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FEASIBLE SETS FOR VERTEX COLORINGS OF P_4-DESIGNS

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A P_4-design of order v is a system $\Sigma = (X, \mathcal{B})$ where X has v vertices and $\mathcal{B}$ is a set of copies of P_4, called blocks, decomposing the complete graph K_v on X. A BP_4-design is a P_4-design Σ endowed with a coloring of the vertices of Σ in such a way that in any block there are at least two vertices with the same color and two vertices with different colors. The feasible set $\Omega(\Sigma)$ of Σ is the set of integers k for which Σ is k-colorable. The minimum and maximum of $\Omega(\Sigma)$ are, respectively, the lower and upper chromatic number of Σ. In this paper the study of BP_4-designs is initiated, giving a lower and upper bound for feasible sets of BP_4-designs and showing the existence, for any admissible integer v, of BP_4-designs of order v with the largest possible feasible set. Some results are obtained for small values of v.

1. Introduction

Let $G = (V, E)$ a graph on n vertices and let K_v be the complete graph on v vertices. A G-design of order v is a couple $\Sigma = (X, \mathcal{B})$, where X is a set of v vertices and $\mathcal{B}$ is a set of copies of G, called *blocks*, decomposing the complete graph K_v having X as set of vertices. The *spectrum* of G-designs is the set of all integers v for which there exists a G-design of order v.

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In this paper we will consider P_4-designs, whose spectrum is well known (see [1]): a P_4-design of order v exists if and only if $v \equiv 0,1 \mod 3$, $v \geq 4$. We initiate the study of vertex colorings of P_4-designs, in the sense due to Voloshin's work on mixed hypergraphs (see [16–18]). A k-coloring of a G-design $\Sigma = (X, \mathcal{B})$ is a map $\varphi \colon X \to C$, where C is a set of k colors. A k-coloring is *strict* if exactly k colors are used. From now on, we assume that all our colorings are strict. It is possible to consider these type of colorings:

- colorings such that any block of $\mathcal{B}$ contains at least two vertices of a common color; if Σ admits such a coloring, we call it a CG-design;
- colorings such that any block of $\mathcal{B}$ contains at least two vertices of different colors; if Σ admits such a coloring, we call it a DG-design (this corresponds to the usual definition of a coloring of a design);
- colorings for which Σ is, at the same time, a CG and a DG-design; if Σ admits such a coloring, we call it a BG-design: this is the type of coloring we are interested in.

Given a G-design $\Sigma = (X, \mathcal{B})$, the *feasible set* of Σ is:

$$\Omega(\Sigma) = \{k \mid \exists \text{ a } k\text{-coloring of } \Sigma\}.$$

The system Σ is *uncolorable* if $\Omega(\Sigma) = \emptyset$. If Σ is colorable, the minimum and the maximum of $\Omega(\Sigma)$ are the *lower* and *upper chromatic number of* Σ and we denote them by, respectively, $\chi(\Sigma)$ and $\overline{\chi}(\Sigma)$.

Given a colorable G-design $\Sigma = (X, \mathcal{B})$ and denoted by X_i the set of vertices of X with color i for any $i = 1, \dots, k$, X_i is called *color class*.

Voloshin colorings has been considered for P_3-designs (see [6–8]), Steiner Triple Systems (see [4, 5, 11, 13–15]), $S(2,4,v)$ (see [9]), $SQS(v)$ (see [12, 14]) and also in the case of hypergraph designs (see [2, 3]).

In this paper we determine sharp bounds for the lower and upper chromatic numbers of colorable BP_4-designs and we show that, for all admissible values v, there exists a BP_4-design with the largest possible feasible set. In the last section we also prove a few results for BP_4-designs of small orders.

2. Feasible sets for BP_4-designs

Let $T = \{a, b, c, d\}$. We denote by $\langle a, b, c, d \rangle$ the path P_4 with T as set of vertices and with edges $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$.

For any $v \in \mathbb{N}$ let:

$$\chi_v = \begin{cases} \left\lfloor \dfrac{3v+2}{4} \right\rfloor & \text{if } v \equiv 0,1,3 \mod 4 \\ \dfrac{3v-2}{4} & \text{if } v \equiv 2 \mod 4. \end{cases}$$

Theorem 2.1. *Let $\Sigma = (X, \mathcal{B})$ be a colorable BP_4-design of order v. Then $2 \leq \chi(\Sigma) \leq \overline{\chi}(\Sigma) \leq \chi_v$.*

Proof. Let $X_1, \ldots, X_s$ be color classes in a coloring of Σ and let $|X_i| = n_i$ for $i = 1, \ldots, s$. Suppose that $n_i = 1$ for $i = 1, \ldots, r$ and $n_i \geq 2$ for $i = r+1, \ldots, s$, for some $r < s$, so that $v = r + \sum_{i=r+1}^{s} n_i$. Let also $X_i = \{x_i\}$ for $i = 1, \ldots, r$.

Any block of $\mathcal{B}$ must contain at most one of the edges $\{x_i, x_j\}$, for $i, j = 1, \ldots, r$, $i \neq j$. So, in any block containing one of these edges there exist $y, z \in X_k$, $y \neq z$, for some $k \in \{r+1, \ldots, s\}$, such that the block is of one of the following types:

1. $< x_i, x_j, y, z >$,

2. $< y, x_i, x_j, z >$.

Let a denote the number of blocks of type 1 and b denote the number of blocks of type 2, for any i, j. Then we have the conditions:

$$a + b = \binom{r}{2}$$
$$a \leq \sum_{i=r+1}^{s} \binom{n_i}{2}$$
$$b \leq \frac{r(v-r)}{2}.$$

This clearly implies that:

$$\binom{r}{2} \leq \sum_{i=r+1}^{s} \binom{n_i}{2} + \frac{r(v-r)}{2}. \tag{1}$$

Now, given $m_1, \ldots, m_n$ integers such that $n \geq 1$ and $m_i \geq 2$ for any i, it is easy to prove by induction the following inequality:

$$\sum_{i=1}^{n} \binom{m_i}{2} \leq \binom{\sum_{i=1}^{n} m_i - 2n + 2}{2} + n - 1.$$

So by equation (1) we get:

$$\binom{r}{2} \leq \binom{\sum_{i=r+1}^{s} n_i - 2(s-r)+2}{2} + s - r - 1 + \frac{r(v-r)}{2} =$$
$$= \binom{v+r-2s+2}{2} + s - r - 1 + \frac{r(v-r)}{2}.$$

