LE MATEMATICHE
Vol. LXXX (2025) – Issue II, pp. 557–592
doi: 10.4418/2025.80.2.4

COMPARING HILBERT DEPTH OF I WITH HILBERT DEPTH OF S/I

A. I. BORDIANU - M. CIMPOEAŞ

Let $S = K[x_1,\dots,x_n]$ be the ring of polynomials over a field K and let I be a monomial ideal of S. We prove that the following are equivalent: (i) I is principal, (ii) $\operatorname{hdepth}(I) = n$, (iii) $\operatorname{hdepth}(S/I) = n-1$.

If I is squarefree, we prove that if $\operatorname{hdepth}(S/I) \leq 3$ or $n \leq 5$, then $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)+1$. Also, we prove that if $\operatorname{hdepth}(S/I) \leq 5$ or $n \leq 7$, then $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)$.

1. Introduction

Let K be a field and $S = K[x_1,\dots,x_n]$ be the polynomial ring over K. Given a finitely generated graded S-module M, the *Hilbert depth* of M, denoted by $\operatorname{hdepth}(M)$, is the maximal depth of a finitely generated graded S-module N with the same Hilbert series as M; see [1] and [7] for further details.

One would expect that it is easy to compute the Hilbert depth of a module, once its Hilbert function is known. But it turns out that even for the powers of the maximal ideal, the computation of the Hilbert depth leads to difficult numerical computations; see [2]. Another argument for studying this invariant is the fact that the Hilbert depth of a finitely generated $\mathbb{Z}^n$-graded S-module M is an upper bound for the Stanley depth of M; for further details on this topic we refer the reader to [5].

Received on August 14, 2024

AMS 2010 Subject Classification: 05A18, 06A07, 13C15, 13P10, 13F20
Keywords: Monomial ideal, Squarefree monomial ideal, Hilbert depth, Depth

Let $0 \subset I \subsetneq J \subset S$ be two squarefree monomial ideals. In [3], the authors presented a new method for computing the Hilbert depth of J/I, as follows: We consider the poset

$$P_{J/I} = \{A \subset [n] \,:\, x_A = \prod_{j \in A} x_j \in J \setminus I\} \subset 2^{[n]}.$$

We let

$$\alpha_j(J/I) = |\{A \in P_{J/I} \,:\, |A| = j\}|, \text{ for } 0 \leq j \leq n.$$

For all $0 \leq q \leq n$ and $0 \leq k \leq q$, we consider the integers

$$\beta_k^q(J/I) := \sum_{j=0}^{k} (-1)^{k-j} \binom{q-j}{k-j} \alpha_j(J/I). \tag{1.1}$$

Note that, using an inverse formula, from (1.1) we deduce that

$$\alpha_k(J/I) = \sum_{j=0}^{k} \binom{q-j}{k-j} \beta_j^q(J/I), \text{ for all } 0 \leq k \leq q \leq n. \tag{1.2}$$

With the above notations, we have the following result:

Theorem 1.1. ([3, Theorem 2.4]) The Hilbert depth of J/I is:

$$\operatorname{hdepth}(J/I) = \max\{q \,:\, \beta_k^q(J/I) \geq 0 \text{ for all } 0 \leq k \leq q\}.$$

From the proof of the above theorem, we note the fact:

Corollary 1.1. *If* $\operatorname{hdepth}(J/I) = q$*, then* $\beta_k^{q'}(J/I) \geq 0$ *for all* $0 \leq k \leq q' \leq q$*.*

If $I \subset J \subset S$ are two monomial ideals, then we consider their polarizations $I^p \subset J^p \subset R$, where R is a new ring of polynomials obtained from S by adding N new variables. The following proposition shows that we can reduce the study of the Hilbert depth of a quotient of monomial ideals to the squarefree case:

Proposition 1.2. *([3, Proposition 2.6]) The Hilbert depth of J/I is*

$$\operatorname{hdepth}(J/I) = \operatorname{hdepth}(J^p/I^p) - N.$$

Note that, if $I \subset J \subset S$ are two monomial ideals, then

$$\operatorname{depth}(J/I) \leq \operatorname{hdepth}(J/I) \leq \dim(J/I). \tag{1.3}$$

