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FAT POINTS ON A GRID IN $\mathbb{P}^2$

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We study homogeneous schemes of fat points in $\mathbb{P}^2$ whose support is either a complete intersection (CI for short) constructed on an $a \times b$ grid or a CI minus a point, i.e. $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$ and $\mathbb{Y}_{grid} = \{CI_{grid}(a, b) \setminus P_{ab}; m\}$ respectively.

We study the connections between the above fat point schemes and particular varieties of simple points called partial intersections (p.i. for short). We prove that a homogeneous fat point scheme of type $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$ has the same graded Betti numbers, and hence, the same Hilbert function of a particular p.i. depending only on a, b, m. Moreover, a scheme of double points of type $\mathbb{Y}_{grid} = \{CI_{grid}(a, b) \setminus P_{ab}; 2\}$ has the same Hilbert function of another particular p.i. depending on a, b, m.

We also describe an alternative approach to the problem by considering the Gröbner basis of $I_{\mathbb{Y}_{grid}}$.

Introduction.

Let $P_1, \dots, P_s$ be s distinct points in $\mathbb{P}^n$ and $m_1, \dots, m_s$ a list of positive integers. Let $I_Z = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s} \subseteq R = k[x_0, \dots, x_n]$, where k is an algebraically closed field and $\wp_i$ is the prime ideal corresponding to P_i in R. Let Z denote the subscheme defined by I_Z. Sometimes Z is denoted as $Z = \{P_1, \dots, P_s; m_1, \dots, m_s\}$. The scheme Z is called a scheme of *fat points*. If $m_i = m$ for all $i = 1, \dots, s$, then Z is called a *homogeneous scheme of fat points*.

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The Hilbert function of I_Z has been studied in $\mathbb{P}^2$ by many authors, (cf. [6], [8], [9], [13], [14], [15], [16]) but much remains conjectural for points in generic position. We know very few results for schemes of fat points in $\mathbb{P}^n$. Important work in this area is due to Alexander and Hirschowitz in [2], [3], and [4]. Other papers that have results about fat points in $\mathbb{P}^n$ are [7], where the authors find an algorithm to compute the Hilbert function of fat points whose support lies on a rational normal curve, and [10], where the author studies fat point schemes on a smooth quadric of $\mathbb{P}^3$.

We specialize to the case $n = 2$ and Z is a homogeneous fat point scheme whose support Z_{red} is either a complete intersection (CI for short) or a CI minus a point. The philosophy behind this approach is that complete intersections (and their subsets) have more properties and structure than general sets of points. We study the following problem:

Problem 1. *Let $\mathbb{X} = \{P_1, \dots, P_{ab}; m_1, \dots, m_{ab}\}$ be a scheme of fat points in $\mathbb{P}^2$ such that $\mathbb{X}_{red} = \{P_1, \dots, P_{ab}\}$ is a CI of type (a, b). Furthermore, assume that $m = m_1 = \cdots = m_{ab}$. Let $\mathbb{Y} \subseteq \mathbb{X}$ be defined by removing one point from the support of $\mathbb{X}$. What are the possible Hilbert functions of $\mathbb{Y} \subseteq \mathbb{P}^2$?*

In the case that the underlying CI is a grid in $\mathbb{P}^2$, we were partially successful in answering the question. Our main contribution is to show that in this restricted case there is a connection between the Hilbert function of fat points whose reduced scheme is a CI and the Hilbert function of a partial intersection (defined in Section 3). This connection is the content of Proposition 3.2 and Proposition 4.7. Moreover, the latter proposition shows that the Hilbert function of schemes of fat points whose support is a CI or a CI minus a point does not depend on the forms of degree a and b that generate the CI, but only on the numbers a, b and m.

Our paper is structured as follows. In the first section we set our notation. Next, we quickly examine what the current literature says in connection to our problem. In Section 3 we examine the homogeneous scheme $\mathbb{X}_{grid} = \{CI(a, b); m\}$ whose support $\mathbb{X}_{red} = CI(a, b)$ is constructed on a grid. We also introduce the notion of a partial intersection. In the following section, we discuss the connection to our problem and the Hilbert functions of partial intersections. In the last section, we describe an alternative approach to the problem by considering the Gröbner basis of $I_{\mathbb{X}_{grid}}$.

Some results of this paper are part of the Ph. D. thesis of the second author ([11]).

All the results of this paper are contained in the survey [5]. We would like to thank all those involved at Pragmatic Summer School of Research at the Università di Catania, especially A. Ragusa, A.V. Geramita, J. Migliore,

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1. Preliminaries and setup.

We fix $R = k[x_0, x_1, x_2]$, where k is an algebraically closed field of characteristic zero. Let $Z = \{P_1, \dots, P_s; m_1, \dots, m_s\}$ be a scheme of fat points in $\mathbb{P}^2$ whose support is $Z_{red} = \{P_1, \dots, P_s\}$. If $m_i = m$ for all $i = 1, \dots, s$, then Z is called a *homogeneous scheme of fat points*.

Let $\mathbb{X}'$ be a complete intersection of type (a, b) in $\mathbb{P}^2$. We write $\mathbb{X}' = CI(a, b)$. Suppose $\mathbb{X}' = \{P_1, \dots, P_{ab}\}$ are the distinct ab points in this CI. If $\wp_i$ is the prime ideal of R that corresponds to P_i, then for any positive integer m, we let

$$I_{\mathbb{X}} = \wp_1^m \cap \cdots \cap \wp_{ab}^m.$$

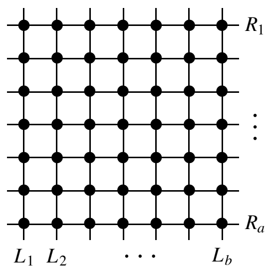
We denote the homogeneous scheme defined by $I_{\mathbb{X}}$ by $\mathbb{X} = \{CI(a, b); m\}$. Note that $\mathbb{X}_{red} = \mathbb{X}' = CI(a, b)$. We also assume that $a \leq b$.

Let $\mathbb{Y}'$ denote the scheme defined by removing one point from $\mathbb{X}' = CI(a, b)$, i.e., $\mathbb{Y}' = CI(a, b) \setminus \{P\}$. We are interested in determining the Hilbert function of $R/I_{\mathbb{Y}}$ where $I_{\mathbb{Y}} = \wp_1^m \cap \cdots \cap \wp_{ab-1}^m$ is the ideal corresponding to the homogeneous scheme $\mathbb{Y}$ of fat points whose support is $\mathbb{Y}'$.

From now on we will write $\mathbb{X} \setminus P$ to denote the set $\mathbb{X} \setminus \{P\}$.

In this paper we focus our attention to the schemes of fat points whose support is a CI constructed on an $a \times b$ grid. We describe this construction below.

Method 1.1. We construct $\mathbb{X} = \{CI(a, b); m\}$ by taking $\mathbb{X}_{red}$ to be a CI generated by two " totally reducible " forms of degree a and b, that is, the CI is given by the intersection of two sets of lines in $\mathbb{P}^2$ as an $a \times b$ grid. We can visualize this as



where the lines $R_1, R_2, \ldots, R_a$ and $L_1, L_2, \ldots, L_b$ which define the grid in $\mathbb{P}^2$ are chosen generically. Once we have picked the lines defining the grid, we can determine $\{P_1, \ldots, P_{ab}\}$ and compute the prime ideal $\wp_i$ corresponding to P_i. We denote the homogeneous scheme $\mathbb{X}$ by $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$ and its defining ideal by $I_{\mathbb{X}_{grid}} = \wp_1^m \cap \cdots \cap \wp_{ab}^m$. The homogeneous scheme $\mathbb{Y}$ is denoted by $\mathbb{Y}_{grid} = \{CI_{grid}(a, b) \setminus P_{ij}; m\}$ for some $1 \leq i \leq a$ and $1 \leq j \leq b$. By renumbering the lines R_i or L_j, we can always suppose to remove P_{ab}. We denote its defining ideal by $I_{\mathbb{Y}_{grid}} = \wp_1^m \cap \cdots \cap \wp_{ab-1}^m$.

