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RENORMALIZED SOLUTIONS FOR SOME NON-COERCIVE QUASILINEAR ELLIPTIC PROBLEMS IN MUSIELAK-ORLICZ SPACES

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In this paper, we study the existence of renormalized solutions for the following non-coercive quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = f - \operatorname{div}(\phi(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the Musielak-Orlicz-Sobolev space $W_0^1 L_\varphi(\Omega)$, where $-\operatorname{div}\, a(x,u,\nabla u)$ is a degenerate Leary Lions operator and $g(x,u)$ is a Carathéodory function that satisfies the sign condition with $\phi(\cdot) \in C^0(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$ and $f \in L^1(\Omega)$. The Musielak-Orlicz function $\varphi(x,t)$ is regular and does not necessarily satisfying the Δ_2-condition.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^{\mathbb{N}}$, $(N \geq 2)$. In [12], Boccardo et al. have studied the existence and regularity of renormalized solutions for the following nonlinear problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = f - \operatorname{div}(\phi(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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in the Sobolev spaces $W_0^{1,p}(\Omega)$, where $-\operatorname{div}(a(x,u,\nabla u))$ is a Leray-Lions operator and the lower order term $g(x,u)$ is a Carathéodory function that verifies some conditions, with $f \in W^{-1,p'}(\Omega)$ and $\phi \in C^0(\mathbb{R},\mathbb{R}^{\mathbb{N}})$. In the case of $\phi = 0$ and $f \in L^1(\Omega)$, Rakotoson in [26] has proved the existence and uniqueness of solutions for the problem (1). We refer the reader to [3] and [17] for more details.

The concept of renormalized solution was originally introduced by DiPerna and Lions in [16], in their study of the Boltzmann equation, and was later adapted by Boccardo et al. in [13] for some elliptic problems with L^1 data.

In the Orlicz Sobolev spaces framework. Aharouch et al. have studied in [5] the existence of renormalized solutions for the elliptic equations (1), where $-\operatorname{div} a(x,u,\nabla u)$ is a Leray-Lions operator and $f \in L^1(\Omega)$. Kozhevnikova has proved in [22] the existence of entropy and renormalized solutions for the following quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) + g(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

in the Musielak-Orlicz Sobolev space $W_0^1 L_\varphi(\Omega)$, where $f \in L^1(\Omega)$ and $g(x,u)$ is a Carathéodory function that verifies some conditions, with the Musielak-Orlicz function φ satisfies the log-Hölder condition. For more results, we refer the reader to [8], [9], [15], [18] and [24].

In the present paper, we study the existence of renormalized solutions for the following non-coercive quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = f - \operatorname{div}(\phi(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $Au = -\operatorname{div}(a(x,u,\nabla u))$ is a degenerate Leray-Lions operator acting from $D(A) \subset W_0^1 L_\varphi(\Omega)$ into $W^{-1}L_{\bar{\varphi}}(\Omega)$, the perturbing function $g(x,u)$ satisfying the sign condition, with $\phi(\cdot) \in C^0(\mathbb{R},\mathbb{R}^{\mathbb{N}})$ and $f \in L^1(\Omega)$. The Musielak-Orlicz function $\varphi(x,t)$ satisfies the fundamental regularity conditions and its conjugate function $\bar{\varphi}(x,t)$ satisfies the Δ_2-condition.

This paper is organized as follows: In section 2 we present some definitions and results related to Musielak-Orlicz Sobolev spaces. In section 3 we present the essential assumptions under which our non-coercive elliptic problem has at least one renormalized solution in the Musielak-Orlicz Sobolev spaces $W_0^1 L_\varphi(\Omega)$. In section 4 we present some technical lemmas required to establish our main result. The last section focuses on the proof of the main theorem.

2. Preliminary

Let Ω be an open domain in $\mathbb{R}^N(N \geq 2)$, and let $\varphi(x,t) : \Omega \times \mathbb{R}^+ \longmapsto \mathbb{R}^+$ be a function such that:

(i) The function $\varphi(x,\cdot)$ is an N-function, i.e. convex, continuous, strictly increasing with $\varphi(x,0) = 0$, $\operatorname{ess\,inf}_{x\in\Omega} \varphi(x,t) > 0$ for any $t > 0$ and such that

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = +\infty. \tag{4}$$

(ii) The function $\varphi(\cdot,t)$ is measurable for any $t \geq 0$.

A function $\varphi(x,t)$ which satisfies the conditions (i) and (ii) is called a Musielak-Orlicz function.
If a Musielak-Orlicz function doesn't depend on x ($i.e.$ $\varphi(x,t) = \varphi(t)$), then this function is called an Orlicz function.
The Musielak-Orlicz function $\bar{\varphi}(x,t)$ complementary to (or conjugate of) $\varphi(x,t)$ is defined by

$$\bar{\varphi}(x,t) = \sup_{s\geq 0}\{st - \varphi(x,s)\} \quad \text{a.e. in } \Omega, \tag{5}$$

and we have the Fenchel-Young's inequality

$$st \leq \bar{\varphi}(x,s) + \varphi(x,t) \quad \text{for any } s,t \geq 0 \quad \text{and a.e. } x \in \Omega. \tag{6}$$

A Musielak-Orlicz function $\varphi(x,t)$ increases essentially more slowly than a Musielak-Orlicz function $\gamma(x,t)$ and we write $\gamma \prec\prec \varphi$, if for every positive constant c we have

$$\lim_{t\to\infty} \sup_{x\in\Omega} \left(\frac{\gamma(x,t)}{\varphi(x,ct)} \right) = 0.$$

A Musielak-Orlicz function $\varphi(x,t)$ satisfies the Δ_2-condition, if there exist $k > 0$ and a nonnegative function $h(\cdot) \in L^1(\Omega)$ such that

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \quad \text{for any } t \geq 0 \quad \text{and a.e. } x \in \Omega.$$

A Musielak-Orlicz function $\varphi(x,t)$ is called locally integrable if for each $t \geq 0$ the function $\varphi(\cdot,t)$ belongs to $L^1_{loc}(\Omega)$, and is called integrable if for each $t \geq 0$ the function $\varphi(\cdot,t)$ belongs to $L^1(\Omega)$.
We consider the following fundamental regularity assumptions on the Musielak-Orlicz function $\varphi(x,t)$.
($\mathcal{M}_1$) There exists a function $\phi : [0,\frac{1}{2}] \times [0,\infty) \to [0,\infty)$ such that $\phi(\cdot,s)$ and

$\phi(r,\cdot)$ are nondecreasing functions, and for all $s \geq 0$ and all $x,y \in \bar{\Omega}$ with $|x-y| \leq \frac{1}{2}$ such that

$$\varphi(x,s) \leq \phi(|x-y|,s)\varphi(y,s), \quad \text{with} \quad \limsup_{\varepsilon\to 0^+} \phi\left(\varepsilon, c\varepsilon^{-N}\right) < C,$$

for any constant $c > 0$ and for some real constant $C > 0$.
$(\mathcal{M}_2)$ The Musielak-Orlicz function $\varphi(x,t)$ is said to satisfy the Y-condition on a segment $[a,b]$ of the real line $\mathbb{R}$, if either

$$(Y_0)\begin{cases} \text{there exists } t_0 \in \mathbb{R}^+ \text{and } 1 \leq i \leq N \text{ such that the partial function} \\ x_i \in [a,b] \mapsto \varphi(x,t) \text{ changes constantly its monotony on both} \\ \text{sides of } t_0 \text{ (that is for } t \geq t_0 \text{ and } t < t_0), \end{cases}$$

or

$$(Y_\infty)\begin{cases} \text{there exists } 1 \leq i \leq N \text{ such that for all } t \geq 0, \text{ the partial function} \\ x_i \in [a,b] \mapsto \varphi(x,t) \text{ is monotone on } [a,b]. \end{cases}$$

Here, x_i is the i^{th} component of $x = (x_1, x_2, \ldots, x_N) \in \Omega$.

