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FAT POINTS ON A GENERIC ALMOST COMPLETE INTERSECTION

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We study homogeneous schemes of fat points in $\mathbb{P}^2$ whose support is either a complete intersection (CI for short) generated by two generic forms or a CI minus a point, i.e., $\mathbb{X}_{gen} = \{CI_{gen}(a, b); m\}$ and $\mathbb{Y}_{gen} = \{CI_{gen}(a, b) \setminus P; m\}$.

We prove that $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$ whose support is on an $a \times b$ grid and $\mathbb{X}_{gen} = \{CI_{gen}(a, b); m\}$ have the same graded Betti numbers, and hence, the same Hilbert function. Moreover, if $m = 2$, then $\mathbb{Y}_{gen} = \{CI_{gen}(a, b) \setminus P; 2\}$ and $\mathbb{Y}_{grid} = \{CI_{grid}(a, b) \setminus P_{ab}; 2\}$ have the same Hilbert function, but they may not have the same graded Betti numbers.

Introduction.

This paper can be regarded as a continuation of [2]. Hence, we will rely on results, definitions, terminology and notation that we have already set in [2]. As in [2] we are concerned with studying the graded Betti numbers, and hence, the Hilbert function of the fat point schemes whose support is a CI or a CI minus a point, i.e., $\mathbb{X} = \{CI(a, b); m\}$ or $\mathbb{Y} = \{CI(a, b) \setminus P; m\}$.

In this paper we focus our attention on fat point schemes whose support is a generic CI. We show that $\mathbb{X}_{gen} = \{CI_{gen}(a, b); m\}$ has the same graded Betti numbers as $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$, and hence, as a partial intersection $\mathbb{X}_{p.i.}$ of type $(\underline{p}, \underline{q})$ where $\underline{p} = (mb, (m-1)b, (m-2)b, \dots, 2b, b)$ and,

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$\underline{q} = (a, a, \ldots, a)$. This connection is the content of Theorem 4.1 and Corollary 4.2. In the case that $m = 2$, we show that $\mathbb{Y}_{gen} = \{CI_{gen}(a, b) \setminus P; 2\}$ and $\mathbb{Y}_{grid} = \{CI_{grid}(a, b) \setminus P_{ab}; 2\}$ have the same Hilbert function (Proposition 4.4); in general, $\mathbb{Y}_{gen}$ and $\mathbb{Y}_{grid}$ may not have the same graded Betti numbers. With these results we show that the Hilbert function of $\mathbb{X}_{gen} = \{CI_{gen}(a, b); m\}$ and $\mathbb{Y}_{gen} = \{CI_{gen}(a, b) \setminus P; 2\}$ does not depend on the forms of degree a and b that generate $CI_{gen}(a, b)$ but only on the numbers a, b and m.

Our paper is structured as follows. In the first section we set our notation. Next, we recall what the current literature says in connection to our problem. In the third section, we try to determine the first integer t such that $H_{\mathbb{Y}}(t) < \binom{t+2}{2} = \dim_k R_t$. In Section 4 we examine the homogeneous scheme $\mathbb{X}_{gen} = \{CI_{gen}(a, b); m\}$ whose support $\mathbb{X}_{red} = CI_{gen}(a, b)$ is constructed on a generic CI. Moreover, we discuss the connection between our problem and the Hilbert functions of partial intersections.

Some results of this paper are part of the Ph. D. thesis of the second author ([6]).

All the results of this paper are contained in the survey [1]. We would like to thank all those involved at Pragmatic, especially A. Ragusa, A.V. Geramita, J. Migliore, C. Peterson and A. Bigatti, for introducing us to this problem. The third author would also like to acknowledge his financial support from NSERC.

1. Preliminaries.

We fix $R = k[x_0, x_1, x_2]$, where k is an algebraically closed field of characteristic zero.