2. Known results.

Let $\mathbb{X} = \{CI(a, b); m\}$ for some a, b, and m with $a \leq b$. In this section we look at the case $a = 1$. If $a = 1$, and thus $\mathbb{X} = \{CI(1, b); m\}$, then $\mathbb{X}$ is a collection of b fat points on a line in $\mathbb{P}^2$. When we remove a point from $\mathbb{X}_{red}$ to construct $\mathbb{Y}$, the resulting scheme is simply $\mathbb{Y} = \{CI(1, b - 1); m\}$. In other words $\mathbb{Y}$ is the scheme of $b - 1$ fat points on line.

We can now use a result of [8] to compute the Hilbert function of $\mathbb{Y}$.

Proposition 2.1. *Let $\mathbb{X} = \{CI(1, b); m\}$ and $\mathbb{Y}$ the homogeneous scheme of fat points whose support is $\mathbb{Y}_{red} = \mathbb{X}_{red} \setminus P$. Then $\mathbb{Y} = \{CI(1, b - 1); m\}$. Furthermore, set $t_i = m + i(b - 2)$ for $0 \leq i \leq m$. Then*

$$\Delta H_{R/I_{\mathbb{Y}}}(t) = \begin{cases} t+1 & 0 \leq t < m \\ m-i & t_i \leq t < t_{i+1} \\ 0 & t_m \leq t \end{cases}$$

where $\Delta H_{R/I_{\mathbb{Y}}}(t) := H_{R/I_{\mathbb{Y}}}(t) - H_{R/I_{\mathbb{Y}}}(t - 1)$.

Proof. The formula is an application of Proposition 3.3 of [8]. □

3. Results on $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$.

In this section we will prove some results concerning the Hilbert function of $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$ and special varieties of points called partial intersections that depend on a, b and m.

Let us define partial intersections in $\mathbb{P}^2$. Fix two sets of lines of $\mathbb{P}^2$, say $\{R'_i\}$ for $i = 1, \ldots, a$ and $\{L'_j\}$ for $j = 1, \ldots, b$ such that no three of them have a common point, and denote $P_{i,j} = R'_i \cap L'_j$. Let $\underline{p} = (p_1, \ldots, p_r)$ and $\underline{q} = (q_1, \ldots, q_r)$ be two sets of r positive integers with $b = p_1 > \ldots > p_r > 0$, $q_1 + \ldots + q_r = a$. Put $r(i) = inf\{s \in \mathbb{N} | \sum_{j=1}^{s} q_j \geq i\}$, for $i = 1, \ldots, a$.

With this notation, we consider the variety V consisting of the points $P_{i,j(i)}$, where $i = 1, \dots, a$ and $j(i) = 1, \dots, p_{r(i)}$. (Note that $r(i)$ takes the meaning of the subscript of the q corresponding to the line R'_i.) Every variety constructed in this way will be called a *partial intersection of type* $(\underline{p}, \underline{q})$, or simply, a p.i.

We have

Lemma 3.1. *Let V be a partial intersection of type $(\underline{p}, \underline{q})$, with $\underline{p} = (p_1, \dots, p_r)$ and $\underline{q} = (q_1, \dots, q_r)$. Put $b = p_1$ and $a = \sum_{i=1}^{r} q_i$. Then $\mathcal{O}_V$ has a minimal free resolution of the following form*

$$0 \to \bigoplus_{t=1}^{r} \mathcal{O}_{\mathbb{P}^2}(-b_t) \to \bigoplus_{t=0}^{r} \mathcal{O}_{\mathbb{P}^2}(-a_t) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_V \to 0 \tag{1}$$

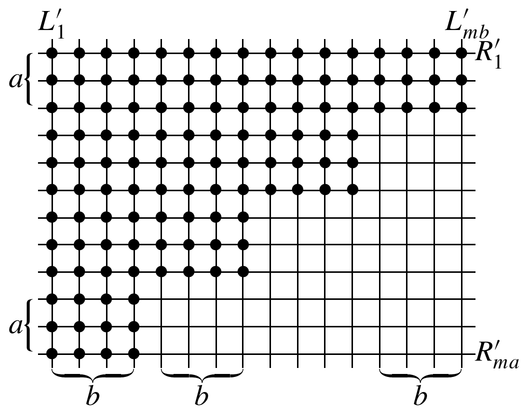
where $a_0 = a$, $a_t = p_t + \sum_{k=0}^{t-1} q_k$ $(q_0 = 0)$, and $b_t = p_t + \sum_{k=1}^{t} q_k$, for $t = 1, \dots, r$.

Proof. See Proposition 2.1 in [17] or [18]. □

Let a, b and m be three positive integers with $a \leq b$. Let $\underline{p} = (p_1, \dots, p_m)$ and $\underline{q} = (q_1, \dots, q_m)$ be two m-tuples of positive integers such that $p_k = (m - k + 1)b$, and $q_k = a$ for $k = 1, \dots, m$. That is, $\underline{p} = (mb, (m-1)b, (m-2)b, \dots, 2b, b)$, and $\underline{q} = (a, a, \dots, a)$. We define $\mathbb{X}_{p.i.}$ to be the partial intersection of type $(\underline{p}, \underline{q}) = ((mb, (m-1)b, (m-2)b, \dots, 2b, b), (a, a, \dots, a))$, i.e.,

$$\mathbb{X}_{p.i.} = \{P'_{i,j(i)} = R'_i \cap L'_{j(i)} \quad | \quad i = 1, \dots, ma \quad \text{and} \quad j(i) = 1, \dots, p_{r(i)}\}.$$

We can visualize this scheme as the following scheme of simple points:



Using Lemma 3.1 to compute the minimal resolution of $\mathbb{X}_{p.i.}$, we have

$$0 \to \bigoplus_{t=1}^{m} \mathcal{O}_{\mathbb{P}^2}(-b_t) \to \bigoplus_{t=0}^{m} \mathcal{O}_{\mathbb{P}^2}(-a_t) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{X}_{p.i.}} \to 0 \tag{2}$$

where $a_0 = ma$, $a_t = (m-t+1)b + (t-1)a$, and $b_t = (m-t+1)b + ta$, for $t = 1, \ldots, m$. The generators of $\mathbb{X}_{p.i.}$ are of type:

$$G'_{k+1} = R'_1 R'_2 \ldots R'_{(m-k)a} L'_1 \ldots L'_{kb} \quad \text{for} \quad k = 0, \ldots, m.$$

From the construction of $\mathbb{X}_{p.i.}$, we can also deduce that

$$\Delta HF(\mathbb{X}_{p.i.}, t) = \Delta HF(C_1, t) + \Delta HF(C_2, t-a) + \ldots + \Delta HF(C_m, t-(m-1)a)$$

where $C_k = CI(a, (m-k+1)b)$ for $k = 1, \ldots, m$.