This implies:

$$r^2 + r(4s-3v-2) \leq v^2 + 4s^2 - 4sv + 3v - 4s. \tag{2}$$

Suppose that $s \geq \frac{3v+2}{4}$. Then, since $v = r + \sum_{i=r+1}^{s} n_i$ and $n_i \geq 2$ for $i = r+1,\dots,n$, we have $v \geq 2s - r$, so that $r \geq 2s - v > 0$ and equation (2) implies:

$$(2s-v)^2 + (4s-3v-2)(2s-v) \leq v^2 + 4s^2 - 4sv + 3v - 4s$$
$$\Rightarrow 8s^2 - 10sv + 3v^2 - v \leq 0.$$

Since $s \geq \frac{3v+2}{4}$ and $\frac{3v+2}{4} > \frac{5}{8}v$, we get easily a contradiction:

$$8\left(\frac{3v+2}{4}\right)^2 - 5\frac{3v+2}{2}v + 3v^2 - v \leq 0 \Rightarrow 2 \leq 0.$$

This shows that it must be $s < \frac{3v+2}{4}$, which proves the statement. □

Now we prove that the above bounds are sharp. More precisely, we prove that:

Theorem 2.2. *For any $v \in \mathbb{N}$, $v \equiv 0,1 \mod 3$ and $v \geq 4$, there exists a BP_4-design of order v having as feasible set the complete interval of integers $[2, \chi_v]$.*

Proof. Let $\mathbf{v = 12h}$, for some $h \in \mathbb{N}$, $h \geq 1$. Let $X = \{x_1,\dots,x_{6h}\}$ and $Y = \{y_1,\dots,y_{6h}\}$ be disjoint sets. We want to construct a BP_4 design of order $12h$ having $X \cup Y$ as vertex set and satisfying the conditions of the statement.

Let $\mathcal{B}_{12h}$ be the set of the following blocks:

- $\langle y_{i+k}, x_i, x_{i+2k}, y_{i+k+3h}\rangle$ for $i = 1,\dots,6h$ and $2k \in \{1,\dots,3h-1\}$;
- $\langle y_{i+k+3h}, x_i, x_{i+2k+1}, y_{i+k}\rangle$ for $i = 1,\dots,6h$ and $2k+1 \in \{1,\dots,3h-1\}$;
- $\langle y_i, x_i, x_{i+3h}, y_{i+3h}\rangle$ for $i = 1,\dots,3h$;
- $\langle x_{i-\bar{k}}, y_i, y_{i+3h}, x_{i+3h-\bar{k}}\rangle$ for $i = 1,\dots,3h$ and with

$$\bar{k} = \begin{cases} \dfrac{3h}{2} & \text{if } h \text{ is even} \\ \dfrac{9h-1}{2} & \text{if } h \text{ is odd} \end{cases}$$

- $\langle y_{i+k}, y_i, y_{i+k+h}, y_{i+3h}\rangle$ for $i = 1,\ldots,6h$ and $k = h+1,\ldots,2h-1$ (in the case $h \geq 2$);
- $\langle y_i, y_{i+h}, y_{i+3h}, y_{i+2h}\rangle$ for $i = 1,\ldots,2h$;
- $\langle y_{i+5h}, y_{i+3h}, y_{i+4h}, y_i\rangle$ for $i = h+1,\ldots,2h$;
- $\langle y_{i+h}, y_{i+3h}, y_{i+4h}, y_i\rangle$ for $i = 2h+1,\ldots,3h$.

It is easy to see that $\Sigma = (X \cup Y, \mathcal{B}_{12h})$ is a P_4-design of order $12h$. Moreover, it is s-colorable for any $s \in [2, \chi_{12h}] = [2, 9h]$. Indeed, it is not difficult to see that:

- for $s = 2$ we can take as color classes the sets $\{x_1,\ldots,x_{6h}, y_1,\ldots,y_{3h}\}$ and $\{y_{3h+1},\ldots,y_{6h}\}$;
- for $s = 3$ we can take as color classes $\{x_1,\ldots,x_{6h}\}$, $\{y_1,\ldots,y_{3h}\}$ and $\{y_{3h+1},\ldots,y_{6h}\}$;
- for $s = 4,\ldots,9h$ we can take as color classes $A_1,\ldots,A_m$, $B_1,\ldots,B_p$, $C_1,\ldots,C_q$, $D_1,\ldots,D_r$, with $m \in \{1,\ldots,6h\}$, $p,q,r \in \{1,\ldots,h\}$ and $m+p+q+r = s$, where:
  - $A_1,\ldots,A_m$ is any partition of $\{x_1,\ldots,x_{6h}\}$;
  - $B_1,\ldots,B_p$ is any partition of $\{y_1,\ldots,y_h\} \cup \{y_{3h+1},\ldots,y_{4h}\}$ such that $\{y_i, y_{3h+i}\}$ for any $i = 1,\ldots,h$ is contained in some of the B_j;
  - $C_1,\ldots,C_q$ is any partition of $\{y_{h+1},\ldots,y_{2h}\} \cup \{y_{4h+1},\ldots,y_{5h}\}$ such that $\{y_i, y_{3h+i}\}$ for any $i = h+1,\ldots,2h$ is contained in some of the C_j;
  - $D_1,\ldots,D_r$ is any partition of $\{y_{2h+1},\ldots,y_{3h}\} \cup \{y_{5h+1},\ldots,y_{6h}\}$ such that $\{y_i, y_{3h+i}\}$ for any $i = 2h+1,\ldots,3h$ is contained in some of the D_j.

Let $\mathbf{v} = \mathbf{12h+1}$, for some $h \in \mathbb{N}$, $h \geq 1$. Let $X = \{x_1,\ldots,x_{6h+1}\}$ and $Y = \{y_1,\ldots,y_{6h}\}$ be disjoint sets. We want to construct a BP_4 design of order $12h+1$ having $X \cup Y$ as vertex set and satisfying the conditions of the statement.

Let $\mathcal{B}_{12h+1}$ be the set of the following blocks:

- $\langle y_k, x_i, x_{i+k}, y_{k+3h}\rangle$ for $i = 1,\ldots,6h+1$ and $k = 1,\ldots,3h$;
- $\langle y_{i+k}, y_i, y_{i+k+h}, y_{i+3h}\rangle$ for $i = 1,\ldots,6h$ and $k = h+1,\ldots,2h-1$ (in the case $h \geq 2$);
- $\langle y_{i+2h}, y_i, y_{i+3h}, y_{i+5h}\rangle$ for $i = 1,\ldots,3h$;
- $\langle y_i, y_{i+h}, y_{i+2h}, y_{i+3h}\rangle$ for $i = 1,\ldots,h$ and $i = 3h+1,\ldots,4h$.