Remark 2.1. Let Ω be a bounded Lipschitz domain, and $\varphi(x,t)$ be a Musielak-Orlicz function that satisfies $(\mathcal{M}_1)$. Then, $\varphi(x,t)$ is integrable over Ω. Moreover, for any Musielak-Orlicz function $\gamma(x,t)$ that verifying $\gamma \prec\prec \varphi$, we have: for any $\varepsilon > 0$ there exists $h_\varepsilon(x) \in L^1(\Omega)$ such that

$$\gamma(x,t) \leq \varphi(x,\varepsilon t) + h_\varepsilon(x) \quad \text{for any } t \geq 0 \quad \text{and a.e. } x \in \Omega.$$

Let $\varphi(x,t)$ be a Musielak-Orlicz function, and $u : \Omega \mapsto \mathbb{R}$ be a measurable function. We define the modular

$$\rho_\varphi(u) = \int_\Omega \varphi(x,|u(x)|)\, dx,$$

and the convex set

$$K_\varphi(\Omega) = \left\{u : \Omega \longmapsto \mathbb{R} \text{ measurable } / \rho_\varphi(u) < +\infty\right\}.$$

The set $K_\varphi(\Omega)$ is called the Musielak-Orlicz class (the generalized Orlicz class). We define the Musielak-Orlicz space $L_\varphi(\Omega)$ by the vector space

$$L_\varphi(\Omega) = \left\{u : \Omega \longmapsto \mathbb{R} \text{ measurable } / \quad \frac{u}{\lambda} \in K_\varphi(\Omega) \quad \text{for some } \lambda > 0\right\}.$$

equipped by the Luxemburg norm

$$\|u\|_\varphi = \inf\left\{\lambda > 0/ \quad \rho_\varphi\left(\frac{u}{\lambda}\right) \leq 1\right\}. \tag{7}$$

Note that, the norm (7) satisfies the inequality

$$\|u\|_\varphi \leq \rho_\varphi(u) + 1 \quad \text{for any } u \in K_\varphi(\Omega). \tag{8}$$

The generalized Hölder's inequality is giving by

$$\left| \int_\Omega uv\, dx \right| \leq 2\|u\|_\varphi \|v\|_{\bar{\varphi}} \quad \text{for any} \quad u \in L_\varphi(\Omega) \text{ and } v \in L_{\bar{\varphi}}(\Omega). \tag{9}$$

The closure in $L_\varphi(\Omega)$ of bounded measurable functions with compact support in Ω is denoted by $E_\varphi(\Omega)$. It is a separable space and $(E_\varphi(\Omega))^* = L_{\bar{\varphi}}(\Omega)$.
We have $E_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if $\varphi(x,t)$ verifies the Δ_2-condition.
The space $L_\varphi(\Omega)$ is reflexive if and only if $\varphi(x,t)$ and $\bar{\varphi}(x,t)$ verify the Δ_2-condition.

A sequence $(u_n)_n \subset L_\varphi(\Omega)$ is called converge to u in $L_\varphi(\Omega)$ for the modular topology if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_\varphi\left(\frac{u_n - u}{\lambda}\right) = 0.$$

The Musielak-Orlicz-Sobolev spaces $W^1L_\varphi(\Omega)$ and $W^1E_\varphi(\Omega)$ are defined by

$$W^1L_\varphi(\Omega) = \left\{u \in L_\varphi(\Omega), \quad \text{with } |\nabla u| \in L_\varphi(\Omega)\right\},$$

and

$$W^1E_\varphi(\Omega) = \left\{u \in E_\varphi(\Omega), \quad \text{with } |\nabla u| \in E_\varphi(\Omega)\right\}.$$

The space $W^1L_\varphi(\Omega)$ is endowed with the norm

$$\|u\|_{1,\varphi} = \|u\|_\varphi + \|\nabla u\|_\varphi. \tag{10}$$

The vector space $(W^1L_\varphi(\Omega), \|\cdot\|_{1,\varphi})$ is a Banach space not necessarily reflexive. A sequence of functions $(u_n)_n \subset W^1L_\varphi(\Omega)$ is called converges to u in $W^1L_\varphi(\Omega)$ for the modular topology, if there exists $\lambda > 0$ such that

$$\left[\rho_\varphi\left(\frac{u_n - u}{\lambda}\right) + \rho_\varphi\left(\frac{|\nabla u_n - \nabla u|}{\lambda}\right)\right] \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

The spaces $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) can be identified to a subspace of the product of $N+1$ copies of $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$), denoting this product by $\Pi L_\varphi(\Omega)$ (resp. $\Pi E_\varphi(\Omega)$). We will use the following weak topology $\sigma\left(\Pi L_\varphi(\Omega), \Pi E_{\bar{\varphi}}(\Omega)\right)$ and $\sigma\left(\Pi L_\varphi(\Omega), \Pi L_{\bar{\varphi}}(\Omega)\right)$.

The space $W_0^1E_\varphi(\Omega)$ is defined as the closure of the Schwartz space $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,\varphi}$ in $W^1E_\varphi(\Omega)$, and the space $W_0^1L_\varphi(\Omega)$ as the weak $\sigma\left(\Pi L_\varphi(\Omega), \Pi E_{\bar{\varphi}}(\Omega)\right)$ closure of $C_0^\infty(\Omega)$ in $W^1L_\varphi(\Omega)$.

The dual space of $W_0^1L_\varphi(\Omega)$ is given by

$$W^{-1}L_{\bar{\varphi}}(\Omega) = \left\{ v = f - \operatorname{div} F, \quad \text{with } f \in L_{\bar{\varphi}}(\Omega) \text{ and } F \in \left(L_{\bar{\varphi}}(\Omega)\right)^N \right\},$$

and the dual space of $W_0^1E_\varphi(\Omega)$ is defined by

$$W^{-1}E_{\bar{\varphi}}(\Omega) = \left\{ v = f - \operatorname{div} F, \quad \text{with } f \in E_{\bar{\varphi}}(\Omega) \text{ and } F \in \left(E_{\bar{\varphi}}(\Omega)\right)^N \right\}.$$

The below lemma gives the modularly density of $C_0^\infty(\Omega)$ in $W_0^1L_\varphi(\Omega)$.

Lemma 2.2 (see [6], Theorem 3)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ ($N \geq 2$), and let φ be a Musielak-Orlicz function that satisfying the condition $(\mathcal{M}_1)$. Then, $C_0^\infty(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular topology. That is, for any $u \in W_0^1L_\varphi(\Omega)$ there exists a sequence of functions $(u_n)_n \subset C_0^\infty(\Omega)$ such that*

$$u_n \longrightarrow u \quad \text{modularly in } W_0^1L_\varphi(\Omega) \quad \text{as} \quad n \to \infty.$$

Remark 2.3. Let Ω be a bounded domain in $\mathbb{R}^N$, and let $\varphi(x,t)$ and $\bar{\varphi}(x,t)$ be a pair of complimentary a Musielak-Orlicz functions. Then, the assumption (4) on $\varphi(x,t)$ implies that $\bar{\varphi}(x,t)$ is integrable over Ω.