Let $\mathbb{X}_{grid} = \{CI(a, b); m\}$ be the fat point scheme constructed on an $a \times b$ grid, where a, b, and m are the same integers used to construct $\mathbb{X}_{p.i.}$ above, that is

$$\mathbb{X}_{grid} = \{P_{ij} = R_i \cap L_j; \quad m_{ij} = m \quad \forall i = 1, \ldots, a \quad \text{and} \quad j = 1, \ldots, b\}.$$

We then have the following result:

Proposition 3.2. *Let a, b and m be positive integers such that $a \le b$. Then $\mathbb{X}_{grid} = \{CI(a, b); m\}$ and the partial intersection $\mathbb{X}_{p.i.}$ of type $((mb, (m-1)b, (m-2)b, \ldots, 2b, b), (a, a, \ldots, a))$ have the same graded Betti numbers, and hence, the same Hilbert function.*

Proof. If $a = 1$, then $\mathbb{X}_{grid} = \{CI(1, b); m\}$, i.e., $\mathbb{X}_{grid}$ is a set of b fat points on a line. We know that $\{mb, (m-1)b + 1, \ldots, m-1+b, m\}$ is a list of the degrees of the generators for $\mathbb{X}_{grid}$. Moreover, by Proposition 2.1 we have

$$\Delta H_{I_{\mathbb{X}_{grid}}}(t) = H_{I_{\mathbb{X}_{gen}}}(t) = \begin{cases} t+1 & 0 \le t < m \\ m-i & t_i \le t < t_{i+1} \\ 0 & t_m \le t \end{cases}$$

where $t_i = m + i(b-1)$ for $0 \le i \le m$.

From Lemma 1.1 in [17], we know that the syz-degrees of $\mathbb{X}_{grid}$ are of type $(m-k)b + k + 1$ for $k = 0, \ldots, m-1$, hence it has a minimal resolution of type (2) and then it has the same graded Betti numbers of a partial intersection of type $((mb, (m-1)b, (m-2)b, \ldots, 2b, b), (1, 1, \ldots, 1))$.

Let us suppose $a > 1$, and the thesis is true for homogeneous fat points schemes $\overline{\mathbb{X}} = \{CI(\overline{a}, \overline{b}); m\}$ with $\overline{a} < a$ and $\overline{b} \leq b$ whose support $CI(\overline{a}, \overline{b})$ is constructed on an $\overline{a} \times \overline{b}$ grid. Define the following two homogeneous fat points schemes:

$$\mathbb{X}_1 := \{CI(a-1, b); m\} = \{P_{ij} = R_i \cap L_j; m = m_{ij} \quad \text{for} \quad i = 1, \dots, a-1,\ j = 1, \dots, b\}$$

and

$$\mathbb{X}_2 := \{CI(1, b); m\} = \{P_{aj} = R_a \cap L_j; m = m_{aj} \quad \text{for} \quad j = 1, \dots, b\}$$

We have $\mathbb{X}_{grid} = \mathbb{X}_1 \cup \mathbb{X}_2$.

By induction $\mathbb{X}_1$ has the same graded Betti numbers as the partial intersection V_1 of type $(\underline{p'}, \underline{q'})$ where $\underline{p'} = (mb, \dots, b)$ and $\underline{q'} = (\underbrace{a-1, \dots, a-1}_{m})$.

That is,

$$V_1 := \{P'_{i,j(i)} = R'_i \cap L'_{j(i)} \quad | \quad i = 1, \dots, m(a-1), \quad j(i) = 1, \dots, p''_{r(i)}\}$$

Similarly, $\mathbb{X}_2$ has the same graded Betti numbers as the partial intersection V_2 of type $(\underline{p''}, \underline{q''})$ where $\underline{p''} = (mb, (m-1)b, \dots, b)$ and $\underline{q''} = (\underbrace{1, 1, \dots, 1}_{m})$.

That is,

$$V_2 := \{P'_{i,j(i)} = R'_i \cap L'_{j(i)} | \quad i = m(a-1)+1, \dots, ma, \quad j(i) = 1, \dots, p''_{r(i)}\}.$$

But

$$\begin{array}{llll} p'_{r(t)} = p''_{r(ma-m+2)} = mb & \text{for} & t = 1, \dots, a-1 \\ p'_{r(t)} = p''_{r(ma-m+2)} = (m-1)b & \text{for} & t = a, \dots, 2(a-1) \\ \quad\vdots & & \\ p'_{r(t)} = p''_{r(ma)} = b & \text{for} & t = m(a-1)-a+2, \dots, m(a-1) \end{array}$$

If we now renumber the lines of type R'_i, we get

$$\begin{aligned} V &= V_1 \cup V_2 \\ &= \{P'_{i,j(i)} = R'_i \cap L'_{j(i)} | \quad i = 1, \dots, ma, \quad j(i) = 1, \dots, p_{r(i)}\} \end{aligned}$$

where

$$p_{r(i)} = \begin{cases} mb & \text{if} \quad 1 \leq i \leq a \\ (m-1)b & \text{if} \quad a+1 \leq i \leq 2a \\ \vdots & \\ b & \text{if} \quad ma-a+1 \leq i \leq ma \end{cases}$$

Hence, V is a partial intersection that is equal to $\mathbb{X}_{p.i.}$ and thus, it has a minimal resolution of type (2). Furthermore, since the generators of $\mathbb{X}_1$ are of type

$$G^{(1)}_{k+1} = R_1^{m-k} \cdots R_{a-1}^{m-k} L_1^k \cdots L_b^k$$

for $k = 0, \dots, m$, then the generators of $\mathbb{X}_{grid} = \mathbb{X}_1 \cup \mathbb{X}_2$ are of type

$$G^{(1)}_{k+1} R_a^{m-k} = R_1^{m-k} \cdots R_{a-1}^{m-k} R_a^{m-k} L_1^k \cdots L_b^k$$

for $k = 0, \dots, m$. Hence $\mathbb{X}_{grid}$ has the same graded Betti numbers as $\mathbb{X}_{p.i.}$. $\square$

Remark 3.3. We notice that the generators of

$$\begin{aligned}\mathbb{X}_{grid} &= \{CI(a, b)\ ;\ m\}\\ &= \{P_{ij} = R_i \cap L_j; \quad m_{ij} = m\ \forall\ i = 1, \dots, a \quad \text{and} \quad j = 1, \dots, b\}\end{aligned}$$

are of the type

$$G_{k+1} = R_1^{m-k} \cdots R_a^{m-k} L_1^k \cdots L_b^k \quad \text{for} \quad k = 0, \dots, m.$$

4. Schemes of double points and connections with Partial Intersections.

In this section we show how partial intersections are connected with Problem 1 in the case that $m = 2$. We continue to assume that $a \leq b$.

Let us consider the following partial intersection $\mathbb{X}_{p.i.}$ of type $(\underline{p}, \underline{q})$ where $\underline{p} = (2b, b)$ and $\underline{q} = (a, a)$. For such a set $\mathbb{X}_{p.i.}$, we know that the minimal resolution is

$$(3) \qquad 0 \to \mathcal{O}_{\mathbb{P}^2}(-a-2b) \oplus \mathcal{O}_{\mathbb{P}^2}(-2a-b) \to$$

$$\to \mathcal{O}_{\mathbb{P}^2}(-2a) \oplus \mathcal{O}_{\mathbb{P}^2}(-2b) \oplus \mathcal{O}_{\mathbb{P}^2}(-a-b) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{X}_{p.i.}} \to 0$$

and the Hilbert function is

$$(4) \qquad \Delta HF(\mathbb{X}pi, t) = \Delta HF(C_1, t) + \Delta HF(C_2, t-a)$$

where $C_1 = CI(a, 2b)$ and $C_2 = CI(a, b)$. More generally, if $b > 2a - 1$, then by using (4) and Theorem 3.2, we have

$$\Delta H_{R/I_{\mathbb{X}_{grid}}}(t) = \Delta H_{R/I_{\mathbb{X}_{p.i.}}}(t) = \begin{cases} t+1 & 0 \leq t \leq 2a-1 \\ 2a & 2a \leq t \leq a+b-1 \\ 3a+b-t-1 & a+b \leq t \leq 2a+b-2 \\ a & 2a+b-1 \leq t \leq 2b-1 \\ a+2b-t-1 & 2b \leq t \leq a+2b-2 \\ 0 & i \geq a+2b-1 \end{cases}$$

If $b \leq 2a-1$, then

$$\Delta H_{R/I_{\mathbb{X}_{grid}}}(t)=\Delta H_{R/I_{\mathbb{X}_{p.i.}}}(t)=\begin{cases} t+1 & 0 \leq t \leq 2a-1 \\ 2a & 2a \leq t \leq a+b-1 \\ 3a+b-t-1 & a+b \leq t \leq 2b-1 \\ 3a+3b-2t-2 & 2b \leq t \leq 2a+b-2 \\ a+2b-t-1 & 2a+b-1 \leq t \leq a+2b-2 \\ 0 & t \geq a+2b-1. \end{cases}$$

In the cases $a = 1$ and $a = 2$, we recover the same formulas of [8] and [6] respectively.