The aim of our paper is to continue the study of the Hilbert depth of monomial ideals, using the above method. In Theorem 2.3 we prove that the following are equivalent for a monomial ideal $I \subset S$:

$$(i)\ I \text{ is principal},\ (ii)\ \operatorname{hdepth}(I) = n \text{ and } (iii)\ \operatorname{hdepth}(S/I) = n-1.$$

Note that $\operatorname{hdepth}(I) = \operatorname{hdepth}(S/I) + 1$ if I is principal. More generally, if S/I is Cohen-Macaulay, then, according to [4, Theorem 2.8], we have

$$\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I) + 1.$$

It is natural to ask if this equality remains true in the non Cohen-Macaulay case. In general, the answer is no. However, we will see that in some special cases, we can still compare $\operatorname{hdepth}(S/I)$ with $\operatorname{hdepth}(I)$. In order to do that, we rely heavily on the famous Kruskal-Katona Theorem; see Theorem 3.1.

In Theorem 3.9 we prove that if $I \subset S$ is squarefree with $\operatorname{hdepth}(S/I) \leq 3$, then

$$\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I) + 1.$$

We also show that this inequality holds if $I \subset K[x_1, \ldots, x_5]$ is squarefree; see Corollary 3.1. The above results are sharp, in the sense that there we provide an example of a squarefree ideal $I \subset K[x_1, \ldots, x_6]$ with $\operatorname{hdepth}(S/I) = \operatorname{hdepth}(I) = 4$; see Example 3.10.

In Theorem 3.12 we prove that if $I \subset S$ is squarefree with $\operatorname{hdepth}(S/I) = 4$, then $\operatorname{hdepth}(I) \geq 4$. Also, in Corollary 3.3 we show that if $I \subset K[x_1, \ldots, x_6]$ is squarefree, then $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)$.

In Example 3.13, we provide a squarefree monomial ideal $I \subset K[x_1, \ldots, x_{13}]$ with $\operatorname{hdepth}(I) = 7 < \operatorname{hdepth}(S/I) = 8$. In Example 3.14, we provide a squarefree monomial ideal $I \subset K[x_1, \ldots, x_{14}]$ with $\operatorname{hdepth}(I) = 6 < \operatorname{hdepth}(S/I) = 7$. In Example 3.15, we provide a squarefree monomial ideal $I \subset K[x_1, \ldots, x_{10}]$ with $\operatorname{hdepth}(I) = 6 < \operatorname{hdepth}(S/I) = 7$.

This yields us to conjecture that $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)$ if $\operatorname{hdepth}(S/I) \leq 6$ or $n \leq 9$; see Conjecture 3.4 and Conjecture 3.5. In the last section we tackle the case $\operatorname{hdepth}(S/I) = 5$ and we show that if $\operatorname{hdepth}(S/I) = 5$ or $n = 7$, then $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)$; see Theorem 4.4 and Corollary 4.1.

2. Preliminary results

First, we prove the following combinatorial identity:

Lemma 2.1. For all $0 \leq k \leq q \leq n$ we have that

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{q-j}{k-j} \binom{n}{j} = \binom{n-q+k-1}{k}.$$

Proof. By the Chu-Vandermonde summation, we have

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{q-j}{k-j} \binom{n}{j} = \sum_{j=0}^{k} \binom{-q+k-1}{k-j} \binom{n}{j} = \binom{n-q+k-1}{k},$$

as required. □

Lemma 2.2. Let $I \subset S$ be a squarefree monomial ideal. The following are equivalent:

(1) $I = (u)$, where $u \in S$ is a squarefree monomial of degree m with $1 \leq m \leq n$.

(2) $\alpha_k(I) = \binom{n-m}{k-m}$ for all $0 \leq k \leq n$.

(3) $\alpha_k(S/I) = \binom{n}{k} - \binom{n-m}{k-m}$ for all $0 \leq k \leq n$.

(4) $\beta_k^n(I) = \delta_{km} = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$.

(5) $\beta_k^{n-1}(S/I) = \begin{cases} 1, & 0 \leq k \leq m-1 \\ 0, & m \leq k \leq n \end{cases}$.