Recall that we construct $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$ by taking $\mathbb{X}_{red}$ to be a CI generated by two "totally reducible" forms of degree a and b, that is, the CI is given by the intersection of two sets of generic lines in $\mathbb{P}^2$ as an $a \times b$ grid. Let us consider the homogeneous scheme $\mathbb{Y}_{grid} = \{CI_{grid}(a, b) \setminus P_{ij}; m\}$ for some $1 \leq i \leq a$ and $1 \leq j \leq b$. By renumbering the lines R_i or L_j, we can always assume $P_{ij} = P_{ab}$. We denote $\mathbb{Y}_{grid}$'s defining ideal by $I_{\mathbb{Y}_{grid}} = \mathcal{P}_1^m \cap \cdots \cap \mathcal{P}_{ab-1}^m$.

Let us describe how to construct a generic CI.

Method 1.1. To construct $\mathbb{X}_{red} = CI_{gen}(a, b)$ we pick two generic forms F and G of degrees a and b respectively which meet in ab distinct points. This implies that $\text{GCD}(F, G) = 1$ and $\{F, G\}$ is a regular sequence. So, $I_{\mathbb{X}} = (F, G)^m$. If $P \in \mathbb{X}_{red}$ and I_P is its defining ideal, then

$$I_{\mathbb{Y}} = I_{\mathbb{X}} : (I_P)^m$$

is the ideal defining the homogeneous scheme $\mathbb{Y} = \{\mathbb{X}_{red} \setminus P; m\}$. We denote this scheme by $\mathbb{Y}_{gen}$.

From now on, we will simply write $\mathbb{X}$ or $\mathbb{Y}$, when we want to mean both $\mathbb{X}_{grid}$ and $\mathbb{X}_{gen}$ or $\mathbb{Y}_{grid}$ and $\mathbb{Y}_{gen}$ respectively.

We observe that the Hilbert function of the above schemes depends on how the underlying CI is constructed. The following two examples with $m > 1$ show that $\mathbb{Y}_{grid}$ and $\mathbb{Y}_{gen}$ do not behave the same.

Example 1.2. We set $a = 3, b = 4$, and $m = 3$. Thus $\mathbb{X} = \{CI(3, 4); 3\}$. When we computed the Hilbert function of $\mathbb{Y}_{grid}$ using Bezout's Theorem we found

t	:	0	1	2	3	4	5	6	7	8	9	10	11	12	
$H_{\mathbb{Y}_{grid}}(t)$	:	1	3	6	10	15	21	28	36	45	54	62	65	66	$\to$

Using CoCoA, we found an example where the Hilbert function of $\mathbb{Y}_{gen}$ is

t	:	0	1	2	3	4	5	6	7	8	9	10	11	12
$H_{\mathbb{Y}_{gen}}(t)$	:	1	3	6	10	15	21	28	36	45	54	62	66	$\to$

The Hilbert function of the two schemes agree except at $d = 11$. This example demonstrates that the two different constructions may result in different Hilbert functions.

When we tried examples of the type $\mathbb{X} = \{CI(a, b); 2\}$, i.e., if $m = 2$, our examples had the property that $H_{\mathbb{Y}_{gen}}(t) = H_{\mathbb{Y}_{grid}}(t)$ for all t. We prove this result in Section 4.

2. Known results.

If $a = 1$, then $\mathbb{X}_{gen} = \mathbb{X}_{grid} = \{CI(1, b); m\}$ is a collection of b fat points on a line in $\mathbb{P}^2$. When we remove a point from $\mathbb{X}_{red}$ to construct $\mathbb{Y}$, the resulting scheme is simply $\mathbb{Y} = \{CI(1, b - 1); m\}$, that is, $\mathbb{Y}$ is a scheme of $b - 1$ fat points on line. We use Proposition 3.3 of [4] to compute the Hilbert function of $\mathbb{Y}$ (see also Proposition 3.1, [2]). We can also derive a formula for $H_{\mathbb{Y}}$ in the case that $\mathbb{X} = \{CI(2, b); m\}$ and the conic on which $\mathbb{X}_{red}$ is contained is non-singular. In particular, for $m = 2$ we have the following result.