It is easy to see that $\Sigma = (X \cup Y, \mathcal{B}_{12h+1})$ is a P_4-design of order $12h+1$. Moreover, it is s-colorable for any $s \in [2, \chi_{12h+1}] = [2, 9h+1]$. Indeed, it is not difficult to see that:

- for $s = 2$ we can take as color classes the sets $\{x_1, \ldots, x_{6h+1}, y_1, \ldots, y_{3h}\}$ and $\{y_{3h+1}, \ldots, y_{6h}\}$;
- for $s = 3$ we can take as color classes $\{x_1, \ldots, x_{6h+1}\}$, $\{y_1, \ldots, y_{3h}\}$ and $\{y_{3h+1}, \ldots, y_{6h}\}$;
- for $s = 4, \ldots, 9h+1$ we can take as color classes $A_1, \ldots, A_m$, $B_1, \ldots, B_p$, $C_1, \ldots, C_q$, $D_1, \ldots, D_r$, with $m \in \{1, \ldots, 6h+1\}$, $p, q, r \in \{1, \ldots, h\}$ and $m + p + q + r = s$, where:
  - $A_1, \ldots, A_m$ is any partition of $\{x_1, \ldots, x_{6h+1}\}$;
  - $B_1, \ldots, B_p$ is any partition of $\{y_1, \ldots, y_h\} \cup \{y_{3h+1}, \ldots, y_{4h}\}$ such that $\{y_i, y_{3h+i}\}$ for any $i = 1, \ldots, h$ is contained in some of the B_j;
  - $C_1, \ldots, C_q$ is any partition of $\{y_{h+1}, \ldots, y_{2h}\} \cup \{y_{4h+1}, \ldots, y_{5h}\}$ such that $\{y_i, y_{3h+i}\}$ for any $i = h+1, \ldots, 2h$ is contained in some of the C_j;
  - $D_1, \ldots, D_r$ is any partition of $\{y_{2h+1}, \ldots, y_{3h}\} \cup \{y_{5h+1}, \ldots, y_{6h}\}$ such that $\{y_i, y_{3h+i}\}$ for any $i = 2h+1, \ldots, 3h$ is contained in some of the D_j.

Let $\mathbf{v} = \mathbf{12h+3}$, for some $h \in \mathbb{N}$, $h \geq 1$. Let $X = \{x_1, \ldots, x_{6h+1}\}$ and $Y = \{y_1, \ldots, y_{6h}\}$ be disjoint sets and let $\Sigma = (X \cup Y, \mathcal{B}_{12h+1})$ be the previous BP_4-design of order $12h+1$. Let us consider two elements $\infty_1, \infty_2 \notin X \cup Y$, $\infty_1 \neq \infty_2$. Let $\mathcal{C}$ be the set of the following blocks:

- $\langle y_i, \infty_1, y_{i+h}, \infty_2 \rangle$ for $i = 1, \ldots, h$ and $i = 3h+1, \ldots, 4h$;
- $\langle y_i, \infty_2, y_{i+2h}, \infty_1 \rangle$ for $i = 1, \ldots, h$ and $i = 3h+1, \ldots, 4h$;
- $\langle x_{3i+1}, \infty_1, x_{3i+2}, \infty_2 \rangle$ for $i = 0, \ldots, 2h-2$;
- $\langle x_{3i+1}, \infty_2, x_{3i+3}, \infty_1 \rangle$ for $i = 0, \ldots, 2h-2$;
- $\langle x_{6h-2}, \infty_1, x_{6h-1}, \infty_2 \rangle$;
- $\langle x_{6h+1}, \infty_2, x_{6h}, \infty_1 \rangle$;
- $\langle x_{6h+1}, \infty_1, \infty_2, x_{6h-2} \rangle$.

Then it is easy to see that $\Sigma' = (X \cup Y \cup \{\infty_1, \infty_2\}, \mathcal{B}_{12h+1} \cup \mathcal{C})$ is a P_4-design of order $12h+3$. Moreover, it is s-colorable for any $s \in [2, \chi_{12h+3}] = [2, 9h+2]$. Indeed, it is not difficult to see that:

- for $s = 2$ we can take as color classes $\{x_1,\dots,x_{6h+1},y_1,\dots,y_{3h},\infty_1\}$ and $\{y_{3h+1},\dots,y_{6h},\infty_2\}$;
- for $s = 3$ we can take as color classes $\{x_1,\dots,x_{6h+1}\}$, $\{y_1,\dots,y_{3h},\infty_1\}$ and $\{y_{3h+1},\dots,y_{6h},\infty_2\}$;
- for $s = 4,\dots,9h+2$ we can take as color classes $A_1,\dots,A_m$, $B_1,\dots,B_p$, $C_1,\dots,C_q$, $D_1,\dots,D_r$, with $m \in \{1,\dots,6h+1\}$, $p \in \{1,\dots,h+1\}$ $q,r \in \{1,\dots,h\}$ and $m+p+q+r = s$, where:
  - $A_1,\dots,A_m$ is any partition of $\{x_1,\dots,x_{6h+1}\}$;
  - $B_1,\dots,B_p$ is any partition of $\{y_1,\dots,y_h,\infty_1\} \cup \{y_{3h+1},\dots,y_{4h},\infty_2\}$ such that $\{y_i,y_{3h+i}\}$ for any $i = 1,\dots,h$ and $\{\infty_1,\infty_2\}$ are contained in some of the B_j;
  - $C_1,\dots,C_q$ is any partition of $\{y_{h+1},\dots,y_{2h}\} \cup \{y_{4h+1},\dots,y_{5h}\}$ such that $\{y_i,y_{3h+i}\}$ for any $i = h+1,\dots,2h$ is contained in some of the C_j;
  - $D_1,\dots,D_r$ is any partition of $\{y_{2h+1},\dots,y_{3h}\} \cup \{y_{5h+1},\dots,y_{6h}\}$ such that $\{y_i,y_{3h+i}\}$ for any $i = 2h+1,\dots,3h$ is contained in some of the D_j.

Let $\mathbf{v} = \mathbf{12h+4}$, for some $h \in \mathbb{N}$, $h \geq 0$.

Let $h = 0$, so that $v = 4$. In this case, it is easy to see that any P_4-design of order 4 is a BP_4-design which is s-colorable for any $s \in [2,\chi_4] = [2,3]$.

Let $h \geq 1$ and let $X = \{x_1,\dots,x_{6h+2}\}$ and $Y = \{y_1,\dots,y_{6h+2}\}$ be disjoint sets. We want to construct a BP_4 design of order $12h+4$ having $X \cup Y$ as vertex set and satisfying the conditions of the statement.

