	$(a-4) \cdot 2$
$\ldots$	$\ldots$	$\ldots$	$\ldots$
$(0, 1)$	$(b, -b)$	$(a-b-4, 0)$	$(b+1) \cdot (a-b-3)$
	$(b-1, -b+1)$	$(a-b-3, 0)$	$b \cdot (a-b-2)$

so that the same computation as before gives

$$b\binom{a-b-1}{2}+\binom{a-b-1}{3}=\frac{(a-b-1)(a-b-2)(a+2b-3)}{6}$$

Hence we have

$$dim_k S_{(a,-b)} = \frac{(a-b+2)(a-b+1)(a+2b+3)}{6} -$$
$$-\frac{(a-b-1)(a-b-2)(a+2b-3)}{6} =$$
$$= 2a^2 - b^2 - ab + 2 = h^0(aH - bL)$$

2^{nd} *case*. Curves of bidegree (a, b) with $a, b \geq 0$. Of course we can assume that $a \geq 2b$; in fact, otherwise every section of $H^0(aH + bL)$ contains the factor x the right number of times.

As before we determine the number of monomials of type (a, b), but in this case we separately count monomials of $S_{(a,b)}$ and monomials of $(\oplus_{n>0} S' t^n)_{(a,b)}$. For the first ones we start counting their number in $k[x, y_1, y_2, z_1, z_2]$.

	type of monomials		# of monomials
$(0, a+b)$	$(a, -a)$		$a+1$
$(0, a+b-1)$	$(a-1, -a+1)$	$(1, 0)$	$a \cdot 2$
...	...	...	...
$(0, b+1)$	$(1, -1)$	$(a-1, 0)$	$2 \cdot a$
$(0, b)$		$(a, 0)$	$a+1$
			$\binom{a+3}{3}$

Then, to complete the computation, we need the number of monomials having degree $(a-4, b+1)$ in $k[x, y_1, y_2, z_1, z_2]$ since these give the elements of $(\sigma)_{(a,b)}$:

	type of monomials		# of monomials
$(0, a+b-3)$	$(a-4, -a+4)$		$a-3$
$(0, a+b-4)$	$(a-5, -a+5)$	$(1, 0)$	$(a-4) \cdot 2$
...	...	...	...

$(0, b+2)$	$(1, -1)$	$(a-5, 0)$	$2 \cdot (a-4)$
$(0, b+1)$		$(a-4, 0)$	$a-3$
			$\binom{a-1}{3}$

Finally, we compute $dim_k(\oplus_{n>0} S' t^n)_{(a,b)}$.

	type of monomials		# of monomials
$(0, b-1)$	$(2, 1)$	$(a-2, 0)$	$a-1$
$(0, b-2)$	$(4, 2)$	$(a-4, 0)$	$a-3$
...	...	...	...
$(0, 1)$	$(2b-2, b-1)$	$(a-2b+2, 0)$	$a-2b+3$
	$(2b, b)$	$(a-2b, 0)$	$a-2b+1$
			$b(a-b)$

Hence summing all we have

$$dim_k R_{(a,b)} = \binom{a+3}{3} + b(a-b) - \binom{a-1}{3} = a^2 - b^2 + ab + 2 = h^0(aH + bL)$$

$\square$

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