Proof. $(1) \Rightarrow (2)$. We can assume that $u = x_{n-m+1} \cdots x_n$. It follows that a squarefree monomial v of degree k belongs to I if and only if $v = u \cdot w$, where $w \in K[x_1, \ldots, x_{n-m}]$ is squarefree with $\deg(w) = k - m$. Thus, we are done.

$(2) \Rightarrow (1)$. Since $\alpha_m(I) = 1$, it follows that there exists a squarefree monomial $u \in I$, of degree m. It follows that $L := (u) \subset I$. From $(1) \Rightarrow (2)$, it follows that $\alpha_k(L) = \alpha_k(I)$ for all $0 \leq k \leq n$. Hence, $I = (u)$, as required.

$(2) \Leftrightarrow (3)$. It follows from the obviously fact: $\alpha_k(I) = \binom{n}{k} - \alpha_k(S/I)$.

$(2) \Rightarrow (4)$. Since $\alpha_k(I) = 0$ for $0 \leq k \leq m-1$, it follows that $\beta_k^n(I) = 0$ for $0 \leq k \leq m-1$. Also, $\beta_m^n(I) = \alpha_m(I) = \binom{n-m}{0} = 1$. Now, assume that $k > m$. From Lemma 2.1, using the substitution $\ell = j - m$, we deduce that

$$\beta_k^n(I) = \sum_{j=m}^{k} (-1)^{k-j} \binom{n-j}{k-j} \binom{n-m}{j-m} = \sum_{\ell=0}^{k-m} (-1)^{k-m-\ell} \binom{(n-m)-\ell}{(k-m)-\ell} \binom{n-m}{\ell}$$
$$= \binom{(n-m)-(n-m)+k-m-1}{k-m} = \binom{k-m-1}{k-m} = 0.$$

$(4) \Rightarrow (2)$. Since $\alpha_k(I) = \sum_{j=0}^{k} \binom{n-j}{k-j} \beta_j^n(I)$ for all $0 \leq k \leq n$, by (4) it follows that $\alpha_k(I) = 0$ for $k < m$ and $\alpha_k(I) = \binom{n-m}{k-m}$ for $k \geq m$. Note that $\binom{n-m}{k-m} = 0$ for $k < m$.

$(3) \Rightarrow (5)$. For any $0 \leq k \leq n-1$, we have that

$$\beta_k^{n-1}(S/I) = \sum_{j=0}^{k} (-1)^{k-j} \binom{n-1-j}{k-j} \binom{n}{j} - \sum_{j=0}^{k} (-1)^{k-j} \binom{n-1-j}{k-j} \binom{n-m}{j-m} \quad (2.1)$$

Using the substitution $\ell = j - m$ we deduce that

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{n-1-j}{k-j} \binom{n-m}{j-m} = \sum_{\ell=0}^{k-m} (-1)^{(k-m)-\ell} \binom{n-m-1-\ell}{k-m-\ell} \binom{n-m}{\ell} \quad (2.2)$$

Using Lemma 2.1, from (2.1) and (2.2) we deduce that

$$\beta_k^{n-1}(S/I) = \binom{n-(n-1)+k-1}{k} - \binom{n-m-(n-m-1)+k-m-1}{k-m} =$$
$$= \binom{k}{k} - \binom{k-m}{k-m} = \begin{cases} 1, & 0 \le k \le m-1 \\ 0, & m \le k \le n-1 \end{cases},$$

as required.

(5) $\Rightarrow$ (3). If $0 \le k \le m-1$, then

$$\alpha_k(S/I) = \sum_{j=0}^{k} \binom{n-1-j}{k-j} \beta_j^{n-1}(S/I) = \sum_{j=0}^{k} \binom{n-1-j}{k-j} = \binom{n}{k}.$$

The last equality follows by induction on k. Indeed, if $k=0$, then there is nothing to prove. Assuming, $k \ge 1$, by induction hypothesis we have:

$$\sum_{j=0}^{k} \binom{n-1-j}{k-j} = \binom{n-1}{k} + \sum_{j=1}^{k} \binom{n-1-j}{k-j} = \binom{n-1}{k} + \sum_{j=0}^{k-1} \binom{n-2-j}{k-1-j}$$
$$= \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k},$$

as required. On the other hand, if $m \le k \le n$, then

$$\alpha_k(S/I) = \sum_{j=0}^{m-1} \binom{n-1-j}{k-j} = \binom{n}{k} - \binom{n-m}{k-m}.$$

Since $\binom{n-m}{k-m} = 0$ for $k < m$, we are done. □

Theorem 2.3. Let $(0) \ne I \subset S$ be a proper monomial ideal. The following are equivalent:

(1) I is principal.