Let $\mathcal{B}_{12h+4}$ be the set of the following blocks:

- $\langle y_{i+k},x_i,x_{i+2k+1},y_{i+k+3h+1}\rangle$ for $i = 1,\dots,6h+2$ and $2k+1 \in \{1,\dots,3h\}$;
- $\langle y_{i+k+3h+1},x_i,x_{i+2k},y_{i+k}\rangle$ for $i = 1,\dots,6h+2$ and $2k \in \{1,\dots,3h\}$;
- $\langle y_{i+3h+1},x_i,x_{i+3h+1},y_i\rangle$ for $i = 1,\dots,3h+1$;
- $\langle x_{i-\bar{k}},y_i,y_{i+3h+1},x_{i+3h+1-\bar{k}}\rangle$ for $i = 1,\dots,3h+1$ and with:

$$\bar{k} = \begin{cases} \dfrac{3}{2}h & \text{if } h \text{ is even} \\[2ex] \dfrac{9h+3}{2} & \text{if } h \text{ is odd} \end{cases}$$

- $\langle y_{i+k},y_i,y_{i+k+h},y_{i+3h+1}\rangle$ for $i = 1,\dots,6h+2$ and $k = h+1,\dots,2h$.

It is easy to see that $\Sigma = (X \cup Y, \mathcal{B}_{12h+4})$ is a P_4-design of order $12h$. Moreover, it is s-colorable for any $s \in [2, \chi_{12h+4}] = [2, 9h+3]$. Indeed, it is not difficult to see that:

- for $s = 2$ we can take as color classes $\{x_1, \ldots, x_{6h+2}, y_1, \ldots, y_{3h+1}\}$ and $\{y_{3h+2}, \ldots, y_{6h+2}\}$;
- for $s = 3$ we can take as color classes $\{x_1, \ldots, x_{6h+2}\}$, $\{y_1, \ldots, y_{3h+1}\}$ and $\{y_{3h+2}, \ldots, y_{6h+2}\}$;
- for $s = 4, \ldots, 9h+3$ we can take as color classes $A_1, \ldots, A_m$, $B_1, \ldots, B_p$, $C_1, \ldots, C_q$, $D_1, \ldots, D_r$, with $m \in \{1, \ldots, 6h+2\}$, $p \in \{1, \ldots, h+1\}$ $q, r \in \{1, \ldots, h\}$ and $m + p + q + r = s$, where:
  - $A_1, \ldots, A_m$ is any partition of $\{x_1, \ldots, x_{6h+2}\}$;
  - $B_1, \ldots, B_p$ is any partition of $\{y_1, \ldots, y_{h+1}\} \cup \{y_{3h+2}, \ldots, y_{4h+2}\}$ such that $\{y_i, y_{3h+1+i}\}$ for any $i = 1, \ldots, h+1$ is contained in some of the B_j;
  - $C_1, \ldots, C_q$ is any partition of $\{y_{h+2}, \ldots, y_{2h+1}\} \cup \{y_{4h+3}, \ldots, y_{5h+2}\}$ such that $\{y_i, y_{3h+1+i}\}$ for any $i = h+2, \ldots, 2h+1$ is contained in some of the C_j;
  - $D_1, \ldots, D_r$ is any partition of $\{y_{2h+2}, \ldots, y_{3h+1}\} \cup \{y_{5h+3}, \ldots, y_{6h+2}\}$ such that $\{y_i, y_{3h+1+i}\}$ for any $i = 2h+2, \ldots, 3h+1$ is contained in some of the D_j.

Let $\mathbf{v = 12h + 6}$, for some $h \in \mathbb{N}$. If $h = 0$, then consider the system $\Sigma = (\{0, 1, 2, 3, 4\} \cup \{\infty\}, \mathcal{B})$, where $\infty \notin \{0, 1, 2, 3, 4\}$ and $\mathcal{B}$ is the set of blocks $\langle \infty, i, i+1, i+3 \rangle$ for $i = 0, 1, 2, 3, 4$. Then, Σ is clearly 2 and 3-colorable. It is also 4-colorable, since we can take $\{\infty, 0, 1\}$, $\{2\}$,$\{3\}$ and $\{4\}$ as color classes.

Let $h \geq 1$. Let $X = \{x_1, \ldots, x_{6h+2}\}$ and $Y = \{y_1, \ldots, y_{6h+2}\}$ be disjoint sets and let $\Sigma = (X \cup Y, \mathcal{B}_{12h+4})$ be the previous BP_4-design of order $12h+4$. Let us consider two elements $\infty_1, \infty_2 \notin X \cup Y$, $\infty_1 \neq \infty_2$. Let $\mathcal{C}$ be the set of the following blocks:

- $\langle y_i, \infty_1, y_{i+h+1}, \infty_2 \rangle$ for $i = 1, \ldots, h$ and $i = 3h+2, \ldots, 4h+1$ (in the case $h \geq 1$);
- $\langle y_i, \infty_2, y_{i+2h+1}, \infty_1 \rangle$ for $i = 1, \ldots, h$ and $i = 3h+2, \ldots, 4h+1$ (in the case $h \geq 1$);
- $\langle x_i, \infty_1, x_{i+h+1}, \infty_2 \rangle$ for $i = 1, \ldots, h$ and $i = 3h+2, \ldots, 4h+1$ (in the case $h \geq 1$);

- $\langle x_i, \infty_2, x_{i+2h+1}, \infty_1 \rangle$ for $i = 1, \ldots, h$ and $i = 3h+2, \ldots, 4h+1$ (in the case $h \geq 1$);
- $\langle x_{h+1}, \infty_1, \infty_2, x_{4h+2} \rangle$;
- $\langle \infty_1, y_{h+1}, \infty_2, x_{h+1} \rangle$;
- $\langle \infty_2, y_{4h+2}, \infty_1, x_{4h+2} \rangle$.

Then it is easy to see that $\Sigma' = (X \cup Y \cup \{\infty_1, \infty_2\}, \mathcal{B}_{12h+4} \cup \mathcal{C})$ is a P_4-design of order $12h+6$. Moreover, it is s-colorable for any $s \in [2, \chi_{12h+6}] = [2, 9h+4]$. Indeed, it is not difficult to see that:

- for $s = 2$ we can take as color classes $\{x_1, \ldots, x_{6h+2}, y_1, \ldots, y_{3h+1}, \infty_1\}$ and $\{y_{3h+2}, \ldots, y_{6h+2}, \infty_2\}$;
- for $s = 3$ we can take as color classes $\{x_1, \ldots, x_{6h+2}\}$, $\{y_1, \ldots, y_{3h+1}, \infty_1\}$ and $\{y_{3h+2}, \ldots, y_{6h+2}, \infty_2\}$;
- for $s = 4, \ldots, 9h+4$ we can take as color classes $A_1, \ldots, A_m$, $B_1, \ldots, B_p$, $C_1, \ldots, C_q$, $D_1, \ldots, D_r$, with $m \in \{1, \ldots, 6h+2\}$, $p, q \in \{1, \ldots, h+1\}$, $r \in \{1, \ldots, h\}$ and $m+p+q+r = s$, where:
  - $A_1, \ldots, A_m$ is any partition of $\{x_1, \ldots, x_{6h+2}\}$;
  - $B_1, \ldots, B_p$ is any partition of $\{y_1, \ldots, y_{h+1}\} \cup \{y_{3h+2}, \ldots, y_{4h+2}\}$ such that $\{y_i, y_{3h+1+i}\}$ for any $i = 1, \ldots, h+1$ is contained in some of the B_j;
  - $C_1, \ldots, C_q$ is any partition of

  $$\{y_{h+2}, \ldots, y_{2h+1}, \infty_1\} \cup \{y_{4h+3}, \ldots, y_{5h+2}, \infty_2\}$$

  such that $\{y_i, y_{3h+1+i}\}$ for any $i = h+2, \ldots, 2h+1$ and $\{\infty_1, \infty_2\}$ are contained in some of the C_j;
  - $D_1, \ldots, D_r$ is any partition of $\{y_{2h+2}, \ldots, y_{3h+1}\} \cup \{y_{5h+3}, \ldots, y_{6h+2}\}$ such that $\{y_i, y_{3h+1+i}\}$ for any $i = 2h+2, \ldots, 3h+1$ is contained in some of the D_j.

Let $\mathbf{v} = \mathbf{12h+7}$, for some $h \in \mathbb{N}$. Let $h = 0$ and let $X = \{x_1, x_2, x_3\}$ and $Z = \{z_1, z_2, z_3, z_4\}$ be disjoint sets. Let $\Sigma = (Z, \mathcal{B})$ a P_4-design of order 4 and let $\mathcal{C}$ be the set of the following blocks:

- $\langle z_1, x_1, x_2, z_2 \rangle$;
- $\langle z_2, x_1, x_3, z_1 \rangle$;

- $\langle z_1, x_2, x_3, z_2 \rangle$;
- $\langle x_1, z_3, x_2, z_4 \rangle$;
- $\langle x_1, z_4, x_3, z_3 \rangle$.

Then $\Sigma' = (X \cup Z, \mathcal{B} \cup \mathcal{C})$ is a P_4-design which is r-colorable for $r = 2,3,4,5$. Indeed, we can take as color classes:

- $\{x_1, z_1, z_2\}$ and $\{x_2, x_3, z_3, z_4\}$ for $r = 2$;
- $\{x_1, x_2, x_3\}$, $\{z_1, z_2\}$ and $\{z_3, z_4\}$ for $r = 3$;
- $\{x_1, x_2\}$, $\{x_3\}$, $\{z_1, z_2\}$, $\{z_3, z_4\}$ for $r = 4$;
- $\{x_1\}$, $\{x_2\}$, $\{x_3\}$, $\{z_1, z_2\}$ and $\{z_3, z_4\}$ for $r = 5$.

Let $h \geq 1$. Let $X = \{x_1, \ldots, x_{6h+3}\}$, $Y = \{y_1, \ldots, y_{6h}\}$ and $Z = \{z_1, z_2, z_3, z_4\}$ be pairwise disjoint sets and let $\Sigma = (Z, \mathcal{B})$ be a P_4-design of order 4. Let $\mathcal{C}$ be the set of the following blocks:

- $\langle y_k, x_i, x_{i+k}, y_{k+3h} \rangle$ for $i = 1, \ldots, 6h+3$ and $k = 1, \ldots, 3h$;
- $\langle z_1, x_i, x_{i+3h+1}, z_2 \rangle$ for $i = 1, \ldots, 6h+3$;
- $\langle x_{3i+1}, z_3, x_{3i+2}, z_4 \rangle$ for $i = 0, \ldots, 2h$;
- $\langle x_{3i+1}, z_4, x_{3i+3}, z_3 \rangle$ for $i = 0, \ldots, 2h$;
- $\langle y_i, z_{2j+1}, y_{i+h}, z_{2j+2} \rangle$ for $j = 0, 1$, $i = 1, \ldots, h$ and $i = 3h+1, \ldots, 4h$;
- $\langle y_i, z_{2j+2}, y_{i+2h}, z_{2j+1} \rangle$ for $j = 0, 1$, $i = 1, \ldots, h$ and $i = 3h+1, \ldots, 4h$;
- $\langle y_{i+k}, y_i, y_{i+k+h}, y_{i+3h} \rangle$ for $i = 1, \ldots, 6h$ and $k = h+1, \ldots, 2h-1$;
- $\langle y_{i+2h}, y_i, y_{i+3h}, y_{i+5h} \rangle$ for $i = 1, \ldots, 3h$;
- $\langle y_i, y_{i+h}, y_{i+2h}, y_{i+3h} \rangle$ for $i = 1, \ldots, h$ and $i = 3h+1, \ldots, 4h$.

Then it is easy to see that $\Sigma' = (X \cup Y \cup Z, \mathcal{B} \cup \mathcal{C})$ is a P_4-design of order $12h+7$. Moreover, it is s-colorable for any $s \in [2, \chi_{12h+7}] = [2, 9h+5]$. Indeed, it is not difficult to see that:

- for $s = 2$ we can take as color classes $\{x_1, \ldots, x_{6h+3}, y_1, \ldots, y_{3h}, z_1, z_3\}$ and $\{y_{3h+1}, \ldots, y_{6h}, z_2, z_4\}$;
- for $s = 3$ we can take as color classes $\{x_1, \ldots, x_{6h+3}\}$, $\{y_1, \ldots, y_{3h}, z_1, z_3\}$ and $\{y_{3h+1}, \ldots, y_{6h}, z_2, z_4\}$;

- for $s = 4, \dots, 9h+5$ we can take as color classes $A_1, \dots, A_m$, $B_1, \dots, B_p$, $C_1, \dots, C_q$, $D_1, \dots, D_r$, with $m \in \{1, \dots, 6h+3\}$, $p, q \in \{1, \dots, h+1\}$, $r \in \{1, \dots, h\}$ and $m+p+q+r = s$, where:
  - $A_1, \dots, A_m$ is any partition of $\{x_1, \dots, x_{6h+3}\}$;
  - $B_1, \dots, B_p$ is any partition of $\{y_1, \dots, y_h\} \cup \{y_{3h+1}, \dots, y_{4h}\} \cup \{z_1, z_2\}$ such that $\{y_i, y_{3h+i}\}$ for any $i = 1, \dots, h$ and $\{z_1, z_2\}$ are contained in some of the B_j;
  - $C_1, \dots, C_q$ is any partition of

$$\{y_{h+1}, \dots, y_{2h}\} \cup \{y_{4h+1}, \dots, y_{5h}\} \cup \{z_3, z_4\}$$

    such that $\{y_i, y_{3h+i}\}$ for any $i = h+1, \dots, 2h$ and $\{z_3, z_4\}$ are contained in some of the C_j;
  - $D_1, \dots, D_r$ is any partition of $\{y_{2h+1}, \dots, y_{3h}\} \cup \{y_{5h+1}, \dots, y_{6h}\}$ such that $\{y_i, y_{3h+i}\}$ for any $i = 2h+1, \dots, 3h$ is contained in some of the D_j.