Proposition 2.1. *Let* $\mathbb{X} = \{CI(2, b); 2\} \subseteq \mathbb{P}^2$ *with* $b \geq 2$ *and suppose that* $\mathbb{X}_{red}$ *is lying on a non-singular conic* $\mathcal{C}$ *in* $\mathbb{P}^2$. *Let* $\mathbb{Y} = \{\mathbb{X}_{red} \setminus P; 2\}$ *be a fat* *point scheme such that* $\mathbb{Y}_{red}$ *lies on* $\mathcal{C}$. *Then the Hilbert function of* $\mathbb{Y}$ *is given by*

$$H_{R/I_{\mathbb{Y}}}(t) = \begin{cases} 0 & t < 0 \\ 1 & t = 0 \\ 3 & t = 1 \\ 4t - 2 & 2 \leq t \leq b + 1 \\ 2t + 2b & b + 1 < t \leq 2b - 2 \\ 6b - 3 & 2b - 1 \leq t \end{cases}$$

Proof. This follows from the more general case given in [3]. □

Remark 2.2. We can use [3] to calculate $H_{R/I_{\mathbb{Y}}}$ if m is arbitrary and $\mathbb{Y}$ is as above. However, the method Catalisano describes is recursive. To compute $\mathbb{Y} = \{CI(2, b) \setminus P; m\}$, it is necessary to compute the Hilbert functions of the schemes $\mathbb{Y}_1 = \{CI(2, b) \setminus P; m-1\}$, $\mathbb{Y}_2 = \{CI(2, b) \setminus P; m-2\}, \ldots, \mathbb{Y}_{m-1} = \{CI(2, b) \setminus P\}$.

To compute the Hilbert function of $\mathbb{Y}_{m-1}$ we need to first note that $\mathbb{X} = \{CI(2, b); 1\}$ is a CI of simple points. Thus $\mathbb{X}$ has the Cayley-Bacharach Property. Recall that a set of s points $\mathbb{X}$ has the Cayley-Bacharach Property if every subset of $s - 1$ points has the same Hilbert function. If $\mathbb{X} = CI(2, b)$, then by [5] we have

$$\Delta H_{R/I_{\mathbb{Y}_{m-1}}}(t) = \begin{cases} \Delta H_{R/I_{\mathbb{X}}}(t) - 1 & \text{if } t = b \\ \Delta H_{R/I_{\mathbb{X}}}(t) & \text{otherwise.} \end{cases}$$

Remark 2.3. The above result also follows from Section 4, Theorem 4.3 of [2].

3. Comments on $\alpha(I_{\mathbb{Y}})$.

Our main goal is to compute the Hilbert function of $R/I_{\mathbb{Y}}$. Rather than trying to determine the complete Hilbert function, we decided to find something even weaker, namely $\alpha(I_{\mathbb{Y}})$. In this section we describe our work on this weaker version of the problem. Recall that if $J \subseteq S = k[x_0, \ldots, x_n]$ is a homogeneous ideal, then we define

$$\alpha(J) := \min\{t | J_t \neq 0\}.$$

Proposition 3.1. *If* $\mathbb{X}_{gen} = \{CI_{gen}(a, b); m\}$, *with* $a \leq b$, *then* $\alpha(I_{\mathbb{X}}) = ma$.

Proof. Let F and G be the two forms of degree a and b (respectively) such that $GCD(F, G) = 1$ and define the underlying CI. Since F and G form a regular sequence,

$$I_{\mathbb{X}} = (F, G)^m = (F^m, F^{m-1}G, F^{m-2}G^2, \dots, FG^{m-1}, G^m).$$

Moreover, this is a list of minimal generators for $I_{\mathbb{X}}$. Now the degrees of the generators of $I_{\mathbb{X}}$ are $\{ma, (m-1)a + b, (m-2)a + 2b, \dots, a + (m-1)b, mb\}$. Since $a \leq b$, ma is the minimal element of this list. The result now follows. □

Remark 3.2. This result is true for all $\mathbb{X} = \{CI(a, b); m\}$, regardless of the construction of $\mathbb{X}_{red}$. This result is used to calculate $\alpha(I_{\mathbb{Y}})$ in some cases.