We now want to calculate the Hilbert function of the following subset of $\mathbb{X}_{p.i.}$

$$\widetilde{\mathbb{Y}}_{p.i.} = \mathbb{X}_{p.i.} \setminus \{P_{r,s},\, P_{a+r,s},\, P_{r,b+s}\}$$

for any $r \in \{1, \dots, a\}$ and $s \in \{1, \dots, b\}$. After renumbering the lines, we can always suppose that $r = a$ and $s = b$. We are therefore interested in the Hilbert function of

$$\widetilde{\mathbb{Y}}_{p.i.} = \mathbb{X}_{p.i.} \setminus \{P_{a,b},\, P_{2a,b},\, P_{a,2b}\}.$$

We note that $\widetilde{\mathbb{Y}}_{p.i.}$ is not a p.i., but we will show that it has the same graded Betti numbers as a particular partial intersection.

We need some definitions.

Definition 4.1. Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct points in $\mathbb{P}^2$. We say that $F \in k[x_0, x_1, x_2]$ is a *separator for* P_i if $F(P_j) = 0$ for all $j \neq i$ and $F(P_i) \neq 0$. We call the *degree of* P_i *in* $\mathbb{X}$ the minimal degree of a separator for P_i.

The following theorem provides a result on the degree of P_i in $\mathbb{X}$ that we will use in the next theorem.

Theorem 4.2. *Let $\mathbb{X}$ be a finite set of distinct points in $\mathbb{P}^2$. Let*

$$0 \to \bigoplus_{j \in B_2} \mathcal{O}_{\mathbb{P}^2}(-j)^{\beta_{2j}} \to \bigoplus_{j \in B_1} \mathcal{O}_{\mathbb{P}^2}(-j)^{\beta_{1j}} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{X}} \to 0.$$

be a minimal free resolution of $\mathcal{O}_{\mathbb{X}}$ with $\beta_{1j}, \beta_{2j} \neq 0$. Then for any point $P \in \mathbb{X}$, the degree of P in $\mathbb{X}$, say α, has the property that $\alpha + 2 \in B_2$.

Proof. See [1]. □

With the above notation, we have

Theorem 4.3.

$$\Delta H_{R/I_{\widetilde{\mathbb{Y}}_{p.i.}}}(t) = \begin{cases} \Delta H_{R/I_{\mathbb{X}_{p.i.}}}(t) - 1 & if\ t = 2a+b-2, \\ & \quad a+2b-3,\ a+2b-2 \\ \Delta H_{R/I_{\mathbb{X}_{p.i.}}}(t) & otherwise. \end{cases}$$

Proof. Let us start by removing the point $P_{a,2b}$ from $\mathbb{X}_{p.i.}$. Since $\mathbb{Y}_1 = \mathbb{X}_{p.i.} \backslash P_{a,2b}$ is a p.i. of type (p', q') where $p' = (2b, 2b-1, b)$ and $q' = (a-1, 1, a)$, then $\mathbb{Y}_1$ has the following minimal resolution

$$0 \to \mathcal{O}^2_{\mathbb{P}^2}(-a-2b+1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2a-b) \to$$

$$\mathcal{O}_{\mathbb{P}^2}(-2a) \oplus \mathcal{O}_{\mathbb{P}^2}(-2b) \oplus \mathcal{O}_{\mathbb{P}^2}(-a-b) \oplus \mathcal{O}_{\mathbb{P}^2}(-a-2b+2) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{Y}_1} \to 0$$

and hence, its Hilbert function is:

$$\Delta H_{R/I_{\mathbb{Y}_1}}(t) = \begin{cases} \Delta H_{R/I_{\mathbb{X}_{p.i.}}}(t) - 1 & \text{if } t = a+2b-2 \\ \Delta H_{R/I_{\mathbb{X}_{p.i.}}}(t) & \text{otherwise.} \end{cases}$$

Let us consider $\mathbb{Y}_2 = \mathbb{Y}_1 \backslash P_{2a,b} = \mathbb{X}_{p.i.} \backslash \{P_{a,2b}, P_{2a,b}\}$. $\mathbb{Y}_2$ is a p.i. of type (p'', q'') where $p'' = (2b, 2b-1, b, b-1)$ and $q'' = (a-1, 1, a-1, 1)$. The minimal resolution of $\mathbb{Y}_2$ is

$$0 \to \mathcal{O}^2_{\mathbb{P}^2}(-a-2b+1) \oplus \mathcal{O}^2_{\mathbb{P}^2}(-2a-b+1) \to \tag{5}$$

$$\mathcal{O}_{\mathbb{P}^2}(-2a) \oplus \mathcal{O}_{\mathbb{P}^2}(-2b) \oplus \mathcal{O}_{\mathbb{P}^2}(-a-b+1) \oplus \mathcal{O}_{\mathbb{P}^2}(-a-2b+2) \oplus \mathcal{O}_{\mathbb{P}^2}(-2a-b+1) \to$$

$$\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{Y}_2} \to 0$$

Hence, the Hilbert function is:

$$\Delta H_{R/I_{\mathbb{Y}_2}}(t) = \begin{cases} \Delta H_{R/I_{\mathbb{Y}_1}}(t) - 1 & \text{if } t = 2a+b-2 \\ \Delta H_{R/I_{\mathbb{Y}_1}}(t) & \text{otherwise.} \end{cases}$$

Using Theorem 4.2 and (5) we can say that the degree of $P_{a,b}$ in $\mathbb{Y}_2$ can be either $a+2b-3$ or $2a+b-3$. Let us show that it is $a+2b-3$.

For $j = b+1, \ldots, b+a-1$, let H_j denote the line passing through $P_{a,j}$ and $P_{i,b}$, i.e., $H_j = \overline{P_{a,j}P_{i,b}}$ where $i = j-b+a$. By this construction, each H_j does not pass through $P_{a,b}$. For $j = b+a, \ldots, 2b-1$, let $H_j = \overline{P_{a,j}}$ be a line passing through $P_{a,j}$ but not $P_{a,b}$. Then the form

$$R_1 \cdots R_{a-1} L_1 \cdots L_{b-1} H_{b+1} \cdots H_{2b-1}$$

defines a curve of degree $a + 2b - 3$ passing through all the points of $\mathbb{Y}_2$ but not $P_{a,b}$.

We claim there is no curve of degree $2a + b - 3$ which passes through $\mathbb{Y}_2 \backslash P_{a,b}$ but not $P_{a,b}$. Indeed, let us consider the set

$$\mathbb{Y}_3 = CI(a, 2b) \setminus P_{a,2b}$$

It is known that the Hilbert function of the set $\mathbb{Y}_4 = \mathbb{Y}_3 \setminus P_{a,b}$ is the following:

$$\Delta H_{R/I_{\mathbb{Y}_4}}(t) = \begin{cases} \Delta H_{R/I_{\mathbb{Y}_3}}(t) - 1 & \text{if } t = a + 2b - 3 \\ \Delta H_{R/I_{\mathbb{Y}_3}}(t) & \text{otherwise.} \end{cases}$$

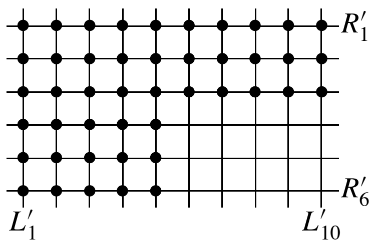
This means that every curve of degree $2a+b-3$ passing through all the points of $\mathbb{Y}_4$ also passes through $P_{a,b}$. Hence, *a fortiori*, every curve of degree $2a+b-3$ passing through all the points of $\mathbb{Y}_2 = \mathbb{Y}_3 \cup \{CI(a, b) \setminus P_{2a,b}\}$ also passes through $P_{a,b}$.