Let $\mathbf{v} = \mathbf{12h+9}$, for some $h \in \mathbb{N}$. Let us consider two disjoint sets $X = \{x_1, \dots, x_{6h+5}\}$ and $Y = \{y_1, \dots, y_{6h+4}\}$. We want to construct a BP_4 design of order $12h+9$ having $X \cup Y$ as vertex set and satisfying the conditions of the statement.

Let $\mathcal{B}_{12h+9}$ be the set of the following blocks:

- $\langle y_k, x_i, x_{i+k}, y_{k+3h+2} \rangle$ for $i = 1, \dots, 6h+5$ and $k = 1, \dots, 3h+2$;
- $\langle y_{i+k}, y_i, y_{i+k+h}, y_{i+3h+2} \rangle$ for $i = 1, \dots, 6h+4$ and $k = h+2, \dots, 2h+1$ (in the case $h \geq 1$);
- $\langle y_{i+h+1}, y_i, y_{i+3h+2}, y_{i+4h+3} \rangle$ for $i = 1, \dots, 3h+2$.

It is easy to see that $\Sigma = (X \cup Y, \mathcal{B}_{12h+9})$ is a P_4-design of order $12h+9$. Moreover, it is s-colorable for any $s \in [2, \chi_{12h+9}] = [2, 9h+7]$. Indeed, it is not difficult to see that:

- for $s = 2$ we can take as color classes $\{x_1, \dots, x_{6h+5}, y_1, \dots, y_{3h+2}\}$ and $\{y_{3h+3}, \dots, y_{6h+4}\}$;
- for $s = 3$ we can take as color classes $\{x_1, \dots, x_{6h+5}\}$, $\{y_1, \dots, y_{3h+2}\}$ and $\{y_{3h+3}, \dots, y_{6h+4}\}$;
- for $s = 4, \dots, 9h+7$ we can take as color classes $A_1, \dots, A_m$, $B_1, \dots, B_p$, $C_1, \dots, C_q$, $D_1, \dots, D_r$, with $m \in \{1, \dots, 6h+5\}$, $p, q \in \{1, \dots, h+1\}$, $r \in \{1, \dots, h\}$ and $m+p+q+r = s$, where:

– $A_1,\dots,A_m$ is any partition of $\{x_1,\dots,x_{6h+5}\}$;
– $B_1,\dots,B_p$ is any partition of $\{y_1,\dots,y_{h+1}\}\cup\{y_{3h+3},\dots,y_{4h+3}\}$ such that $\{y_i,y_{3h+2+i}\}$ for any $i=1,\dots,h+1$ is contained in some of the B_j;
– $C_1,\dots,C_q$ is any partition of $\{y_{h+2},\dots,y_{2h+2}\}\cup\{y_{4h+4},\dots,y_{5h+4}\}$ such that $\{y_i,y_{3h+2+i}\}$ for any $i=h+2,\dots,2h+2$ is contained in some of the C_j;
– $D_1,\dots,D_r$ is any partition of $\{y_{2h+3},\dots,y_{3h+2}\}\cup\{y_{5h+5},\dots,y_{6h+4}\}$ such that $\{y_i,y_{3h+2+i}\}$ for any $i=2h+3,\dots,3h+2$ is contained in some of the D_j.

Let $\mathbf{v}=\mathbf{12h+10}$, for some $h\in\mathbb{N}$. Let us consider three pairwise disjoint sets, $X=\{x_1,\dots,x_{6h+4}\}$, $Y=\{y_1,\dots,y_{6h+4}\}$ and $Z=\{z_1,z_2\}$. Let $\mathcal{B}$ be the set of the following blocks:

- $\langle y_{i+k},x_i,x_{i+2k},y_{i+k+3h+2}\rangle$ for $i=1,\dots,6h+4$ and $2k\in\{1,\dots,3h+1\}$ (in the case $h\geq 1$);
- $\langle y_{i+k+3h+2},x_i,x_{i+2k+1},y_{i+k}\rangle$ for $i=1,\dots,6h+4$ and $2k+1\in\{1,\dots,3h+1\}$;
- $\langle y_i,x_i,x_{i+3h+2},y_{i+3h+2}\rangle$ for $i=1,\dots,3h+2$;
- $\langle x_{i-\bar{k}},y_i,y_{i+3h+2},x_{i+3h+2-\bar{k}}\rangle$ for $i=1,\dots,3h+2$ and with:

$$\bar{k}=\begin{cases}\dfrac{3}{2}h+1 & \text{if } h \text{ is even}\\[2ex] \dfrac{9h+5}{2} & \text{if } h \text{ is odd}\end{cases}$$

- $\langle y_{i+k},y_i,y_{i+k+h+1},y_{i+3h+2}\rangle$ for $i=1,\dots,6h+4$ and $k=h+1,\dots,2h$ (in the case $h\geq 1$);
- $\langle z_1,y_i,y_{i+2h+1},z_2\rangle$ for $i=1,\dots,6h+4$;
- $\langle x_{3i+1},z_1,x_{3i+2},z_2\rangle$ for $i=0,\dots,2h-1$ (in the case $h\geq 1$);
- $\langle x_{3i+1},z_2,x_{3i+3},z_1\rangle$ for $i=0,\dots,2h-1$ (in the case $h\geq 1$);
- $\langle x_{6h+1},z_1,z_2,x_{6h+2}\rangle$;
- $\langle x_{6h+2},z_1,x_{6h+3},z_2\rangle$;
- $\langle x_{6h+1},z_2,x_{6h+4},z_1\rangle$.