Lemma 2.4 (see [7], Theorem 1.4)**.** *Under the same conditions of the previous lemma 2.2, we have*

$$W_0^1L_\varphi(\Omega) = W_0^{1,1}(\Omega) \cap W^1L_\varphi(\Omega) = \left\{ u \in W^1L_\varphi(\Omega); \quad u_{/\partial\Omega} = 0 \right\}. \tag{11}$$

Lemma 2.5 (see [7], Theorem 1.1)**.** *(Poincaré's inequality) Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$, and let $\varphi(x,t)$ be a Musielak-Orlicz function that satisfies $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$. Then, there exists two constants $d_1 > 0$ and $d_2 > 0$ depending only on Ω and φ such that*

$$\int_\Omega \varphi(x,|u|)dx \leq d_1 \int_\Omega \varphi\left(x, d_2|\nabla u|\right) dx \quad \text{for any } u \in W_0^1L_\varphi(\Omega). \tag{12}$$

In addition, there exists a constant $C > 0$ depends only on Ω and φ such that

$$\|u\|_\varphi \leq C\|\nabla u\|_\varphi \qquad \text{for any} \quad u \in W_0^1L_\varphi(\Omega). \tag{13}$$

Remark 2.6.

- We can prove the Poincaré's inequality (13) by following the same technique used in [2] without assuming the condition $(\mathcal{M}_2)$ on $\varphi(x,t)$.
- The Poincaré's inequality (13) implies that, the two norms $\|\nabla \cdot\|_\varphi$ and $\|\cdot\|_{1,\varphi}$ are equivalents in $W_0^1L_\varphi(\Omega)$.

3. Essential Assumptions

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$, $(N \geq 2)$. Let $\varphi(x,t)$ be a Musielak-Orlicz function that satisfies $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$, and we assume that the complementary function of $\varphi(x,t)$ denote by $\bar{\varphi}(x,t)$ satisfying the Δ_2-condition, and then we get $(L_{\bar{\varphi}}(\Omega) = E_{\bar{\varphi}}(\Omega))$.
We consider the following non-coercive quasilinear elliptic problem

$$\begin{cases} Au + g(x,u) = f - \operatorname{div}(\phi(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

We shall now give the essential assumptions for each term of our problem (14). The mapping $A : \mathcal{D}(A) \subset W_0^1 L_\varphi(\Omega) \longmapsto W^{-1} L_{\bar{\varphi}}(\Omega)$ is a Leray-Lions operator defined by

$$Au = -\operatorname{div} a(x,u,\nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ is a Carathéodory function that satisfying the following conditions

$$|a(x,s,\xi)| \leq a_0(x) + k_1 \bar{\varphi}^{-1}(x, \gamma(x, k_2|s|)) + k_1 \bar{\varphi}^{-1}(x, \varphi(x, k_3|\xi|)), \tag{15}$$

$$\big(a(x,s,\xi) - a(x,s,\xi')\big) \cdot (\xi - \xi') > 0 \quad \text{for all } \xi \neq \xi', \tag{16}$$

for almost all $x \in \Omega$ and for any $(s,\xi,\xi') \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, where $a_0(x)$ is a nonnegative function lying in $L_{\bar{\varphi}}(\Omega)$, and $\gamma(x,t)$ is a Musielak-Orlicz function such that $\gamma \prec\prec \varphi$, the constants k_1, k_2 and k_3 are non negatives.
The degenerate coercivity condition

$$a(x,s,\xi) \cdot \xi \geq \alpha(|s|)\varphi(x,|\xi|), \tag{17}$$

where $\alpha(|s|)$ is a nonnegative decreasing function such that $\alpha(|s|) \geq \dfrac{|s|+1}{\lambda(|s|+1)}$, and $\lambda(\cdot)$ is an Orlicz function that verifies $\lambda \prec\prec \varphi$.

Under the assumption (17) and the continuity of the function $a(x,s,\cdot)$ with respect to ξ, we have

$$a(x,s,0) = 0.$$

For the perturbing function $g(x,s) : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$, we assume the usual conditions.

$$\sup_{|s| \leq k} |g(x,s)| \in L^1(\Omega) \quad \text{for any } k > 0, \tag{18}$$

and

$$g(x,s)s \geq 0 \quad \text{for any } s \in \mathbb{R}. \tag{19}$$

Finally, we assume that

$$f \in L^1(\Omega) \quad \text{and} \quad \phi \in C^0(\mathbb{R}, \mathbb{R}^{\mathbb{N}}). \tag{20}$$

4. Some technical Lemmas

Lemma 4.1 (see [19], Theorem 13.47)**.** *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

(i) $u_n \longrightarrow u$ *a.e. in* Ω,

(ii) $u_n \geq 0$ *a.e. in* Ω,

(iii) $\int_\Omega u_n\, dx \longrightarrow \int_\Omega u\, dx$ *as* $n \to \infty$,

then $u_n \longrightarrow u$ *strongly in* $L^1(\Omega)$ *as* $n \to \infty$.

Lemma 4.2 (see [10], Lemma 1)**.** *Let $u \in L_\varphi(\Omega)$ and $(u_n)_n$ be a uniformly bounded sequence in $L_\varphi(\Omega)$. If $u_n \longrightarrow u$ a.e. in Ω, then $u_n \rightharpoonup u$ weakly in $L_\varphi(\Omega)$ for $\sigma\left(L_\varphi(\Omega), E_{\bar{\varphi}}(\Omega)\right)$.*

Lemma 4.3 (see [10], Lemma 4)**.** *Let $F : \mathbb{R} \mapsto \mathbb{R}$ be uniformly Lipschitz function, with $F(0) = 0$. If $u \in W_0^1 L_\varphi(\Omega)$, then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of $F'(\cdot)$ is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \dfrac{\partial u}{\partial x_i} & a.e \text{ in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & a.e \text{ in } \{x \in \Omega : u(x) \in D\}. \end{cases} \tag{21}$$

For any $k > 0$, we define the truncation function by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\frac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

Also, we define the continuous function

$$S_k(s) = 1 - |T_{k+1}(s) - T_k(s)|.$$

Remark 4.4. Let $k > 0$, it's clear that the function $T_k(\cdot)$ verifying the assumptions of the Lemma 4.3, then $T_k(u) \in W_0^1 L_\varphi(\Omega)$ for any $u \in W_0^1 L_\varphi(\Omega)$. Moreover, we have

$$\frac{\partial T_k(u)}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i} & \text{for } |u| < k, \\ 0 & \text{for } |u| \geq k. \end{cases}$$

Lemma 4.5. *Let $(u_n)_n$, be a sequence in $L_\varphi(\Omega)$ such that*

$$u_n \longrightarrow u \quad a.e. \text{ in } \Omega \quad \text{as } n \to \infty,$$

and

$$\varphi(x, u_n) \leq v \in L^1(\Omega) \quad \text{for any } n \in \mathbb{N}.$$

Then,

$$u \in L_\varphi(\Omega) \text{ and } \quad u_n \longrightarrow u \quad \text{modularly in } L_\varphi(\Omega) \quad \text{as } n \to \infty.$$

Lemma 4.6. *Let $(u_n)_n$ be a bounded sequence in $L^\infty(\Omega)$ and u be a measurable function, with*

$$u_n \longrightarrow u \quad \text{a.e. in } \Omega \quad \text{as } n \to \infty.$$

Then, $u \in L^\infty(\Omega)$ and $u_n \rightharpoonup u$ weak$-*$ in $L^\infty(\Omega)$ as $n \to \infty$.
In addition, if $v \in L_\varphi(\Omega)$ (resp, $v \in E_\varphi(\Omega)$), then

$$u_n v \longrightarrow uv \quad \text{modularly in } L_\varphi(\Omega) \big(\text{ resp, strongly in } E_\varphi(\Omega)\big) \quad \text{as} \quad n \to \infty.$$

The proofs of Lemma 4.5 and Lemma 4.6 are based on the Vitali's theorem.