(2) $\operatorname{hdepth}(I) = n$.

(3) $\operatorname{hdepth}(S/I) = n-1$.

Proof. Using polarization, we can assume that I is squarefree.

(1) $\Rightarrow$ (2). Since I is principal, from Lemma 2.2 it follows that $\beta_k^n(I) \ge 0$ for all $0 \le k \le n$. Therefore, $\operatorname{hdepth}(I) = n$.

(2) $\Rightarrow$ (1). Since $I \ne (0)$ and $\operatorname{hdepth}(I) = n$, it follows that

$$1 = \alpha_n(I) = \sum_{j=0}^{n} \beta_j^n(I) \text{ and } \beta_j^n(I) \ge 0 \text{ for all } 0 \le j \le n. \tag{2.3}$$

Also, since $1 \notin I$, we have $\beta_0^n(I) = \alpha_0(I) = 0$. Hence, from (2.3) it follows that there exists some integer m with $1 \leq m \leq n$ such that $\beta_m^n(I) = 1$ and $\beta_k^n(I) = 0$ for all $k \neq m$. By Lemma 2.2, we get the required result.

$(1) \Rightarrow (3)$. Also, from Lemma 2.2 it follows that $\beta_k^{n-1}(S/I) \geq 0$ for all $0 \leq k \leq n$ and thus $\operatorname{hdepth}(S/I) \geq n-1$. On the other hand, as $I \neq (0)$, $\alpha_n(S/I) = 0$ and thus $\operatorname{hdepth}(S/I) \leq n-1$. Hence, we are done.

$(3) \Rightarrow (1)$. Since $I \neq (0)$ and $\operatorname{hdepth}(S/I) = n-1$, it follows that

$$0 < m := \alpha_{n-1}(S/I) = \sum_{j=0}^{n-1} \beta_j^{n-1}(S/I) \text{ and } \beta_j^{n-1}(S/I) \geq 0 \text{ for all } 0 \leq j \leq n-1. \tag{2.4}$$

We claim that

$$\alpha_k(S/I) = \binom{n}{k} \text{ for all } 0 \leq k \leq m-1. \tag{2.5}$$

Indeed, if this is not the case, then there exists a squarefree monomial $v \in I$ with $\deg(v) = \ell < m$, let's say $v = x_1 x_2 \cdots x_\ell$.

It follows that $v_k := x_1 \cdots x_{k-1} x_{k+1} \cdots x_n \in I$ for all $\ell+1 \leq k \leq n$ and therefore $\alpha_{n-1}(S/I) \leq \ell < m$, a contradiction. Hence (2.5) is true.

From (2.5), by straightforward computations we get that

$$\beta_k^{n-1}(S/I) = 1 \text{ for all } 0 \leq k \leq m-1.$$

Hence, from (2.4) it follows that $\beta_k^{n-1}(S/I) = 0$ for all $m \leq k \leq n-1$. The required conclusion follows from Lemma 2.2. $\square$

Lemma 2.4. Let $I \subset S$ be a monomial ideal. Let $J := (I, y_1, \ldots, y_m) \subset R := S[y_1, \ldots, y_m]$, where $m \geq 1$. We have that

$$\operatorname{hdepth}(J) \geq \min\{\operatorname{hdepth}(S/I) + 1, \operatorname{hdepth}(I) + m\}.$$

Moreover, if $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)$, then $\operatorname{hdepth}(J) \geq \operatorname{hdepth}(R/J) + 1$.