Then it is easy to see that $\Sigma' = (X \cup Y \cup Z, \mathcal{B})$ is a P_4-design of order $12h+10$. Moreover, it is s-colorable for any $s \in [2, \chi_{12h+10}] = [2, 9h+7]$. Indeed, it is not difficult to see that:

- for $s = 2$ we can take as color classes $\{x_1, \dots, x_{6h+4}, y_1, \dots, y_{3h+2}, z_1\}$ and $\{y_{3h+3}, \dots, y_{6h+4}, z_2\}$;
- for $s = 3$ we can take as color classes $\{x_1, \dots, x_{6h+4}\}$, $\{y_1, \dots, y_{3h+2}, z_1\}$ and $\{y_{3h+3}, \dots, y_{6h+4}, z_2\}$;
- for $s = 4, \dots, 9h+7$ we can take as color classes $A_1, \dots, A_m$, $B_1, \dots, B_p$, $C_1, \dots, C_q$, $D_1, \dots, D_r$, with $m \in \{1, \dots, 6h+4\}$, $p, q, r \in \{1, \dots, h+1\}$ and $m+p+q+r = s$, where:
  - $A_1, \dots, A_m$ is any partition of $\{x_1, \dots, x_{6h+4}\}$;
  - $B_1, \dots, B_p$ is any partition of $\{y_1, \dots, y_{h+1}\} \cup \{y_{3h+3}, \dots, y_{4h+3}\}$ such that $\{y_i, y_{3h+2+i}\}$ for any $i = 1, \dots, h+1$ is contained in some of the B_j;
  - $C_1, \dots, C_q$ is any partition of $\{y_{h+2}, \dots, y_{2h+2}\} \cup \{y_{4h+4}, \dots, y_{5h+4}\}$ such that $\{y_i, y_{3h+2+i}\}$ for any $i = h+2, \dots, 2h+2$ is contained in some of the C_j;
  - $D_1, \dots, D_r$ is any partition of
    $$\{y_{2h+3}, \dots, y_{3h+2}\} \cup \{y_{5h+5}, \dots, y_{6h+4}\} \cup \{z_1, z_2\}$$
    such that $\{y_i, y_{3h+2+i}\}$ for any $i = 2h+3, \dots, 3h+2$ and $\{z_1, z_2\}$ are contained in some of the D_j.

□

3. BP_4-designs of small orders

In this section we get a few results in the case that the P_4-designs have small orders. Let us recall that, given two sets X and Y with $X \subset Y$ and $|X| = s$, X is called an s-subset of Y.

In the following proposition we make a few easy remarks:

Proposition 3.1. *Let $v \in \mathbb{N}$, with $v \equiv 0, 1 \mod 3$ and $v \geq 4$, and let $s \in [2, \chi_v]$. If Σ is a BP_4-design of order v, then:*

1. *if Σ is 2-colorable, it is also 3-colorable;*
2. *if Σ is a s-colorable, with $s > \frac{v}{2}$, then Σ is also $(s-1)$-colorable.*

Proof. It is obvious that a 2-colorable P_4-design is also 3-colorable. If Σ is a is s-colorable for some $s > \frac{v}{2}$, then in any s-coloring of Σ there exist at least two color classes X_1 and X_2 such that $2 \leq |X_1 \cup X_2| \leq 3$. Then Σ admits an $(s-1)$-coloring with color classes $X_1 \cup X_2$, $X_3, \ldots, X_s$. □

Proposition 3.2. *Let $v \in \mathbb{N}$, with $v \equiv 0,1 \mod 3$ and $v \geq 4$, and let $s \in [2, \chi_v]$. Then:*

1. *for $v = 4$ and $v = 6$ any BP_4-design of order v is s-colorable, for any s;*

2. *for $v = 7$ any BP_4-design of order v is s-colorable for $s = 2,3,4$ and there exists a BP_4-design which is not 5-colorable;*

3. *for $v = 9$ any BP_4-design of order v is s-colorable for $s = 2,3,4$ and there exists a BP_4-design which is not s-colorable for $s = 6,7$;*

4. *for $v = 10$ any BP_4-design of order v is s-colorable for $s = 2,3,4$.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be any P_4-design of order v. We denote by $r(x,y)$ the number of blocks of $B \in \mathcal{B}$ having x and y among its set of vertices.

If $\mathbf{v} = \mathbf{4}$, then the statement is trivial. If $\mathbf{v} = \mathbf{6}$, the cases $s = 2$ and $s = 3$ are immediate. We just need to prove the case $s = 4$. Let $X = \{0,1,2,3,4,5\}$. If there is a triple, say $\{0,1,2\}$, which is not contained in any block, then $\{0\}$, $\{1\}$, $\{2\}$ and $\{3,4,5\}$ are the color classes of a 4-coloring of Σ. Otherwise, since $|\mathcal{B}| = 5$, any triple in X is contained in exactly just one block of $\mathcal{B}$. So, if $\{0,1,2,3\} = V(B)$ for some $B \in \mathcal{B}$, then $\{0\}$, $\{1\}$, $\{2,3\}$ and $\{4,5\}$ are the color classes of a 4-coloring of Σ.

Let $\mathbf{v} = \mathbf{7}$. Then $|\mathcal{B}| = 7$ and so, clearly, there exists a 4-subset X_1 of X which is not the set of vertices of some block of $\mathcal{B}$. This means that X_1 and $X \setminus X_1$ provide a 2-coloring of Σ. Moreover, any partition of X in 3 nonempty subsets, all of cardinality at most 3, provides a 3-coloring of Σ.

Now we want to show that Σ is 4-colorable. Let $X = \{0,1,\ldots,6\}$. We know that:

$$\sum_{x,y \in X, x \neq y} r(x,y) = 6 \cdot 7 = 42.$$

Then, we can suppose that $r(0,1) \leq 2$. If $\{0,1,2,3\}$ and $\{0,1,4,5\}$ are the set of vertices of blocks containing 0 and 1, then $\{0\}$, $\{1\}$, $\{2,3\}$ and $\{4,5,6\}$ are the color classes of a 4-coloring of Σ. This holds also if only $\{0,1,2,3\}$ is the set of vertices of the only block containing 0 and 1. If $\{0,1,2,3\}$ and $\{0,1,2,4\}$ are the set of vertices of two blocks containing 0 and 1, then $\{0\}$, $\{1\}$, $\{2,3,4\}$ and $\{5,6\}$ are the color classes of a 4-coloring of Σ.

Now, let $Y = \{0,1,\ldots,6\}$ and let $\Sigma' = (Y, \mathcal{C})$ be P_4-design of order 7 with base block $\langle 0,1,3,6 \rangle$. It is easy to see that any 5-subset of Y contains a triple

$i, i+1, i+2$ for some $i = 0, 1, \ldots, 6$. From this we easily get that any 5-subset of Y contains some block of $\mathcal{C}$, which implies that Σ' is not 5-colorable.

Let $\mathbf{v} = \mathbf{9}$. Then $|\mathcal{B}| = 12$, $\binom{9}{4} = \binom{9}{5} = 126$ and there exist at most 60 5-subsets of X containing the set of vertices of at least one block of $\mathcal{B}$. So, clearly, there exists a 5-subset X_1 of X which does not contain the set of vertices of some block of $\mathcal{B}$ and such that $X \setminus X_1$ is not the set of vertices of some block of $\mathcal{B}$. This means that X_1 and $X \setminus X_1$ provide a 2-coloring of Σ. Moreover, any partition of X in 3-subsets, all of cardinality 3, provides a 3-coloring of Σ.