Lemma 4.7 (see [21], lemma 4.10)**.** *Under the assumptions* (15)$-$(17)*, let $(u_n)_n$ be a sequence in $W_0^1L_\varphi(\Omega)$ such that, $(u_n)_n$ is uniformly bounded in $L^\infty(\Omega)$, $u_n \rightharpoonup u$ weakly in $W_0^1L_\varphi(\Omega)$, $(a(x,u_n,\nabla u_n))_n$ is bounded in $\big(L_{\bar\varphi}(\Omega)\big)^N$ and*

$$\int_\Omega D_n^\tau \, dx = \int_\Omega \left(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi_\tau)\right)\left(\nabla u_n - \nabla u\chi_\tau\right) \, dx \longrightarrow 0, \quad (22)$$

as n then τ tending to ∞, where χ_τ is the characteristic function of the set

$$K_\tau = \{x \in \Omega; \quad |\nabla u(x)| \leq \tau\}.$$

Then,

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega.$$
$$u_n \longrightarrow u \quad \text{modularly in } W_0^1L_\varphi(\Omega).$$
$$a(x,u_n,\nabla u_n)\cdot\nabla u_n \longrightarrow a(x,u,\nabla u)\cdot\nabla u \quad \text{strongly in } L^1(\Omega).$$

5. Main result

Definition 5.1. A measurable function u defined on Ω is called a renormalized solution of problem (14). If $T_k(u) \in W_0^1L_\varphi(\Omega)$ for any $k > 0$, and $g(x,u) \in L^1(\Omega)$ with

$$\lim_{k\to\infty} \int_{\{k \leq |u| \leq k+1\}} a(x,u,\nabla u)\cdot\nabla u \, dx = 0.$$

In addition, the function u satisfies the following equality

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\cdot\big(\nabla u S'(u)v + \nabla v S(u)\big) \, dx + \int_\Omega g(x,u) v S(u) \, dx \\ = \int_\Omega f(x) v S(u) \, dx + \int_\Omega \phi(u)\cdot\big(\nabla u S'(u) v + \nabla v S(u)\big) \, dx,\end{aligned} \quad (23)$$

for any $v \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$ and any $S(\cdot) \in W^{1,\infty}(\mathbb{R})$ with a compact support.

Now, we shall prove the following existence result.

Theorem 5.2. *Under the assumptions* (15)$-$(20)*, there exists at least one renormalized solution for the non-coercive quasilinear elliptic problem* (14).

Proof of Theorem 5.2

Step 1: Approximate problems

Let $n \in \mathbb{N}^*$. We set $f_n(x) = T_n(f(x))$, then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$ as $n \to \infty$. Let $g_n(x,s) = T_n(g(x,s))$ and $\phi_n(s) = \phi(T_n(s))$. Note that

$$|g_n(x,s)| \le n \quad \text{and} \quad |g_n(x,s)| \le |g(x,s)|. \tag{24}$$

Moreover, since $\phi(\cdot) \in C^0(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$ then

$$|\phi_n(s)| \le \sup_{|s| \le n} |\phi(s)| < \infty. \tag{25}$$

We consider the approximate problem of (14) giving by

$$\begin{cases} -\text{div}\,(a(x, T_n(u_n), \nabla u_n)) + g_n(x, u_n) = f_n - \text{div}\,(\phi_n(u_n)) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{26}$$

We consider the two operators A_n and G_n acting from $W_0^1 L_\varphi(\Omega)$ into $W^{-1} L_{\bar{\varphi}}(\Omega)$, defined by

$$\langle A_n u, v \rangle = \int_\Omega (a(x, T_n(u), \nabla u) - \phi_n(u)) \cdot \nabla v \, dx,$$

and

$$\langle G_n u, v \rangle = \int_\Omega g_n(x,u) v \, dx.$$

The assumptions (15)$-$(17) and (24), (25) imply that the operator $B_n = A_n + G_n$ satisfying the conditions $(i) - (iv)$ in [20].
Indeed, for the degenerate coercivity condition we have

$$a(x, T_n(u), \nabla u) \cdot \nabla u \ge \alpha(n) \varphi(x, |\nabla u|) \quad \text{for any } u \in \mathcal{D}(A). \tag{27}$$

In view of Young's inequality and Poincaré's inequality, we conclude that

$$\{ u \in \mathcal{D}(A), \quad < B_n u,\ u - f_n > \le 0 \ \}$$

is bounded in $W_0^1 L_\varphi(\Omega)$, thus the condition (iv) is verified with $\bar{u} = 0$. Then, by Proposition 1 in [20], the problem (26) has at least one solution $u_n \in W_0^1 L_\varphi(\Omega)$, i.e.

$$\begin{aligned} \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla v \, dx + \int_\Omega g_n(x, u_n) v \, dx \\ = \int_\Omega f_n v \, dx + \int_\Omega \phi_n(u_n) \cdot \nabla v \, dx, \end{aligned} \tag{28}$$

for any $v \in W_0^1 L_\varphi(\Omega)$.

Step 2: Weak convergence of $(T_k(u_n))_n$

Let $k>0$, by taking $v=T_k(u_n)$ as a test function for the approximate problem (26), we obtain

$$\begin{aligned}\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n)\,dx+\int_\Omega g_n(x,u_n)T_k(u_n)\,dx\\ =\int_\Omega f_nT_k(u_n)\,dx+\int_\Omega \phi_n(u_n)\cdot\nabla T_k(u_n)\,dx.\end{aligned}\tag{29}$$

For the second term on the left-hand side of (29), thanks to (19) we have

$$\int_\Omega g_n(x,u_n)T_k(u_n)\,dx\geq 0.\tag{30}$$

Moreover, we have

$$\left|\int_\Omega f_nT_k(u_n)\,dx\right|\leq\int_\Omega |f_n||T_k(u_n)|\,dx\leq k\|f\|_{L^1(\Omega)}.\tag{31}$$

Concerning the second terms on the right-hand side of (29). We set $\Phi_n(t)=\int_0^t\phi_n(s)\,ds$, thanks to (11) and (20) we have $\Phi_n(u_n)=0$ on $\partial\Omega$. In view of the Green formula, we obtain

$$\begin{aligned}\int_\Omega \phi_n(u_n)\cdot\nabla T_k(u_n)\,dx&=\int_\Omega \phi_n(T_k(u_n))\cdot\nabla T_k(u_n)\,dx\\&=\int_\Omega \operatorname{div}(\Phi_n(T_k(u_n)))\,dx\\&=\int_{\partial\Omega}\Phi_n(T_k(u_n))\cdot\vec{n}\,d\sigma=0,\end{aligned}\tag{32}$$

where $\vec{n}$ is a exterior normal vector on the boundary $\partial\Omega$.
By combining (29) and (30)−(32), we obtain

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k((u_n)\,dx\leq k\|f\|_{L^1(\Omega)}.\tag{33}$$

On the other hand, in view of (17) we have

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k((u_n)\,dx&\geq\int_\Omega \alpha(|T_k(u_n)|)\varphi(x,|\nabla T_k(u_n)|)\,dx\\&\geq\alpha(k)\int_\Omega\varphi(x,|\nabla T_k(u_n)|)\,dx\\&\geq\frac{k+1}{\lambda(k+1)}\int_\Omega\varphi(x,|\nabla T_k(u_n)|)\,dx.\end{aligned}$$

Then, from (33) we deduce that

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx \le \|f\|_{L^1(\Omega)}\lambda(k+1). \tag{34}$$