Proof. Let $J_k := IR + (y_1, \ldots, y_k)R \subset R$ for $0 \leq k \leq m$. From the filtration $IR = J_0 \subset J_1 \subset \cdots \subset J_m = J$, we deduce the following K-vector space decomposition

$$J = J_0 \oplus J_1/J_0 \oplus \cdots \oplus J_m/J_{m-1}. \tag{2.6}$$

From (2.6) it follows that

$$\beta_k^d(J) = \beta_k^d(J_0) + \beta_k^d(J_1/J_0) + \cdots + \beta_k^d(J_m/J_{m-1}) \text{ for all } 0 \leq k \leq d \leq n+m. \tag{2.7}$$

Note that $J_i/J_{i-1} \cong y_i(R/J_{i-1})$ for all $1 \leq i \leq m$. Hence, from [3, Lemma 2.13] and [3, Theorem 2.22] it follows that

$$\operatorname{hdepth}(J_i/J_{i-1}) = \operatorname{hdepth}(R/J_{i-1}) = \operatorname{hdepth}((S/I)[y_i, \ldots, y_m]) =$$

$$= \operatorname{hdepth}(S/I) + m - i + 1 \text{ for all } 1 \leq i \leq m. \tag{2.8}$$

On the other hand, again from [3, Lemma 2.10] we have that

$$\operatorname{hdepth}(J_0) = \operatorname{hdepth}(IR) = \operatorname{hdepth}(I) + m. \tag{2.9}$$

From (2.7), (2.8) and (2.9) we get

$$\operatorname{hdepth}(J) \geq \min\{\operatorname{hdepth}(S/I) + 1, \operatorname{hdepth}(I) + m\},$$

as required. Also, since $R/J \cong S/I$, the last assertion follows immediately. $\square$

Remark 2.5. Let $I \subset S$ be a squarefree monomial ideal with $\operatorname{hdepth}(S/I) = q$. According to Theorem 2.3, if $q = n-1$, then I is principal and, therefore, $\operatorname{hdepth}(I) = n$. Hence, in order to compare $\operatorname{hdepth}(S/I)$ with $\operatorname{hdepth}(I)$ we can assume $q \leq n-2$. Another reduction we can make is to assume that $I \subset \mathfrak{m}^2$, where $\mathfrak{m} = (x_1, \ldots, x_n)$.

Indeed, if this is not the case, then, by reordering the variables, we can write

$$I = (I', x_{m+1}, \ldots, x_n) \text{ where } I' \subset S' = K[x_1, \ldots, x_m] \text{ with } I' \subset \mathfrak{m}'^2 \text{ and}$$

$\mathfrak{m}' = (x_1, \ldots, x_m)S'$. According to Lemma 2.4, if $\operatorname{hdepth}(I') \geq \operatorname{hdepth}(S'/I')$, then

$$\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I) + 1.$$

Note that $I \subset \mathfrak{m}^2$ if and only if $\alpha_0(I) = \alpha_1(I) = 0$, or, equivalently, $\alpha_0(S/I) = 1$ and $\alpha_1(S/I) = n$.

3. Main results

Given two positive integers ℓ, k there is a unique way to expand ℓ as a sum of binomial coefficients, as follows

$$\ell = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_j}{j}, \; n_k > n_{k-1} > \cdots > n_j \geq j \geq 1.$$

This expansion is constructed using the greedy algorithm, i.e. setting n_k to be the maximal n such that $\ell \geq \binom{n}{k}$, replace ℓ with $\ell - \binom{n_k}{k}$ and k with $k-1$ and repeat until the difference becomes zero. We define

$$\ell^{(k)} = \binom{n_k}{k+1} + \binom{n_{k-1}}{k} + \cdots + \binom{n_j}{j+1}.$$

We recall the famous Kruskal–Katona Theorem, which gives a complete characterization of the f-vectors of simplicial complexes.

Theorem 3.1. ([6, Theorem 2.1]) A vector $f = (1 = f_{-1}, f_0, f_1, \dots, f_{d-1})$ is the f-vector of some simplicial complex Δ of dimension $d-1$ if and only if

$$0 < f_i \leq f_{i-1}^{(i-1)} \text{ for all } 1 \leq i \leq d-1.$$

In the following, $(0) \neq I \subset S$ is a proper squarefree monomial ideal with $\operatorname{hdepth}(S/I) = q$, unless otherwise stated. Moreover, according to Remark 2.5, we will assume $q \leq n-2$, $\alpha_0(I) = \alpha_1(I) = 0$, $\alpha_0(S/I) = 1$ and $\alpha_1(S/I) = n$. In other words, the ideal I is not principal and $I \subset \mathfrak{m}^2$.