If we put $\mathbb{Y} = \mathbb{Y}_2 \backslash P_{a,b}$, then we have

$$\Delta H_{R/I_{\mathbb{Y}}}(t) = \begin{cases} \Delta H_{R/I_{\mathbb{Y}_2}}(t) - 1 & \text{if } t = a + 2b - 3 \\ \Delta H_{R/I_{\mathbb{Y}_2}}(t) & \text{otherwise.} \end{cases}$$

We observe that $\mathbb{Y} = \widetilde{\mathbb{Y}}_{p.i.}$ and hence, we are done. $\square$

Example 4.4. Let us consider the case $a = 3$ and $b = 5$ and $m = 2$. Let $\mathbb{X}_{p.i.}$ be the partial intersection of type $(\underline{p}, \underline{q})$, where $\underline{p} = (10, 5)$ and $\underline{q} = (3, 3)$, i.e.,



The minimal resolution is

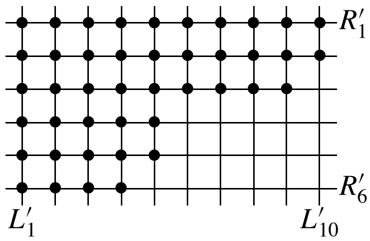
$$0 \to \mathcal{O}_{\mathbb{P}^2}(-13) \oplus \mathcal{O}_{\mathbb{P}^2}(-11) \to \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-8) \oplus \mathcal{O}_{\mathbb{P}^2}(-10) \to$$

$$\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{X}_{p.i.}} \to 0$$

and the Hilbert function is

t	:	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$H_{\mathbb{X}_{p.i.}}(t)$	:	1	3	6	10	15	21	27	33	38	42	44	45	$\rightarrow$	
$\Delta H_{\mathbb{X}_{p.i.}}(t)$	:	1	2	3	4	5	6	6	6	5	4	2	1	0	$\rightarrow$

When we remove the points $P_{3,10}$, $P_{6,5}$ and $P_{3,5}$ from $\mathbb{X}_{p.i.}$ we construct the set $\widetilde{\mathbb{Y}}_{p.i.}$, i.e.



The minimal resolution is

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-12) \oplus \mathcal{O}_{\mathbb{P}^2}(-11) \oplus \mathcal{O}^2_{\mathbb{P}^2}(-10) \rightarrow$$

$$\mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-8) \oplus \mathcal{O}_{\mathbb{P}^2}(-9) \oplus \mathcal{O}^2_{\mathbb{P}^2}(-10) \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\widetilde{\mathbb{Y}}_{p.i.}} \rightarrow 0$$

and the Hilbert function is

t	:	0	1	2	3	4	5	6	7	8	9	10	11	12
$H_{\widetilde{\mathbb{Y}}_{p.i.}}(t)$	:	1	3	6	10	15	21	27	33	38	41	42	$\rightarrow$	
$\Delta H_{\widetilde{\mathbb{Y}}_{p.i.}}(t)$	:	1	2	3	4	5	6	6	6	5	3	1	0	$\rightarrow$

Remark 4.5. We observe that from Theorem 4.3, the degree of $P_{a,b}$ in $\mathbb{Y}_2 = \mathbb{X}_{p.i.} \backslash \{P_{a,2b}, P_{2a,b}\}$ is the same as in $P_{a,2b-1}$. In fact the form

$$C := R'_1 \cdots R'_{a-1} L'_1 \cdots L'_{2b-2}$$

has degree $a + 2b - 3$ and $C(P_{i,j}) = 0$ for all $(i, j) \neq (a, 2b - 1)$, but $C(P_{a,2b-1}) \neq 0$.

Remark 4.6. We notice that from previous remark and from the construction, we have

$$\Delta HF(\widetilde{\mathbb{Y}_{p.i.}}, t) = \Delta HF(C'_1, t) + \Delta HF(C'_2, t - a)$$

where $C'_1 = CI(a, 2b) \backslash \{P_{a,2b-1}, P_{a,2b}\}$ and $C'_2 = CI(a, b) \backslash \{P_{a,b}\}$.

Let us consider the homogeneous schemes of double points $\mathbb{X}_{grid} = \{CI_{grid}(a,b); 2\}$ and $\mathbb{Y}_{grid} = \{CI_{grid} \backslash P_{ab}\ ;\ 2\}$. Using the same a and b, consider the partial intersection $\mathbb{X}_{p.i.}$ of type $(\underline{p}, \underline{q})$ where $\underline{p} = (2b, b)$ and $\underline{q} = (a, a)$. Let

$$\mathbb{Y}_{p.i.} = \mathbb{X}_{p.i.} \backslash \{P_{a,2b}, P_{2a,b}, P_{a,2b-1}\}.$$

In this case, we observe that $\mathbb{Y}_{p.i.}$ is partial intersection of type $\underline{p} = (2b, 2b-2, b, b-1)$, $\underline{q} = (a-1, 1, a-1, 1)$. Furthermore, by the Remark 4.5 and Remark 4.6, $\widetilde{\widetilde{\mathbb{Y}}}_{p.i.}$ and $\mathbb{Y}_{p.i.}$ share the same graded Betti numbers. With this notation, we have

Proposition 4.7. *$\mathbb{Y}_{grid}$ and $\mathbb{Y}_{p.i.}$ have the same graded Betti numbers.*

Proof. We can work in an analogous way as in Proposition 3.2. For $a = 1$ there is nothing to prove. Let us suppose $a > 1$ and the theorem is true for homogeneous schemes of double points of the type $\overline{\mathbb{Y}_{grid}} = \{CI(\overline{a}, \overline{b}) \backslash P_{\overline{a},\overline{b}}\ ;\ 2\}$ with $\overline{a} < a$ and $\overline{b} \leq b$ whose support is $CI(\overline{a}, \overline{b}) \backslash P_{\overline{a},\overline{b}}$ constructed on an $\overline{a} \times \overline{b}$ grid.

Define the following homogeneous schemes of double points:

$$\mathbb{Y}_1 := \{CI(a-1, b); 2\} = \{P_{ij} = R_i \cap L_j; m = 2 \mid i = 1, \ldots, a-1, j = 1, \ldots, b\}$$

and

$$\mathbb{Y}_2 := \{CI(1, b-1)\ ;\ 2\} = \{P_{aj} = R_a \cap L_j;\ m = 2 \mid j = 1, \ldots, b-1\}.$$

Then we have $\mathbb{Y}_{grid} = \mathbb{Y}_1 \cup \mathbb{Y}_2$.

By induction $\mathbb{Y}_1$ has the same graded Betti numbers as the partial intersection V_1 of type $(\underline{p}', \underline{q}')$ where $\underline{p}' = (2b, b)$ and $\underline{q}' = (a-1, a-1)$, and $\mathbb{Y}_2$ has the same graded Betti numbers as the partial intersection V_2 of type $(\underline{p}'', \underline{q}'')$ where $\underline{p}'' = (2(b-1), b-1)$ and $\underline{q}'' = (1, 1)$. Renumbering the lines of type R_i', we get

$$V = \mathbb{Y}_{p.i.} = V_1 \cup V_2$$

and hence, $\mathbb{Y}_{p.i.}$ has the same graded Betti numbers as $\mathbb{Y}_1 \cup \mathbb{Y}_2$. □

Corollary 4.8. *$\mathbb{Y}_{grid}$, $\widetilde{\mathbb{Y}}_{p.i.}$ and $\mathbb{Y}_{p.i.}$ have the same Hilbert function.*

Remark 4.9. Proposition 3.2 and Proposition 4.7 show that the Hilbert functions of the schemes $\mathbb{X}_{grid}$ and $\mathbb{Y}_{grid}$ do not depend on the forms of degree a and b that generate $CI_{grid}(a, b)$ but it depends only on the numbers a, b and m. (see also [12]).