Since $I_{\mathbb{X}} \subseteq I_{\mathbb{Y}}$, we can deduce that $\alpha(I_{\mathbb{Y}}) \leq \alpha(I_{\mathbb{X}})$. Furthermore, let $(P; m)$ be a scheme of one fat point and let $I_{(P;m)} := (I_P)^m$, where I_P is the prime ideal of forms vanishing at P. A point is a trivial CI and thus $\alpha(I_{(P;m)}) = m$. Since $I_{\mathbb{Y}} \subseteq I_{(P;m)}$, we have

$$m = \alpha(I_{(P;m)}) \leq \alpha(I_{\mathbb{Y}}) \leq ma. \tag{1}$$

Recall that $\mathbb{Y}_{grid} \subseteq \mathbb{X}_{grid}$ is constructed by removing a point P from the support of $\mathbb{X}_{grid}$. If we consider $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$, then we can improve the lower bound (1) significantly, and in some cases, completely determine $\alpha(I_{\mathbb{Y}_{grid}})$. The point P lies at the point of intersection of two lines R and L that form the basis of the grid. Suppose that L is one of the b lines and R is one of the a lines that form the underlying CI that is the support of $\mathbb{X}_{grid}$. If we remove the points on L from the support of $\mathbb{X}_{grid}$, we get a new scheme

$$\mathbb{X}_L = \{CI_{grid}(a, b) \setminus \{P\}_{P \in L}; m\} = \{CI_{grid}(a, b-1); m\}.$$

Moreover, $\mathbb{X}_L$ is a subscheme of $\mathbb{Y}_{grid}$, and thus $\alpha(I_{\mathbb{X}_L}) \leq \alpha(I_{\mathbb{Y}_{grid}})$.

Proposition 3.3. *Let $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$, with $a \leq b$, be a scheme of fat points constructed on a grid. If $b = a$, then*

$$m(a-1) < \alpha(I_{\mathbb{Y}_{grid}}) \leq ma.$$

Otherwise, if $b > a$, then

$$\alpha(I_{\mathbb{Y}_{grid}}) = ma.$$

Proof. If $b = a$, then $b - 1 = a - 1 < a$, and hence $\alpha(I_{\mathbb{X}_L}) = m(a-1)$. On the other hand, if $b > a$, then $b - 1 \geq a$, and thus $\alpha(I_{\mathbb{X}_L}) = ma$. The inequality $\alpha(I_{\mathbb{X}_L}) \leq \alpha(I_{\mathbb{Y}_{grid}}) \leq \alpha(I_{\mathbb{X}_{grid}})$ and the fact that $\alpha(I_{\mathbb{X}_{grid}}) = ma$ now give the desired result. □

Remark 3.4. In [1], we formulated the following conjecture:

Conjecture 3.5. *The results of Proposition 3.3. do not depend upon the construction of* $\mathbb{X} = \{CI(a,b); m\}$. *In particular, if* $b > a$, *then* $\alpha(I_{\mathbb{Y}}) = \alpha(I_{\mathbb{X}})$.

In this paper we prove this conjecture in the case $m = 2$. See Proposition 4.8. See [7] for any m.

Since the bound $\alpha(I_{\mathbb{Y}}) \leq \alpha(I_{\mathbb{X}})$ always holds, regardless of the construction, we can ask if there are conditions that force $\alpha(I_{\mathbb{Y}}) < \alpha(I_{\mathbb{X}})$. The following proposition gives one such necessary condition.

Proposition 3.6. *Suppose* $\mathbb{X} = \{CI(a,a); m\}$, $a > 1$ *and* $m > a^2 - a - 1$. *Then* $\alpha(I_{\mathbb{Y}}) < \alpha(I_{\mathbb{X}})$.