Now we want to show that Σ is 4-colorable. Let $x = \{0, 1, \ldots, 8\}$. Since $|\mathcal{B}| = 12$, we can say that there exist two vertices, say 0 and 1, such that $r(0,1) \leq 2$.

Suppose that $\{0,1,2,3\}$ and $\{0,1,4,5\}$ are set of vertices of the two blocks containing 0 and 1. If $\{2,3,4,5\}$ is not the set of vertices of some block, then Σ is 4-colorable, since we can take $\{0\}$, $\{1\}$, $\{2,3,4,5\}$ and $\{6,7,8\}$ as color classes of a 4-coloring.

Suppose that $\{0,1,2,3\}$ and $\{0,1,4,5\}$ are set of vertices of the two blocks containing 0 and 1 and that two of the remaining vertices, say 6 and 7, are not contained in a block with 2 and 3, for example. Then, $\{0\}$, $\{1\}$, $\{2,3,6,7\}$ and $\{4,5,8\}$ are color classes of a 4-coloring of Σ.

If the above conditions don't hold, then we can say that the following sets:

$$\{2,3,4,5\}, \{2,3,6,7\}, \{2,3,6,8\}, \\ \{2,3,7,8\}, \{4,5,6,7\}, \{4,5,6,8\}, \{4,5,7,8\}$$

are sets vertices of some blocks of Σ. This implies that all the edges of $K_{\{2,3,\ldots,8\}}$ are contained in these blocks. So, the remaining 5 blocks must al contain both 0 and 1, which is not possible, because $r(0,1) \leq 2$.

If $\{0,1,2,3\}$ and $\{0,1,2,4\}$ are set of vertices of the two blocks containing 0 and 1, we can note that there exists at least one vertex in $\{5,6,7,8\}$, say 5, that is not contained in any block together with $2,3,4$. So, $\{0\}$, $\{1\}$, $\{2,3,4,5\}$ and $\{6,7,8\}$ are color classes of a 4-coloring of Σ.

If only $\{0,1,2,3\}$ is the set of vertices of some blocks containing 0 and 1, then we take any vertex in $\{4,5,\ldots,8\}$, say 4, and proceed as above.

Now, we show that there exists a BP_4-design of order 9 which is not 6 and 7-colorable. Let $Y = \{0,1,\ldots,6\}$ and we take the P_4-design $\Sigma' = (Y, \mathcal{C})$ of order 7 we base block $\langle 0,1,3,6 \rangle$, which we know is not 5-colorable. Let $\infty_1, \infty_2 \notin Y$, $\infty_1 \neq \infty_2$. Consider the family $\mathcal{D}$ of the following blocks:

$$\langle 0,\infty_1,\infty_2,1 \rangle, \langle 1,\infty_1,2,\infty_2 \rangle, \langle 0,\infty_2,3,\infty_1 \rangle, \langle 4,\infty_1,5,\infty_2 \rangle, \langle 4,\infty_2,6,\infty_1 \rangle.$$

Then $\Sigma'' = (Y \cup \{\infty_1,\infty_2\}, \mathcal{C} \cup \mathcal{D})$ is a a P_4-design of order 9 which is not s-colorable for $s = 7$, because Σ' is not 5-colorable. It is immediate to see that it is not 6-colorable, because in a 6-coloring we must have $\{\infty_1\}$ and $\{\infty_2\}$ as color classes, which implies that $\{0,1,2,3\}$ and $\{4,5,6\}$ are color classes and this is not a 6-coloring.

Let $\mathbf{v} = \mathbf{10}$. Then $|\mathcal{B}| = 15$, so that there exist at most 90 5-subsets of X containing the vertex set of some block in $\mathcal{B}$. Since $\binom{10}{5} = 252$, there exists a partition of X into two 5-subsets X_1, X_2 that don't contain the vertex set of some block in $\mathcal{B}$. This shows that Σ is 2-colorable. By Proposition 3.1 it is also 3–colorable.

Let $X = \{0,1,\dots,9\}$. Now we want to show that Σ is always 4-colorable. We know that:

$$\sum_{x,y \in X, x \neq y} r(x,y) = 6 \cdot 15 = 90.$$

Suppose that $r(0,1) = 1$ and let $\{0,1,2,3\}$ be the set of vertices of the only block containing both 0 and 1. Since $\frac{6\cdot 5}{2\cdot 3} = 5$ and $\binom{6}{4} = 15$, there exist at least 10 4-subsets of $\{4,5,\dots,9\}$ that are not the set of vertices of some block in $\mathcal{B}$. Since it must be $r(2,3) \leq 8$, we can suppose that both $\{2,3,4,5\}$ and $\{6,7,8,9\}$ are not sets of vertices of some blocks in $\mathcal{B}$. This means that the sets $\{0\}$, $\{1\}$, $\{2,3,4,5\}$ and $\{6,7,8,9\}$ provide a 4-coloring of Σ.

We can suppose, now, that $r(x,y) = 2$ for any $x,y \in X$, $x \neq y$.

Suppose that $\{0,1,2,3\}$ and $\{0,1,4,5\}$ are the set of vertices of the blocks containing both 0 and 1. Since $r(x,y) = 2$ for any $x,y \in X$, $x \neq y$, we can suppose that $\{2,3,6,7\}$ and $\{4,5,8,9\}$ are not the set of vertices of some blocks of Σ. This means that the sets $\{0\}$, $\{1\}$, $\{2,3,6,7\}$ and $\{4,5,8,9\}$ provide a 4-coloring of Σ.

Suppose that $\{0,1,2,3\}$ and $\{0,1,2,4\}$ are the set of vertices of the blocks containing both 0 and 1.

If there exists the block with set of vertices $\{2,3,4,5\}$ (and only one block of this type can exist), since $r(x,y) = 2$ for any $x,y \in X$, $x \neq y$, then we can suppose that $\{2,3,4,6,7\}$ doesn't contain any block of Σ. So, $\{0\}$, $\{1\}$, $\{2,3,4,6,7\}$ and $\{5,8,9\}$ provide a 4-coloring of Σ.

Otherwise, we can anyway take, as above, 6 and 7 in such a way that $\{2,3,4,6,7\}$ doesn't contain any block of Σ. And again $\{0\}$, $\{1\}$, $\{2,3,4,6,7\}$ and $\{5,8,9\}$ provide a 4-coloring of Σ.

In the case that $\{0,1,2,3\}$ is the only set of vertices of the blocks containing 0 and 1 we take 4 in such a way that the triple $\{2,3,4\}$ is not contained in any block and we proceed as above. □

Remark 3.3. In [10] (see note at page 156) it is stated the existence of Steiner systems $S(2,4,25)$ with lower chromatic number 3. This, together with the existence of a P_4-design of order 4, imply the existence of BP_4-designs of order 25 which are 3-colorable, but not 2-colorable.

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