According to (8) and Poincaré's inequality there exists a constant $C(k)>0$ depending on k and doesn't depending on n such that

$$\|T_k(u_n)\|_{1,\varphi} \le C(k). \tag{35}$$

Thus, the sequence $(T_k(u_n))_n$ is uniformly bounded in $W_0^1L_\varphi(\Omega)$. In view of the Banach-Alaoglu-Bourbaki's theorem, there exists a measurable function $v_k \in W_0^1L_\varphi(\Omega)$ such that

$$T_k(u_n) \rightharpoonup v_k \ \text{ weakly in } W_0^1L_\varphi(\Omega) \quad \text{as} \quad n\to\infty. \tag{36}$$

Moreover, by the compact embedding $W_0^1L_\varphi(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$ we obtain

$$T_k(u_n) \longrightarrow v_k \ \text{ strongly in } L^1(\Omega) \quad \text{and a.e. on } \Omega \text{ as } n\to\infty, \text{ for a subsequence.} \tag{37}$$

In view of the Poincaré's inequality, we conclude that

$$\begin{aligned}
\inf_{x\in\Omega}\left(\varphi\left(x,\frac{k}{d_2}\right)\right)\text{meas}(|u_n|>k) &\le \int_{\{|u_n|>k\}} \varphi\left(x,\frac{|T_k(u_n)|}{d_2}\right)\,dx \\
&\le \int_\Omega \varphi\left(x,\frac{|T_k(u_n)|}{d_2}\right)\,dx \\
&\le d_1\int_\Omega \varphi\left(x,|\nabla T_k(u_n)|\right)\,dx \\
&\le d_1\|f\|_{L^1(\Omega)}\lambda(k+1).
\end{aligned}$$

Having in mind that $\lambda \prec\prec \varphi$, we obtain

$$\text{meas}(|u_n|>k) \le C_1 \sup_{x\in\Omega}\left(\frac{\lambda(k+1)}{\varphi\left(x,\frac{k}{d_2}\right)}\right) \longrightarrow 0 \quad \text{as } k\to\infty. \tag{38}$$

It follows that : for any $\varepsilon>0$, thanks to (38), there exists a positive constant large enough $k_0(\varepsilon)>0$ such that

$$\text{meas}(|u_n|>k) \le \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}(|u_m|>k) \le \frac{\varepsilon}{3} \quad \text{for any } k>k_0(\varepsilon). \tag{39}$$

Moreover, in view of (37) we have $(T_k(u_n))_n$ is a Cauchy sequence in measure on Ω, then for any $k>0$ and $\delta,\varepsilon>0$, there exists $n_0(k,\delta,\varepsilon)>0$ such that

$$\text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\} \le \frac{\varepsilon}{3} \quad \text{for all} \quad m,n\ge n_0(k,\delta,\varepsilon). \tag{40}$$

By combining (39) and (40), we conclude that : for any $\delta, \varepsilon > 0$ there exists $n_1(\delta, \varepsilon) > 0$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \le \varepsilon \quad \text{for all} \quad n, m \ge n_1(\delta, \varepsilon).$$

Hence, $(u_n)_n$ is a Cauchy sequence in measure in Ω, and there exists a subsequence still denoted by $(u_n)_n$ and a measurable function u such that

$$u_n \longrightarrow u \quad \text{a.e. in } \Omega \quad \text{as } n \to \infty. \tag{41}$$

In view of (36), we conclude that

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi(\Omega), \Pi E_{\bar{\varphi}}(\Omega)\right). \tag{42}$$

Moreover, in view of (20) we have $|\phi_n(T_k(u_n))| \le \sup_{|s| \le k}(|\phi(s)|) \in L_{\bar{\varphi}}(\Omega)$, it implies from (41) and Vitali's theorem that

$$\phi_n(T_k(u_n)) \longrightarrow \phi(T_k(u)) \quad \text{strongly in } \left(L_{\bar{\varphi}}(\Omega)\right)^N = \left(E_{\bar{\varphi}}(\Omega)\right)^N \quad \text{as } n \to \infty. \tag{43}$$

Step 3 : Some regularity results

In this step, we will show that

$$\limsup_{n \to \infty} \int_{\{h \le |u_n| \le h+1\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \longrightarrow 0 \quad \text{as } h \to \infty,$$

and

$$g_n(x, u_n) \longrightarrow g(x, u) \quad \text{strongly in } L^1(\Omega) \quad \text{as} \quad n \to \infty.$$

Let $h > 0$, by taking $v = (T_{h+1}(u_n) - T_h(u_n))$ as a test function in (26) we obtain

$$\begin{aligned} &\int_{\{h \le |u_n| \le h+1\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx + \int_\Omega g_n(x, u_n)\left(T_{h+1}(u_n) - T_h(u_n)\right) dx \\ &= \int_\Omega f_n\left(T_{h+1}(u_n) - T_h(u_n)\right) dx + \int_\Omega \phi_n(u_n) \cdot \left(\nabla T_{h+1}(u_n) - \nabla T_h(u_n)\right) dx. \end{aligned} \tag{44}$$

For the second term on the left-hand side of (44), in view of (19) we have

$$\begin{aligned} \int_\Omega g_n(x, u_n)\left(T_{h+1}(u_n) - T_h(u_n)\right) dx &= \int_\Omega |g_n(x, u_n)|\, |T_{h+1}(u_n) - T_h(u_n)| \, dx \\ &\ge \int_{\{|u_n| > h+1\}} |g_n(x, u_n)| \, dx. \end{aligned} \tag{45}$$

Concerning the second term on the right-hand side of (44), similarly as in (32) we have

$$\begin{aligned}
&\int_\Omega \phi_n(u_n)\cdot(\nabla T_{h+1}(u_n)-\nabla T_h(u_n))\ dx\\
&=\int_\Omega \phi_n(u_n)\cdot\nabla T_{h+1}(u_n)\ dx-\int_\Omega \phi_n(u_n)\cdot\nabla T_h(u_n)\ dx\\
&=\int_\Omega \phi_n\left(T_{h+1}(u_n)\right)\cdot\nabla T_{h+1}(u_n)\ dx-\int_\Omega \phi_n\left(T_h(u_n)\right)\cdot\nabla T_h(u_n)\ dx\\
&=0.
\end{aligned}\tag{46}$$

Finally, for the first term on the right-hand side of (44), thanks to (39) we obtain

$$\begin{aligned}
\varepsilon_1(n,h)&=\left|\int_\Omega f_n(x)\left(T_{h+1}(u_n)-T_h(u_n)\right)\right|\ dx\\
&\leq\sup_n\int_{\{|u_n>h|\}}|f(x)|\ dx\longrightarrow 0\quad \text{ as } h\to\infty.
\end{aligned}\tag{47}$$

By combining (44) and (45)−(47), we conclude that

$$\int_{\{h\leq|u_n|\leq h+1\}} a\left(x,T_n(u_n),\nabla u_n\right)\cdot\nabla u_n\ dx+\int_{\{|u_n|>h+1\}}|g_n(x,u_n)|\ dx\leq\varepsilon_1(n,h).$$

According to (17), we get

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{h\leq|u_n|\leq h+1\}} a\left(x,T_n(u_n),\nabla u_n\right)\cdot\nabla u_n\ dx=0,\tag{48}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{|u_n|>h+1\}}|g_n(x,u_n)|\ dx=0.\tag{49}$$

Thus, thanks to (49) we have: for any $\varepsilon>0$, there exists $h(\varepsilon)>0$ such that

$$\int_{\{|u_n|>h\}}|g_n(x,u_n)|\ dx\leq\frac{\varepsilon}{2}\quad\text{ for any } h>h(\varepsilon).\tag{50}$$

On the other hand, let E be a measurable subset of Ω. In view of (18), we have $|g_n(x,T_h(u_n))|\leq\sup_{|s|\leq h}|g(x,s)|\in L^1(\Omega)$. Thus, there exists $\beta(h,\varepsilon)>0$ such that

$$\int_E|g_n(x,T_h(u_n))|\ dx\leq\frac{\varepsilon}{2}\qquad\text{ for any }\quad meas(E)\leq\beta(h,\varepsilon).\tag{51}$$