5. Finding a Gröbner basis for $\mathbb{Y}_{grid}$.

In this section we work over complex projective space ($k = \mathbb{C}$) and restrict our attention to the case that $m = 2$. Suppose G is a set of points in $\mathbb{P}^2$ lying in the affine slice $\mathbb{A}^2_z = \{[x, y, 1] \in \mathbb{A}^2 | x, y \in k\}$. Let $G' = \{P = (x, y) \in \mathbb{A}^2 | [x, y, 1] \in G\}$. If Z is the scheme of double points whose support is G, then its corresponding ideal is given by $I_Z = (\{\text{homogeneous} g \in k[x, y, z] | g(P) = 0, \frac{\partial g}{\partial x}(P) = 0, \frac{\partial g}{\partial y}(P) = 0, \text{ and } \frac{\partial g}{\partial z}(P) = 0 \text{ for all } P \in G\})$. Let $I_{Z'} = \{f \in k[x, y] | f(P') = 0, \frac{\partial f}{\partial x}(P') = 0, \text{ and } \frac{\partial f}{\partial y}(P') = 0 \text{ for all } P' \in G'\}$. The Hilbert function of $k[x, y, z]/I_Z$ is defined by $H_Z(t) = \dim_k k[x, y, z]_t - \dim_k(I_Z)_t$ for any natural number t. We can, with the help of Theorem 6.3 below, compute the Hilbert function of $k[x, y, z]/I_Z$ with reference only to the properties of $I_{Z'}$.

We begin by fixing basic definitions. Let σ denote the degree reverse lexicographical order on the terms in $\mathbb{T}^2 = \{x^i y^j \in k[x, y] | i, j \in \{0, 1, 2, \ldots\}\}$. Notice that this term ordering is degree compatible. For any nonzero $f \in k[x, y]$ we have $deg(f) = deg(\mathrm{LT}_\sigma(f))$. Suppose $B = \{g_1, g_2, \ldots, g_n\}$ is a Gröbner basis of $I_{Z'}$ with respect to the term ordering σ. The leading term set of the ideal $I_{Z'}$ is defined by $\mathrm{LT}_\sigma\{I_{Z'}\} = \{t \in \mathbb{T}^2 | t = \mathrm{LT}_\sigma(f) \text{ for some } f \in I_{Z'}\}$. Since B is a Gröbner basis of $I_{Z'}$, the leading term set of $I_{Z'}$ is also given by $\mathrm{LT}_\sigma\{I_{Z'}\} = \{t\mathrm{LT}_\sigma(g_i) \in \mathbb{T}^2 | t \in \mathbb{T}^2 \text{ and } g_i \in B\}$. For any $d \geq 0$, let $P_d = \{f \in k[x, y] | deg(f) \leq d\}$. P_d is a vector space over k. Let $(I_{Z'})_d$ be the degree *d or less* polynomials in $I_{Z'}$.

Recall that there is a natural vector space isomorphism $\phi_d : P_d \to k[x, y, z]_d$ given by $\phi_d(f(x, y)) = z^d f(\frac{x}{z}, \frac{y}{z})$. and that ϕ_d restricted to $(I_{Z'})_d = I_{Z'} \cap P_d$ is an isomorphism of the k-vector spaces $(I_{Z'})_d$ and $(I_Z)_d$.

Theorem 5.1. *The Hilbert function of $k[x, y, z]/I_Z$ is given by the formula $H(d) = \dim_k P_d - \dim_k(I_{Z'})_d$. If $T_d = \{t \in \mathbb{T}^2 | deg(t) \leq d\}$ and for a particular choice of d, $Q = LT_\sigma\{I_{Z'}\} \cap T_d$, then $\dim_k(I_{Z'})_d = \#Q$.*

Proof. We apply the observations of the previous paragraph and note that if $n = \#Q$, then we can list the distinct terms $t_1, t_2, \ldots, t_n$ appearing in Q in decreasing order with respect to σ. For each term t_i, choose a polynomial $g_i \in I_{Z'}$ such that $\mathrm{LT}(g_i) = t_i$. Since σ is degree compatible, $g_i \in (I_{Z'})_d$. $M = \{g_1, g_2, \ldots, g_n\}$ form a basis for the k-vector space $(I_{Z'})_d$. This is justified by noting that if $L = k_1 g_1 + k_2 g_2 + \cdots + k_n g_n = 0$, then k_1 is the coefficient of the term t_1. But $L = 0$, so the coefficient of t_1 must be 0, i.e. k_1 must be 0. Then we have $L = k_2 g_2 + k_3 g_3 + \cdots + k_n g_n = 0$. k_2 is the coefficient of t_2 and thus must be 0 since $L = 0$. Continuing the same argument, we get $k_i = 0$ for all $i = 1, 2, 3, \ldots, n$. This shows that $\{g_1, g_2, \ldots, g_n\}$ are linearly independent. The fact that $\{g_1, g_2, \ldots, g_n\}$ generate $(I_{Z'})_d$ follows easily from the division algorithm. $\square$

The first difference Hilbert function of $k[x, y, z]/I_Z$ is denoted $\Delta H(d)$ and is defined by $\Delta H(d) = H(d) - H(d-1)$ if $d \geq 1$ and $\Delta H(d) = 1$ if $d = 0$. We can rewrite the above theorem with respect to this new definition as follows.

Corollary 5.2. *For the ring $k[x, y, z]/I_Z$,*

$\Delta H(d) = \#\{t \in \mathbb{T}^2 |\ deg(t) = d$ *and* $t \notin LT_\sigma\{I_{Z'}\}$, *of the ideal* $I_{Z'}\}$.

Proof. The proof is immediate from the definition of ΔH and the above theorem. $\square$

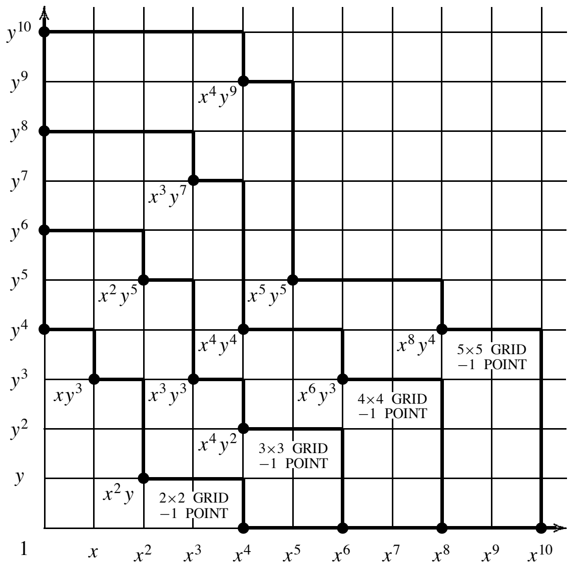


FIGURE 1

Definition 5.3. A set of points, G', in $\mathbb{A}^2$ is a *flush almost complete $N \times M$ grid* if it is a grid made up of N rows and M columns, has sides parallel to the x and y axes respectively (we are assuming the co-ordinates of the points are in $\mathbb{R}$) and is missing the point in the lower left-hand corner. In this event there exists $x_1 < x_2 < \ldots < x_M$ and $y_1 < y_2 < \ldots < y_N$ in $\mathbb{R}$ such that $G' = \{(x_i, y_j)$ such that $i \in \{1, 2, \ldots, M\}$ and $j \in \{1, 2, \ldots, N\}\} - \{(x_1, y_1)\}$. Naturally associated to G' is $G = \{[x_i : y_j : 1] \in \mathbb{P}^2$ such that $(x_i, y_j) \in G'\}$, We want the Hilbert function of the scheme of double points whose support is G. This scheme is the same as the scheme $\mathbb{Y}_{grid}$ of the previous section, except a particular point has been removed for convenience. For the remainder of the paper we tacitly assume we have in the background G', G, and the ideals $I_{Z'}$ and I_Z.