Proof. To show that $\alpha(I_{\mathbb{Y}}) < \alpha(I_{\mathbb{X}}) = ma$, we only need to show that there is a nonzero element of degree $ma - 1$ in $I_{\mathbb{Y}}$. Suppose we can show that $\deg \mathbb{Y} < \binom{(ma-1)+2}{2} = \dim_k R_{ma-1}$. For any zero dimensional scheme $Z \subseteq \mathbb{P}^n$, it is well known that $H_Z(t) \leq \deg Z$ for all t. This implies that $H_{\mathbb{Y}}(ma-1) \leq \deg \mathbb{Y} < \dim_k R_{ma-1}$, and thus, $(I_{\mathbb{Y}})_{ma-1} \neq 0$, as desired.

A simple calculation shows that

$$\deg \mathbb{Y} < \binom{(ma-1)+2}{2} \Leftrightarrow \frac{m^2a^2}{2} + \frac{ma^2}{2} - \frac{m^2}{2} - \frac{m}{2} < \frac{m^2a^2}{2} + \frac{ma}{2}$$

$$\Leftrightarrow ma^2 - ma - (m^2 + m) < 0$$

$$\Leftrightarrow a^2 - a - (m+1) < 0$$

By hypothesis, a is a positive integer within this bound, and thus, the conclusion follows. □

Remark 3.7. We require the hypothesis $a > 1$ to exclude the case that $\mathbb{X}$ is a single fat point. For $m = 1$, we have that $1 < a < \frac{1+\sqrt{9}}{2} = 2$. There is no positive integer within this interval. For $m = 2$, we have that $1 < a < \frac{1+\sqrt{13}}{2} \approx 2.302$. The only positive integer in this interval is $a = 2$. We found that the Hilbert functions of $\mathbb{X} = \{CI(2,2); 2\}$ and $\mathbb{Y} = \{CI(2,2)\backslash P_{22}; 2\}$ (using both constructions) are

$$\begin{array}{lcccccc} H_{\mathbb{X}} & : & 1 & 3 & 6 & 10 & 12 & \rightarrow \\ H_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & \rightarrow & \end{array}$$

In this case, $\alpha(I_{\mathbb{Y}}) = 3 < 4 = \alpha(I_{\mathbb{X}})$.

4. Results on $\mathbb{X}_{gen} = \{CI_{gen}(a, b); m\}$.

In this section we will prove some results concerning the Hilbert function of $\mathbb{X}_{gen} = \{CI_{gen}(a, b); m\}$.

Let $\mathbb{X}_{grid} = \{CI_{grid}(a, b); m\}$ be a homogeneous fat point scheme whose support $\mathbb{X}_{red} = CI_{grid}(a, b)$ is an $a \times b$ grid, and let $\mathbb{X}_{gen} = \{CI_{gen}(a, b); m\}$ be a homogeneous fat point scheme whose support is a generic CI of type (a, b). With the above notation

Theorem 4.1. *$\mathbb{X}_{grid}$ and $\mathbb{X}_{gen}$ have the same graded Betti numbers.*

Proof. We know that if F and G are the two irreducible forms of degree a and b (respectively) that define a $CI(a, b)$, then from Proposition 3.1

$$(F^m, F^{m-1}G, F^{m-2}G^2, \dots, FG^{m-1}, G^m)$$

is a list of minimal generators for $I_{\mathbb{X}_{gen}}$.

The degrees of the generators of $I_{\mathbb{X}_{gen}}$ are $\{ma, (m-1)a + b, (m-2)a + 2b, \dots, a + (m-1)b, mb\}$, so the Hilbert-Burch matrix $\mathcal{A}(\mathbb{X}_{gen})$ of $\mathbb{X}_{gen}$ is:

$$\begin{pmatrix} F & 0 & 0 & \dots & 0 \\ G & F & 0 & \dots & 0 \\ 0 & G & F & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & G \end{pmatrix}$$

and the degree matrix $\partial\mathcal{A}(\mathbb{X}_{gen})$ of $\mathbb{X}_{gen}$ is the following

$$\begin{pmatrix} a & 2a-b & \dots & ma-(m-1)b \\ b & a & \dots & (m-1)a-(m-2)b \\ 2b-a & b & \dots & (m-2)a-(m-3)b \\ \vdots & \vdots & \dots & \vdots \\ mb-(m-1)a & (m-1)b-(m-2)a & \dots & b \end{pmatrix}.$$