By combining (50) and (51), we conclude that: For any $\varepsilon>0$ there exists $\beta(\varepsilon)>0$ such that

$$\int_E g_n\left(x,T_h(u_n)\right)\ dx\leq\int_E\left|g_n\left(x,T_h(u_n)\right)\right|\ dx+\int_{\{|u_n|>h\}}|g_n(x,u_n)|\ dx\leq\varepsilon,\tag{52}$$

for any $E \subset \Omega$ with $\text{meas}(E) \leq \beta(\varepsilon)$. Thus, the sequences $(g_n(x,u_n))_n$ is uniformly equi-integrable. Consequently, in view of (41) and Vitali's theorem we conclude that

$$g_n(x,u_n) \longrightarrow g(x,u) \quad \text{strongly in } L^1(\Omega) \quad \text{as } n \to \infty. \tag{53}$$

Step 4 : Almost everywhere convergence of the gradients

In this step, we will show that the conditions of Lemma 4.7 hold true.

Firstly, we prove that the sequence $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is uniformly bounded in $\left(L_{\bar\varphi}(\Omega)\right)^N$. Indeed, thanks to (16) we have for any $\nu \in \left(E_\varphi(\Omega)\right)^N$

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)) \cdot \nu\, dx \leq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, dx \\ + \int_\Omega a(x,T_k(u_n),\nu) \cdot (\nu - \nabla T_k(u_n))\, dx.\end{aligned} \tag{54}$$

In view of (15), we have

$$\bar\varphi\left(x, \frac{|a(x,T_k(u_n),\nu)|}{1+2k_1}\right) \leq \bar\varphi(x,a_0(x)) + k_1\gamma(x,k_2k) + k_1\varphi(x,k_3|\nu|) \in L^1(\Omega).$$

Thus, the sequence $(a(x,T_k(u_n),\nu))_n$ is uniformly bounded in $L_{\bar\varphi}(\Omega)$, and in view of Hölder's inequality we obtain

$$\begin{aligned}&\int_\Omega a(x,T_k(u_n),\nu) \cdot (\nu - \nabla T_k(u_n))\, dx \\ &\qquad \leq 2\|a(x,T_k(u_n),\nu)\|_{\bar\varphi} \left(\|\nu\|_\varphi + \|\nabla T_k(u_n)\|_\varphi\right).\end{aligned}$$

Having in mind (35) we conclude that

$$\int_\Omega a(x,T_k(u_n),\nu) \cdot (\nu - \nabla T_k(u_n))\, dx \leq C_0(k,\nu). \tag{55}$$

By combining (33) and (54)$-$(55), we conclude that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)) \cdot \nu\, dx \leq C(k,\nu) \quad \text{for any} \quad \nu \in \left(E_\varphi(\Omega)\right)^N,$$

where $C(k,\nu)$ is a finite positive constant that depends only on k and ν.
By using the uniform boundedness principle we deduce that, the sequence $(a(x, T_k(u_n),\nabla T_k(u_n)))_n$ is uniformly bounded in $\left(L_{\bar\varphi}(\Omega)\right)^N$. Hence, there exists a measurable function $\eta_k \in \left(L_{\bar\varphi}(\Omega)\right)^N$ such that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \eta_k \quad \text{weakly in } \left(L_{\bar\varphi}(\Omega)\right)^N \text{ as } n \to \infty, \tag{56}$$

for the weak topology $\sigma\left(\left(L_{\bar{\varphi}}(\Omega)\right)^N, \left(E_{\varphi}(\Omega)\right)^N\right)$.

Now, we will establish that

$$\lim_{\tau\to\infty}\lim_{n\to\infty}\int_\Omega \left(a\left(x,T_k(u_n),\nabla T_k(u_n)\right)-a\left(x,T_k(u_n),\nabla T_k(u)\chi_\tau\right)\right)\cdot\left(\nabla T_k(u_n)-\nabla T_k(u)\chi_\tau\right)\,dx=0.$$

Let $0<k<\tau<h<l<n$; we denote by $\varepsilon_i(n)$ some various functions of real number that goes to 0 as n tends to infinity. Similarly we define $\varepsilon_i(n,l)$, $\varepsilon_i(n,l,h)$ and $\varepsilon_i(n,l,h,\tau)$ for $i=1,2,\ldots$.

In view of (42) we have $T_k(u)\in W_0^1L_\varphi(\Omega)$, thanks to Lemma 2.2 there exists a sequence $(w_l)_{l\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ such that

$$T_k(w_l)\longrightarrow T_k(u)\text{ modularly in }W_0^1L_\varphi(\Omega)\quad\text{ as }l\to\infty. \tag{57}$$

Thus,

$$T_k(w_l)\rightharpoonup T_k(u)\text{ weakly in }W_0^1L_\varphi(\Omega)\quad\text{ as }l\to\infty. \tag{58}$$

We set $\vartheta_{n,l}=T_k(u_n)-T_k(w_l)$ and $\vartheta_l=T_k(u)-T_k(w_l)$. In view of (41) we have

$$\vartheta_{n,l}\longrightarrow\vartheta_l\quad\text{ a.e. in }\Omega\quad\text{ as }n\to\infty, \tag{59}$$

and

$$\vartheta_l\longrightarrow 0\quad\text{ a.e. in }\Omega\quad\text{ as }l\to\infty\quad\text{ for a subsequence.} \tag{60}$$

By taking $S_h(u_n)\vartheta_{n,l}$ as a test function in (26), we obtain

$$J^1_{n,l,h}+J^2_{n,l,h}+J^3_{n,l,h}=J^4_{n,l,h}+J^5_{n,l,h}+J^6_{n,l,h}, \tag{61}$$

where

$$\begin{aligned}
J^1_{n,l,h}&=\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\vartheta_{n,l}S_h(u_n)\,dx,\\
J^2_{n,l,h}&=-\int_{\{h\le|u_n|\le h+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,\mathrm{sign}(u_n)\vartheta_{n,l}\,dx,\\
J^3_{n,l,h}&=\int_\Omega g_n(x,u_n)\vartheta_{n,l}S_h(u_n)\,dx,\\
J^4_{n,l,h}&=\int_\Omega f_n\vartheta_{n,l}S_h(u_n)\,dx,\\
J^5_{n,l,h}&=\int_\Omega \phi_n(u_n)\cdot\nabla\vartheta_{n,l}S_h(u_n)\,dx,\\
J^6_{n,l,h}&=-\int_{\{h\le|u_n|\le h+1\}} \phi_n(u_n)\cdot\nabla u_n\,\mathrm{sign}(u_n)\vartheta_{n,l}\,dx.
\end{aligned}$$

For the first term $J^1_{n,l,h}$, we have $S_h(u_n)=1$ on $\{|u_n|\le k\}$ and $|S_h(u_n)|\le 1$, then

$$\begin{aligned}
J^1_{n,l,h}=&\int_\Omega a\left(x,T_k(u_n),\nabla T_k(u_n)\right)\cdot\nabla\vartheta_{n,l}\,dx\\
&-\int_{\{k<|u_n|\le h+1\}} a\left(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n)\right)\cdot\nabla T_k(w_l)S_h(u_n)\,dx.
\end{aligned}$$