For various choices of G', we get various ideals $I_{Z'}$. Figure 1 displays leading term sets of ideals corresponding to flush almost complete 2×2, 3×3, 4×4, and 5×5 grids. In general, if G' is a particular almost complete $N \times N$ grid with corresponding ideal $I_{Z'}$, the leading term set of $I_{Z'}$ is the collection of terms which are multiples of the terms y^{2N}, x^{2N}, $x^N y^N$, $x^{N-1}y^{2N-1}$, and $x^{2N-2}y^{N-1}$.

We will find explicitly generators of the ideal $I_{Z'}$ with leading terms equal to y^{2N}, x^{2N}, $x^N y^N$, $x^{N-1}y^{2N-1}$, and $x^{2N-2}y^{N-1}$. This set of generators is a Gröbner basis of $I_{Z'}$. Actually, we do more than this. The next theorem gives a Gröbner basis for the ideal corresponding to the almost complete $N \times M$ grid.

Theorem 5.4. *Suppose G' is a flush almost complete $N \times M$ grid with $M \geq N$ and $N, M \geq 2$. Let $H_1, H_2, \ldots, H_N$ in $k[x, y]$ be polynomials corresponding to the horizontal lines of the grid running from top to bottom. Similarly, let $V_1, V_2, \ldots, V_M$ in $k[x, y]$ be polynomials corresponding to the vertical lines of the grid running from left to right. For each $i \in \{2, 3, \ldots, N\}$, let D_k be the polynomial in $k[x, y]$ corresponding to the line passing through (x_1, y_k) and (x_k, y_1). The following polynomials are generators of $I_{Z'}$.*

$$\begin{array}{rcl}
g_1 & = & H_1^2 H_2^2 \ldots H_N^2 \\
g_2 & = & V_1^2 V_2^2 \ldots V_M^2 \\
g_3 & = & H_1 H_2 \ldots H_N V_1 V_2 \ldots V_M \\
g_4 & = & H_1^2 H_2^2 H_3^2 \ldots H_N V_2 V_3 \ldots V_M \\
g_5 & = & D_2 D_3 \ldots D_N H_1 H_2 \ldots H_{N-1} V_2 V_3 \ldots V_N V_{N+1}^2 V_{N+2}^2 \ldots V_M^2
\end{array}$$

In particular, $\{g_1, g_2, g_3, g_4, g_5\}$ is a Gröbner basis of $I_{Z'}$.

Proof. We observe that $\mathrm{LT}_\sigma(H_i) = y$ for all $i \in \{1, 2, \ldots, N\}$. $\mathrm{LT}_\sigma(V_j) = x$ for all $j \in \{1, 2, \ldots, M\}$ and $\mathrm{LT}_\sigma(D_k) = x$ for all $k \in \{2, 3, \ldots, N\}$. Therefore, the leading terms of g_1, g_2, g_3, g_4, are y^{2N}, x^{2M}, $x^M y^N$, $x^{M-1}y^{2N-1}$, and $x^{2M-2}y^{N-1}$ respectively. Let $K = \{t \in \mathbb{T}^2$ that are multiples of y^{2N}, x^{2M}, $x^M y^N$, $x^{M-1}y^{2N-1}$, or $x^{2M-2}y^{N-1}\}$ and $C = \mathbb{T}^2 - K$.

$$C \cup \{x^{M-1}y^{2N-1}, x^{2M-2}y^{N-1}, x^{2M-1}y^{N-1}\} =$$

$$= \{x^p y^q \mid 0 \leq p \leq M-1, 0 \leq q \leq N-1\} \cup$$

$$\cup \{x^{p+M} y^q \mid 0 \leq p \leq M-1, 0 \leq q \leq N-1\} \cup$$

$$\cup\{x^p y^{q+N} \mid 0 \le p \le M-1, 0 \le q \le N-1\}.$$

The latter three sets are disjoint and each have MN elements. C and $\{x^{M-1}y^{2N-1}, x^{2M-2}y^{N-1}, x^{2M-1}y^{N-1}\}$ are also disjoint with the latter set having three elements. Thus $\#C = 3MN - 3 = \dim_k(k[x, y]/I_{Z'})$. So K is the leading term set of $I_{Z'}$. Since the leading terms of the polynomials in $\{g_1, g_2, g_3, g_4, g_5\}$ generate K, $\{g_1, g_2, g_3, g_4, g_5\}$ is a Gröbner basis for $I_{Z'}$. □

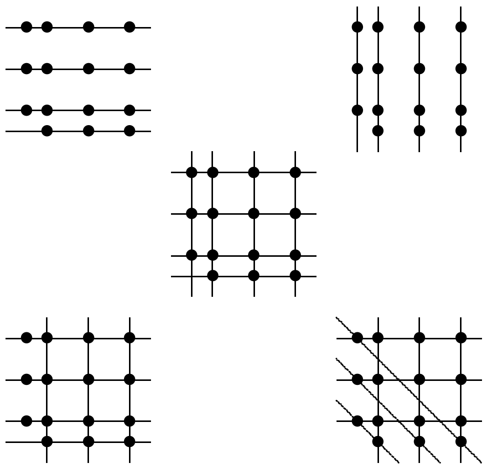


FIGURE 2

To get a sense of the geometric nature of the solution, Figure 2 gives the zero sets of the generators for $I_{Z'}$, the ideal corresponding to the flush almost complete 4×4 grid G'. The following theorem generalizes the previous one in the natural way.

Theorem 5.5. *Suppose $F \in k[x] \subseteq k[x, y]$ has distinct roots $x_1, x_2, \ldots, x_M$ and $H \in k[y] \subseteq k[x, y]$ has distinct roots $y_1, y_2, \ldots, y_N$. Let $G' = \{(x_i, y_j)$ such that $i \in \{1, 2, \ldots, M\}$, $j \in \{1, 2, \ldots, N\}$ and i and j are not both simultaneously $1\}$. If $M \ge N$ and $N, M \ge 2$, then the following polynomials generate $I_{Z'}$:*

$$\begin{aligned}
g_1 &= (y-y_1)^2(y-y_2)^2\ldots(y-y_N)^2\\
g_2 &= (x-x_1)^2(x-x_2)^2\ldots(x-x_M)^2\\
g_3 &= (y-y_1)(y-y_2)\ldots(y-y_N)(x-x_1)(x-x_2)\ldots(x-x_M)\\
g_4 &= (y-y_1)(y-y_2)^2(y-y_3)^2\ldots(y-y_N)^2(x-x_2)\ldots(x-x_M)
\end{aligned}$$

$$\begin{aligned} g_5 &= ((y_2-y_1)x+(x_2-x_1)y+(x_1y_1-x_2y_2))((y_3-y_1)x+(x_3-x_1)y \\ &\quad +(x_1y_1-x_3y_3))\dots((y_N-y_1)x+(x_N-x_1)y+(x_1y_1-x_Ny_N)) \\ &\quad (y-y_2)(y-y_3)\dots(y-y_N)(x-x_2)(x-x_3)\dots(x-x_M) \end{aligned}$$

In particular, $\{g_1, g_2, g_3, g_4, g_5\}$ *is a Gröbner Basis for* $I_{Z'}$.