The syz-degrees are of type $(m-k)a + (k+1)b$ for $0 \le k \le m-1$, and hence $\mathbb{X}_{gen}$ has a resolution of type

$$(2) \qquad 0 \to \bigoplus_{t=1}^{m} \mathcal{O}_{\mathbb{P}^2}(-b_t) \to \bigoplus_{t=0}^{m} \mathcal{O}_{\mathbb{P}^2}(-a_t) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{X}_{gen}} \to 0$$

where $a_0 = ma$, $a_t = (m-t+1)b + (t-1)a$, and $b_t = (m-t+1)b + ta$, for $t = 1, \dots, m$.

From Proposition 3.2 in [2], $\mathbb{X}_{gen}$ and $\mathbb{X}_{grid}$ have the same graded Betti numbers. $\square$

Corollary 4.2. *$\mathbb{X}_{gen}$ and $\mathbb{X}_{grid}$ have the same Hilbert function.*

Before proving the main theorem of this section, we need the following result.

Lemma 4.3. *Let $Z = \{P_1, \ldots, P_s\}$ be a set of s distinct points of $\mathbb{P}^2$ lying on a curve $\mathcal{C}$. Let*

$$\mathbb{X} = \{(P_i; 2) | i = 1, \ldots, s\}$$

be a scheme of double points of degree $3s$ whose support is Z. Then there exists a subscheme $\mathbb{X}'$ of $\mathbb{X}$ contained in $\mathcal{C}$ of degree $2s$.

Proof. We can reduce the problem to the case $s = 1$. In this case we can assume that P is the origin. Now suppose that the curve $\mathcal{C}$ is defined by $g(x, y)+y = 0$, where $g(x, y)$ contains terms of degree greater than or equal to two. If $I_{\mathbb{X}}$ and $I_{\mathcal{C}}$ are the defining ideals of $\mathbb{X}$ and $\mathcal{C}$ respectively, then

$$I_{\mathbb{X}'} = I_{\mathcal{C}} + I_{\mathbb{X}} = (y, x^2)$$

and it defines a subscheme of $\mathbb{X}$ of degree 2. □

The following theorem shows that the Hilbert function of any scheme of double points whose support is a CI minus a point does not depend on how the underlying complete intersection is constructed.

Theorem 4.4. *$\mathbb{Y}_{grid} = \{CI_{grid}(a, b) \setminus P_{ab}; 2\}$ and $\mathbb{Y}_{gen} = \{CI_{gen}(a, b) \setminus P; 2\}$ have the same Hilbert function.*

Proof. For $a = 1$ we can use [4] and for $a = 2$ we can see [3].

Let us suppose $a \geq 3$. It is enough to prove that

$$\Delta H_{R/I_{\mathbb{Y}_{gen}}}(t) = \begin{cases} \Delta H_{R/I_{\mathbb{X}_{gen}}}(t) - 1 & \text{if } t = a + 2b - 2, 2a + b - 2, a + 2b - 3 \\ \Delta H_{R/I_{\mathbb{X}_{gen}}}(t) & \text{otherwise} \end{cases}$$

Since the support of $\mathbb{X}_{gen}$ is a generic $CI(a, b)$ defined by two irreducible forms F and G of degrees a and b respectively, we can find a form H defining a curve of degree $a + b - 2$ that passes through all the points of $CI(a, b)$ but not P. The form GH then defines a curve of degree $a + 2b - 2$ that passes through all the points of $\mathbb{X}_{gen}$ with multiplicity at least two but with multiplicity less than two through P.