Since $\nabla T_k(w_l) \in (E_\varphi(\Omega))^N$, then from (41) and Lemma 4.6 we get $\nabla T_k(w_l) S_h(u_n)\chi_{\{k<|u_n|\le h+1\}} \longrightarrow \nabla T_k(w_l)S_h(u)\chi_{\{k\le|u|\le h+1\}}$ strongly in $(E_\varphi(\Omega))^N$ as $n\to\infty$, in view of (56) we obtain

$$\lim_{n\to\infty}\int_{\{k<|u_n|\le h+1\}} a\left(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n)\right)\cdot\nabla T_k(w_l)S_h(u_n)\,dx$$
$$=\int_{\{k\le|u|\le h+1\}} \eta_{h+1}\cdot\nabla T_k(w_l)S_h(u)\,dx.$$

Having in mind (58) we conclude that

$$\lim_{l\to\infty}\int_{\{k\le|u|\le h+1\}} \eta_{h+1}\cdot\nabla T_k(w_l)S_h(u)dx = \int_{\{k\le|u|\le h+1\}} \eta_{h+1}\cdot\nabla T_k(u)S_h(u)dx = 0.$$

It follows that

$$J^1_{n,l,h} = \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(w_l))dx+\varepsilon_1(n,l). \tag{62}$$

For the second term $J^2_{n,l,h}$, we have $|\vartheta_{n,l}|\le 2k$ and thanks to (48) we obtain

$$\varepsilon_2(n,h) = |J^2_{n,l,h}| \le 2k\int_{\{h\le|u_n|\le h+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx \longrightarrow 0 \quad \text{as } n,h\to\infty. \tag{63}$$

Concerning the third and forth terms $J^3_{n,l,h}$ and $J^4_{n,l,h}$, in view of (59)–(60) we have $|\vartheta_{n,l}| \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ as $n,l\to\infty$, and thanks to (53) we obtain

$$\varepsilon_3(n,l) = |J^3_{n,l,h}| \le \int_\Omega |g_n(x,u_n)||\vartheta_{n,l}|\,dx \longrightarrow 0 \quad \text{as } n,\ l\to\infty. \tag{64}$$

Similarly, we have $f_n(x)\longrightarrow f(x)$ strongly in $L^1(\Omega)$, then

$$\varepsilon_4(n,l) = |J^4_{n,l,h}| \le \int_\Omega |f||\vartheta_{n,l}|\,dx \longrightarrow 0 \quad \text{as } n,l\to\infty. \tag{65}$$

For the fifth term $J^5_{n,l,h}$, we have $|S_h(u_n)|\le 1$ and $\operatorname{supp}(S_h)\subset[-h-1,h+1]$, then

$$|J^5_{n,l,h}| \le \int_\Omega |\phi_n(T_{h+1}(u_n))\cdot\nabla\vartheta_{n,l}||S_h(u_n)|\,dx$$
$$\le \int_\Omega |\phi_n(T_{h+1}(u_n))||\nabla\vartheta_{n,l}|\,dx.$$

By using (42) and (58), we get $|\nabla\vartheta_{n,l}| = |\nabla T_k(u_n)-\nabla T_k(w_l)| \rightharpoonup 0$ weakly in $L_\varphi(\Omega)$ as $n,l\to\infty$, and thanks to (43) we get

$$\varepsilon_5(n,l) = |J^5_{n,l,h}| \le \int_\Omega |\phi_n(T_{h+1}(u_n))||\nabla\vartheta_{n,l}|\,dx \longrightarrow 0 \quad \text{as } n,l\to\infty. \tag{66}$$

Concerning the last term on the right-hand side of (61), we have $h > k$ then

$$\begin{aligned}
J^6_{n,l,h} &= -\int_{\{h\le|u_n|\le h+1\}} \phi_n(u_n)\cdot\nabla u_n|\vartheta_{n,l}|\,dx\\
&= -\int_{\{h\le|u_n|\le h+1\}} \operatorname{div}(\Phi_n(u_n))\,|T_k(u_n)-T_k(w_l)|\,dx\\
&= \int_{\{h\le|u_n|\le h+1\}} \Phi_n(u_n)\cdot\nabla|T_k(u_n)-T_k(w_l)|\,dx\\
&= -\int_{\{h\le|u_n|\le h+1\}} \Phi_n(T_{h+1}(u_n))\cdot\nabla T_k(w_l)\operatorname{sign}(u_n)\,dx,
\end{aligned}$$

where $\Phi_n(t) = \int_0^t \phi_n(s)\,ds \in C^1(\mathbb{R},\mathbb{R}^{\mathbb{N}})$. Since $\Phi_n(T_{h+1}(u_n)) \longrightarrow \Phi(T_{h+1}(u))$ strongly in $\left(E_{\bar\varphi}(\Omega)\right)^N$ as $n\to\infty$, then by (58) we obtain

$$\varepsilon_6(n,l) = J^6_{n,l,h} \to -\int_{\{h\le|u|\le h+1\}} \Phi(T_{h+1}(u))\cdot\nabla T_k(u)\operatorname{sign}(u)\,dx = 0 \ \text{ as } n,l\to\infty. \quad (67)$$

By combining (61) and (62)$-$(67), we deduce that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(w_l))\,dx \le \varepsilon_7(n,l,h).$$

It follows that

$$\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_\tau))\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_\tau)\,dx\\
&\quad\le \varepsilon_7(n,l,h) + \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(w_l)-\nabla T_k(u)\chi_\tau)\,dx\\
&\quad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_\tau)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_\tau)\,dx\\
&\quad = \varepsilon_7(n,l,h) + I_1 + I_2,
\end{aligned} \quad (68)$$

where χ_τ is the characteristic function of the set $\{|\nabla T_k(u)|\le\tau\}$.
We have $(\nabla T_k(w_l)-\nabla T_k(u)\chi_\tau) \in \left(E_\varphi(\Omega)\right)^N$, and having in mind (56) and (58) we obtain

$$\begin{aligned}
\lim_{l\to\infty}\lim_{n\to\infty} I_1 &= \lim_{l\to\infty}\int_\Omega \eta_k\cdot(\nabla T_k(w_l)-\nabla T_k(u)\chi_\tau)\,dx\\
&= \int_\Omega \eta_k\cdot\nabla T_k(u)\chi_{\{|\nabla T_k(u)|>\tau\}}\,dx.
\end{aligned}$$

Since $|\eta_k\cdot\nabla T_k(u)| \in L^1(\Omega)$, by using Lebesgue's dominated convergence theorem, we obtain

$$\lim_{\tau\to\infty}\lim_{l\to\infty}\lim_{n\to\infty}\int_\Omega I_1 = 0. \quad (69)$$

On the other hand, we have $\bar{\varphi}$ satisfies the Δ_2-condition and in view of (15), (41) and Lebesgue dominated convergence theorem, we obtain

$$a(x,T_k(u_n),\nabla T_k(u)\chi_\tau) \longrightarrow a(x,T_k(u),\nabla T_k(u)\chi_\tau) \text{ strongly in } \left(E_{\bar{\varphi}}(\Omega)\right)^N \text{ as } n\to\infty.$$

It follows from (42) that

$$\begin{aligned}\lim_{n\to\infty} I_2 &= -\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_\tau)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_\tau)\ dx\\ &= -\int_\Omega a(x,T_k(u),\nabla T_k(u)\chi_\tau)\cdot\nabla T_k(u)\chi_{\{|\nabla T_k(u)|>\tau\}}\ dx\\ &= \int_\Omega a(x,T_k(u),0)\cdot\nabla T_k(u)\chi_{\{|\nabla T_k(u)|>\tau\}}\ dx\\ &= 0.\end{aligned} \tag{70}$$

By combining (68) and (69)$-$(70), we deduce that

$$\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_\tau))\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_\tau)\ dx \le \varepsilon_7(n,l,h,\tau),$$

where $\varepsilon_7(n,l,h,\tau) \longrightarrow 0$ as $n,\ l,\ h$ and τ respectively tends to infinity. Thus, we conclude that

$$\lim_{\tau\to\infty}\lim_{n\to\infty}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_\tau))\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_\tau)\ dx = 0.$$