Proof. Obvious. □

Theorem 5.6. *If* $M > N$*, the generators* $\{g_1, g_2, g_3, g_4, g_5\}$ *in the previous corollary can be homogenized to give a set of generators* $\{h_1, h_2, h_3, h_4, h_5\}$ *for* I_Z *which are a minimal set of generators.*

Proof. After homogenizing g_1, g_2, g_3, g_4, and g_5, we obtain the following generators for I_Z.

$$\begin{aligned} h_1 &= (y-y_1z)^2(y-y_2z)^2\dots(y-y_Nz)^2 \\ h_2 &= (x-x_1z)^2(x-x_2z)^2\dots(x-x_Mz)^2 \\ h_3 &= (y-y_1z)(y-y_2z)\dots(y-y_Nz)(x-x_1z)(x-x_2z)\dots(x-x_Mz) \\ h_4 &= (y-y_1z)(y-y_2z)^2(y-y_3z)^2\dots(y-y_Nz)^2(x-x_2z)\dots(x-x_Mz) \\ h_5 &= ((y_2-y_1)x+(x_2-x_1)y+(x_1y_1-x_2y_2)z)((y_3-y_1)x+(x_3-x_1)y \\ &\quad +(x_1y_1-x_3y_3)z)\dots((y_N-y_1)x+(x_N-x_1)y+(x_1y_1-x_Ny_N)z) \\ &\quad (y-y_2z)(y-y_3z)\dots(y-y_Nz)(x-x_2z)(x-x_3z)\dots(x-x_Mz) \end{aligned}$$

The degrees of h_1, h_2, h_3, h_4 and h_5 are $2N$, $2M$, $N+M$, $N+M+(N-2)$, and $N+M+(M-3)$ respectively. We want to show that $\{h_1, h_2, h_3, h_4, h_5\}$ is unshortenable. In other words, we want to show for each i, $h_i \notin H_i$ if H_i is the ideal generated by $\{h_1, h_2, h_3, h_4, h_5\} - \{h_i\}$. Suppose $h_1 \in (h_2, h_3, h_4, h_5)$. Then $h_1 = p_2h_2 + p_3h_3 + p_4h_4 + p_5h_5$ for some homogeneous polynomials p_2, p_3, p_4 and p_5. But $(x - x_2z)$ is a factor of $p_2h_2 + p_3h_3 + p_4h_4 + p_5h_5$ and is not a factor of h_1. So $h_1 \notin (h_2, h_3, h_4, h_5)$. Similarly, $h_2 \notin (h_1, h_3, h_4, h_5)$ since any homogeneous element in (h_1, h_3, h_4, h_5) has $(y-y_2z)$ as a factor, but h_2 does not. $h_5 \notin (h_1, h_2, h_3, h_4)$ since any homogeneous element in (h_1, h_2, h_3, h_4) vanishes on $(x_1, y_1, 1)$ and h_5 does not. Suppose $h_4 \in (h_1, h_2, h_3, h_5)$. Since $M > N$, $deg(h_4) \leq deg(h_5)$. So $h_4 = p_1h_1 + p_2h_2 + p_3h_3 + k_5h_5$ for some $k_5 \in K$ and some $p_1, p_2, p_3 \in k[x, y, z]$. k_5 must be 0 since otherwise k_5h_5 is not 0 at $(x_1, y_1, 1)$ and $h_4 - p_1h_1 - p_2h_2 - p_3h_3$ is. This is a contradiction. So $h_4 = p_1h_1 + p_2h_2 + p_3h_3$. $\frac{\partial h_4}{\partial y}(x_1, y_1, 1) \neq 0$, but $\frac{\partial(p_1h_1+p_2h_2+p_3h_3)}{\partial y}(x_1, y_1, 1) = 0$. This is a contradiction. So $h_4 \notin (h_1, h_2, h_3, h_5)$ if $M > N$.

Now consider the possibility that $h_3 \in (h_1, h_2, h_4, h_5)$. If $M \neq 3$, then $deg(h_5) > deg(h_3)$. Also the $deg(h_2) > deg(h_3)$. So $h_3 \in (h_1, h_2, h_4, h_5)$ implies $h_3 = p_1h_1+p_4h_4$. $(y-y_2z)^2$ appears in the factorization of $p_1h_1+p_4h_4$, but not in h_3. This is a contradiction. If $M = 3$, then $N = 2$ since $M > N$. So

$h_3 = L_1h_1 + k_4h_4 + k_5h_5$ for some $k_4, k_5 \in K$ and $L_1 \in (k[x, y, z])_1$. k_5 must be 0 since otherwise k_5h_5 does not vanish at $(x_1, y_1, 1)$ and $h_3 - L_1h_1 - k_4h_4$ does. So $h_3 = L_1h_1 + k_4h_4$. $\frac{\partial k_4h_4}{\partial y}(x_1, y_1, 1) \neq 0$, but $\frac{\partial(h_3-L_1h_1)}{\partial y}(x_1, y_1, 1) = 0$. This is a contradiction. So $h_3 \notin (h_1, h_3, h_4, h_5)$. □

Theorem 5.7. *If $M = N$, $\{g_1, g_2, g_3, g_5\}$ generate $I_{Z'}$ and can be homogenized to give a set of generators $\{h_1, h_2, h_3, h_5\}$ for I_Z which are a minimal set of generators.*

Proof. The polynomial yg^5 has leading term $x^{2N-2}y^N$, but after division by g_3 we get a polynomial $F = yg_5 + pg_3$ (p is determined during the division algorithm) which has leading term $x^{N-1}y^{2N-1}$. Consequently, $\{g_1, g_2, g_3, g_5, F\}$ is a Gröbner basis for $I_{Z'}$. Since $F \in (g_1, g_2, g_3, g_5)$, $\{g_1, g_2, g_3, g_5\}$ generate $I_{Z'}$. The corresponding homogeneous polynomials h_1, h_2, h_3, h_5 generate I_Z, and in fact they are an unshortenable set of generators.

Any element of (h_2, h_3, h_5) has $(x - x_2z)$ as a factor, but h_1 does not, i.e. $h_1 \notin (h_2, h_3, h_5)$.

Similarly, $h_2 \notin (h_1, h_3, h_5)$ since $(y - y_2z)$ is a factor of every element in (h_1, h_3, h_5), but is not a factor of h_2. $h_5 \notin (h_1, h_2, h_3)$ since any element of (h_1, h_2, h_3) vanishes on $(x_1, y_1, 1)$, but h_5 does not. Suppose $h_3 \in (h_1, h_2, h_5)$. If $N \geq 3$, $h_3 \in (h_1, h_2, h_5)$ implies $h_3 = k_1h_1 + k_2h_2 + k_5h_5$ for some constants k_1, k_2, and k_5 (if $N > 3$, k_5 will be 0). $k_5 = 0$ since $h_3 - k_1h_1 - k_5h_5$ vanishes on $(x_1, y_1, 1)$ and h_5 does not. So $h_3 = k_1h_1 + k_2h_2$, which is contradictory since $h_3 - k_1h_1$ has $(y - y_1z)$ as a factor and k_2h_2 does not. If $N = 2$, $h_3 \in (h_1, h_2, h_5)$ implies $h_3 = k_1h_1 + k_2h_2 + L_5h_5$ for some $k_1, k_2 \in K$ and $L_5 \in (k[x, y, z])_1$. $k_1 = 0$ since $(x - x_2z)$ is a factor of $h_3 - k_2h_2 - L_5h_5$ but is not a factor of k_1h_1. $k_2 = 0$ since $(y - y_2z)$ is a factor of $h_3 - k_1h_1 - L_5h_5$ but is not a factor of k_2h_2. This is a contradiction since there is no orm L_5 is $h_3 = L_5h_5$. So $h_3 \notin (h_1, h_2, h_5)$. □

The previous two theorems give us enough information to find the minimal free resolution of I_Z, since we know both the Hilbert function and a minimal set of generators for I_Z (see also [11] and [12]).

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