Thus we have

$$\Delta H_{R/I_{\mathbb{Y}_{gen}}}(a + 2b - 2) \leq \Delta H_{R/I_{\mathbb{X}_{gen}}}(a + 2b - 2) - 1. \quad (3)$$

The form FH defines a curve of degree $2a + b - 2$ that passes through all the points of $\mathbb{X}_{gen}$ with multiplicity at least two but with multiplicity less than two through P. We have

$$\Delta H_{R/I_{\mathbb{Y}_{gen}}}(2a + b - 2) \leq \Delta H_{R/I_{\mathbb{X}_{gen}}}(2a + b - 2) - 1. \tag{4}$$

From (3) and (4) and since deg $(\mathbb{Y}_{gen}) =$ deg $(\mathbb{X}_{gen}) - 3$, if we show that the only permissible value for which we can find a form $L \in (I_{\mathbb{Y}_{gen}})_t \setminus (I_{\mathbb{X}_{gen}})_t$ is exactly $t = a + 2b - 3$, then we get the desired result.

By Lemma 4.3, we do not need to check for $t \leq 2a + b - 2$.

If $2b - 1 \leq 2a + b - 2$, then $\Delta H_{\mathbb{X}_{gen}}(t) = a + 2b - t - 1$ for $2a + b - 1 \leq t \leq a + 2b - 3$. Hence, for such a t, it takes on the values $b - a$ to 2, decreasing by one at each step. If we consider $\Delta H_{\mathbb{Y}_{gen}}(t') = \Delta H_{\mathbb{X}_{gen}}(t') - 1$ for a suitable $t' \in \{2a + b - 1, \dots, a + 2b - 4\}$, then using [8] there exists a curve $\mathcal{C}$ of degree $b - a - k$ for a suitable $k \in \{1, \dots, b - a - 2\}$ that contains a scheme $Z \subset \mathbb{X}_{gen}$ of degree $(b - a - k)(3a + 2k + 1)$.

Let Z^* be the scheme defined by the ideal $I_{Z^*} = (I_{\mathcal{C}}, F)$. From Bèzout's theorem, $|Z^*| = a(b - a - k)$ and a subscheme of $\mathbb{X}_{gen}$ whose support is Z^* has degree at most $3a(b - a - k)$. But

$$3a(b - a - k) < (b - a - k)(3a + 2k + 1) \Leftrightarrow 2k + 1 > 0.$$

This is always true for any $k \in \{1, \dots, b - a - 2\}$. This contradicts the irreducibility of F.

Analogously, if $2b - 1 > 2a + b - 2$, then $\Delta H_{\mathbb{X}_{gen}}(t) = a + 2b - t - 1$ for $2b \leq t \leq a + 2b - 3$. Hence, for such t, it takes on the values $a - 1$ to 2 decreasing by one at each step. However, $\Delta H_{\mathbb{Y}_{gen}}(t) \neq \Delta H_{\mathbb{X}_{gen}}(t) - 1$ for $2a + b - 1 \leq t \leq 2b - 2$, otherwise we will not have an O-sequence. By Lemma 4.3, we do not need to check for $t \leq 2b - 2$. If we consider $\Delta H_{\mathbb{Y}_{gen}}(t') = \Delta H_{\mathbb{X}_{gen}}(t') - 1$ for a suitable $t' \in \{2b - 1, \dots, a + 2b - 4\}$, then using [8] there exists a curve $\mathcal{C}'$ of degree $a - k$ for a suitable $k \in \{1, \dots, a - 2\}$ that contains a scheme $Z' \subset \mathbb{X}_{gen}$ of degree $(a - k)(2b - a + 2k + 1)$.

Let $\overline{Z}$ be the subscheme of $\mathbb{X}_{gen}$ defined by the ideal $I_{\overline{Z}} = (I_{\mathcal{C}'}, F)$. Applying Bezout's theorem, $|\overline{Z}| = a(a - k)$ and a subscheme of $\mathbb{X}_{gen}$ whose support is $\overline{Z}$ has degree at most $3a(a - k)$. But

$$3a(a - k) < (a - k)(2b - a + 2k + 1) \Leftrightarrow 2a < 2b + 2k + 1.$$

But since this is always true for any $k \in \{1, \dots, a - 2\}$, we have a contradiction for the irreducibility of F. This proves the theorem. □

Remark 4.5. In general, $\mathbb{Y}_{gen}$ and $\mathbb{Y}_{grid}$ have different graded Betti numbers, as we show in the next example.