In view of Lemma 4.7, we obtain

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega, \tag{71}$$

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{modularly in } W_0^1L_\varphi(\Omega), \tag{72}$$

and

$$a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n) \longrightarrow a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u) \text{ strongly in } L^1(\Omega). \tag{73}$$

Moreover, thanks to (41), (71) and Lemma 4.2, we deduce that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,T_k(u),\nabla T_k(u)) \text{ weakly in } \left(L_{\bar{\varphi}}(\Omega)\right)^N. \tag{74}$$

Step 5 : Passage to the limit

Let $v \in C_0^\infty(\Omega)$, and $S(\cdot) \in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp} S(\cdot) \subset [-M,M]$ for some $M > 0$ and let $n \geq M$. By taking $vS(u_n)$ as a test function for the approximate problem (14), we obtain

$$\begin{aligned}\int_\Omega a\left(x,T_n(u_n),\nabla u_n\right)\cdot\left(\nabla u_n S'(u_n)v+\nabla v S(u_n)\right)\,dx+\int_\Omega g_n\left(x,u_n\right)vS(u_n)\,dx\\ =\int_\Omega f_n(x)vS(u_n)\,dx+\int_\Omega \phi_n(u_n)\cdot\left(\nabla u_n S'(u_n)v+\nabla v S(u_n)\right)\,dx.\end{aligned} \tag{75}$$

Now, we pass to the limit on each term of the equality (75).
Firstly, we have $\operatorname{supp} S(\cdot) \subset [-M,M]$ then $S(u_n) = S(T_M(u_n))$, it follows that

$$\begin{aligned}&\int_\Omega a\left(x,T_n(u_n),\nabla u_n\right)\cdot\left(\nabla u_n S'(u_n)v+\nabla v S(u_n)\right)\,dx\\ &\quad=\int_\Omega a\left(x,T_M(u_n),\nabla T_M(u_n)\right)\cdot\left(\nabla T_M(u_n)S'(T_M(u_n))v+\nabla v S(T_M(u_n))\right)\,dx,\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \phi_n(u_n)\cdot\left(\nabla u_n S'(u_n)v+\nabla v S(u_n)\right)\,dx\\ &\quad=\int_\Omega \phi\left(T_M(u_n)\right)\cdot\left(\nabla T_M(u_n)S'(T_M(u_n))v+\nabla v S(T_M(u_n))\right)\,dx.\end{aligned}$$

In view of (41) and the Lemma 4.6 we have $S'(T_M(u_n))v \rightharpoonup S'(T_M(u))v$ weak$-*$ in $L^\infty(\Omega)$ and $S(T_M(u_n))\nabla v \longrightarrow S(T_M(u))\nabla v$ strongly in $\left(E_\varphi(\Omega)\right)^N$. By using (43) and (73)$-$(74), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega a\left(x,T_n(u_n),\nabla u_n\right)\cdot\left(\nabla u_n S'(u_n)v+\nabla v S(u_n)\right)\,dx\\ &\quad=\lim_{n\to\infty}\int_\Omega a\left(x,T_M(u_n),\nabla T_M(u_n)\right)\cdot\nabla T_M(u_n)S'(T_M(u_n))v\,dx\\ &\qquad+\lim_{n\to\infty}\int_\Omega a\left(x,T_M(u_n),\nabla T_M(u_n)\right)\cdot\nabla v S(T_M(u_n))\,dx\\ &\quad=\int_\Omega a\left(x,T_M(u),\nabla T_M(u)\right)\cdot\nabla T_M(u)S'(T_M(u))v\,dx\\ &\qquad+\int_\Omega a\left(x,T_M(u),\nabla T_M(u)\right)\cdot\nabla v S(T_M(u))\,dx\\ &\quad=\int_\Omega a\left(x,u,\nabla u\right)\cdot\left(\nabla u S'(u)v+\nabla v S(u)\right)\,dx.\end{aligned} \tag{76}$$

Moreover, we have $\phi\left(T_M(u_n)\right) \longrightarrow \phi\left(T_M(u)\right)$ strongly in $\left(E_{\bar{\varphi}}(\Omega)\right)^N$ and since $\nabla T_M(u_n) \rightharpoonup \nabla T_M(u)$ weakly in $\left(L_\varphi(\Omega)\right)^N$ as $n \to \infty$, then

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \phi(u_n)\cdot\left(\nabla u_n S'(u_n)v + \nabla v S(u_n)\right)\,dx \\
&\quad= \lim_{n\to\infty}\int_\Omega \phi\left(T_M(u_n)\right)\cdot\nabla T_M(u_n) S'(T_M(u_n))v\,dx \\
&\qquad+ \lim_{n\to\infty}\int_\Omega \phi\left(T_M(u_n)\right)\cdot\nabla v S(T_M(u_n))\,dx \\
&\quad= \int_\Omega \phi\left(T_M(u)\right)\cdot\nabla T_M(u) S'(T_M(u))v\,dx + \int_\Omega \phi\left(T_M(u)\right)\cdot\nabla v S(T_M(u))\,dx \\
&\quad= \int_\Omega \phi(u)\cdot\left(\nabla u S'(u)v + \nabla v S(u)\right)\,dx.
\end{aligned} \tag{77}$$

For the others terms of (75), we have $S(u_n)v = S\left(T_M(u_n)\right)v \rightharpoonup S\left(T_M(u)\right)v = S(u)v$ weak$-*$ in $L^\infty(\Omega)$ and since $f_n \longrightarrow f$ strongly in $L^1(\Omega)$, then

$$\lim_{n\to\infty}\int_\Omega f_n v S(u_n)\,dx = \int_\Omega f v S(u)\,dx. \tag{78}$$

Moreover, thanks to (53) we obtain

$$\lim_{n\to\infty}\int_\Omega g_n(x,u_n)\,v S(u_n)\,dx = \int_\Omega g(x,u)\,v S(u)\,dx. \tag{79}$$

By combining (75) and (76)$-$(79) we conclude that

$$\begin{aligned}
&\int_\Omega a(x,u,\nabla u)\cdot\left(\nabla u S'(u)v + \nabla v S(u)\right)\,dx + \int_\Omega g(x,u)v S(u)\,dx \\
&\quad= \int_\Omega f v S(u)\,dx + \int_\Omega \phi(u)\cdot\left(\nabla u S'(u)v + \nabla v S(u)\right)\,dx.
\end{aligned} \tag{80}$$

Remark 5.3. In the last step of this proof, we can take the function $v \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ instead of $C_0^\infty(\Omega)$.
Indeed, for $v \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, there exists a sequence $(v_l) \subset C_0^\infty(\Omega)$ such that $\sup_l \|v_l\|_{L^\infty(\Omega)} \leq C$ and

$$v_l \longrightarrow v \text{ modularly in } W_0^1 L_\varphi(\Omega) \quad \text{as } l \to \infty.$$

Moreover,

$$v_l \rightharpoonup v \text{ weakly in } W_0^1 L_\varphi(\Omega),$$

and

$$v_l \rightharpoonup v \text{ weak}-* \text{ in } L^\infty(\Omega) \quad \text{as } l \to \infty \qquad \text{for a supsequence.}$$

Thus, by taking $v = v_l$ in (80) and passing l to infinity, the inequality (80) remains true for any $v \in W_0^1 L_\varphi(\Omega)) \cap L^\infty(\Omega)$ and any $S(\cdot) \in W^{1,\infty}(\mathbb{R})$ with compact support.

Then, the proof of the Theorem 5.2 is completed.

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