Example 4.6. Let us consider $\mathbb{Y}_{gen} = \{CI_{gen}(2,4) \setminus P; 2\}$ and $\mathbb{Y}_{grid} = \{CI_{grid}(2,4) \setminus P_{24}; 2\}$. Using [3] we get

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-9) \oplus \mathcal{O}^2_{\mathbb{P}^2}(-7) \to$$

$$\to \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}^2_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-7) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{Y}_{gen}} \to 0$$

Using Proposition 4.7 in [2], we get

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-9) \oplus \mathcal{O}^2_{\mathbb{P}^2}(-7) \oplus \mathcal{O}_{\mathbb{P}^2}(-8) \to$$

$$\to \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}^2_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-7) \oplus \mathcal{O}_{\mathbb{P}^2}(-8) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{Y}_{grid}} \to 0$$

Remark 4.7. The formulas of Section 4 in [2] and Theorem 4.4 give us explicit formulas for the Hilbert function of $\mathbb{Y}_{gen} = \{CI_{gen}(a,b) \setminus P; 2\}$ for any a and b with $a \leq b$.

We conclude this section by using the above result to show that Conjecture 3.5 is true if $m = 2$.

Proposition 4.8. *Let* $\mathbb{X} = \{CI(a,b); 2\}$ *and let* $\mathbb{Y} = \{CI(a,b) \setminus P; 2\}$. *If* $a < b$, *then*

$$\alpha(I_{\mathbb{X}}) = \alpha(I_{\mathbb{Y}}) = 2a.$$

Proof. We only need to prove that $\Delta H_{R/I_{\mathbb{Y}}}(2a-1) = 2a = \Delta H_{R/I_{\mathbb{X}}}(2a-1)$. We can use Theorem 4.4 to compute $\Delta H_{R/I_{\mathbb{Y}}}(2a-1)$. There are two cases to consider, $a+1 = b$ and $a+1 < b$. We will show the second case since the first is similar.

To show that $\Delta H_{R/I_{\mathbb{Y}}}(2a-1) = 2a$, we only need to show that $2a-1 \neq 2a+b-2$, $a+2b-3$, or $a+2b-2$. If this is the case, then $\Delta H_{R/I_{\mathbb{Y}}}(2a-1) = \Delta H_{R/I_{\mathbb{X}}}(2a-1)$ as desired. Indeed, if $2a-1 = 2a+b-2$, then $b = -1$ which is clearly a contradiction. If $2a-1 = a+2b-3$, then $a = 2b-2$. But since $a < b-1$, then $2b-2 < b-1$. But this can only happen if $b < 1$ which is again a contradiction. A similar argument shows that $2a-1 \neq a+2b-2$. $\square$

Remark 4.9. A CI of type (a, b) in $\mathbb{P}^2$ is always generated by two homogeneous forms F and G of degree a and b (respectively) such that $\mathrm{GCD}(F, G) = 1$. The CI that we constructed via the two methods, as described in [2] and in this paper, are "extremal" in the sense that in one case F and G are totally reducible, i.e., the product of linear forms, and in the other case, F and G are irreducible forms.

If the $CI(a, b)$ is generated by two forms such that one of F and G is irreducible and the other reducible, from Theorem 4.1 and Corollary 4.2 we can say that also in this case the graded Betti numbers, and hence, the Hilbert function of a homogeneous scheme of fat points for any m, do not depend on how the underlying CI is constructed. In the same case, if the support is a CI minus a point, from Theorem 4.4 we can say that the Hilbert function of a scheme of double points does not depend on the geometry of the support. However, by Remark 4.5, if $\mathbb{Y}$ is a scheme of fat points whose support is a CI minus a point, then the graded Betti numbers depend on how we construct the CI.

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