The following result is a direct consequence of Theorem 3.1 and of the interpretation of I as the Stanley-Reisner ideal associated to a simplicial complex:

Lemma 3.2. We have that:

$$0 < \alpha_k(S/I) \leq \alpha_{k-1}(S/I)^{(k-1)} \text{ for all } 2 \leq k \leq d.$$

In particular, if $\alpha_k(S/I) = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_j}{j}$, where $2 \leq k \leq d$ and $j \geq 1$, then

$$\alpha_{k-1}(S/I) \geq \binom{n_k}{k-1} + \binom{n_{k-1}}{k-2} + \cdots + \binom{n_j}{j-1}.$$

Proof. Let Δ be the Stanley-Reisner simplicial complex associated to I. Let $d = \dim(\Delta) + 1$. It is easy to note that

$$\alpha_j(S/I) = f_{j-1}(S/I), \text{ for all } 0 \leq j \leq d.$$

The conclusion follows easily from Theorem 3.1. □

Lemma 3.3. Let $I \subset S$ be a squarefree monomial ideal. Then:

$$\beta_k^q(I) = \binom{n-q+k-1}{k} - \beta_k^q(S/I), \text{ for all } 0 \leq k \leq q \leq n.$$

Proof. It follows from the obvious fact that $\alpha_j(I) = \binom{n}{j} - \alpha_j(S/I)$, for all $0 \leq j \leq n$, and Lemma 2.1. □

Lemma 3.4. The following are equivalent:

(1) $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I) + 1$.

(2) $\beta_{k+1}^{q+1}(S/I) \leq \binom{n-q+k-1}{k+1}$, for all $0 \leq k \leq q$.

Proof. First, note that $\beta_0^{q+1}(I) = 0$. Also, from Lemma 3.3 it follows that

$$\beta_{k+1}^{q+1}(I) = \binom{n-q+k-1}{k+1} - \beta_{k+1}^{q+1}(S/I), \text{ for all } 0 \leq k \leq q.$$

The result follows immediately. □

In the following, our aim is to prove the condition (2) of Lemma 3.3 for $q \leq 3$ and $0 \leq k \leq q$.

Lemma 3.5. For any $0 \leq k \leq q+1$, we have that

$$\beta_k^k(S/I) \leq \alpha_k(S/I).$$

Proof. First, note that $\beta_0^0(S/I) = \alpha_0(S/I) = 1$.

Assume that $1 \leq k \leq q+1$. From Corollary 1.1 and (1.1) it follows that

$$\beta_k^k(S/I) = \alpha_k(S/I) - \beta_{k-1}^{k-1}(S/I) \leq \alpha_k(S/I),$$

as required. $\square$

Lemma 3.6. If $q \geq 1$, then $\beta_2^{q+1}(S/I) \leq \binom{n-q}{2}$.

Proof. From (1.1), since $\alpha_0(S/I) = 1$, it follows that

$$\beta_2^{q+1}(S/I) = \alpha_2(S/I) - q\alpha_1(S/I) + \binom{q+1}{2}. \tag{3.1}$$

If $\alpha_2(S/I) = \binom{n_2}{2}$ for some integer $2 \leq n_2 \leq n$, then, from Lemma 3.2, it follows that $\alpha_1(S/I) \geq \binom{n_2}{1} = n_2$. Therefore, from (3.1) and Lemma 2.1 we get

$$\beta_2^{q+1}(S/I) \leq \binom{n_2}{2} - qn_2 + \binom{q+1}{2} = \binom{n_2-q}{2} \leq \binom{n-q}{2}. \tag{3.2}$$

If $\alpha_2(S/I) = \binom{n_2}{2} + \binom{n_1}{1}$ for $n > n_2 > n_1 \geq 1$, then, from Lemma 3.2, it follows that $\alpha_1(S/I) \geq n_2 + 1$. As above, we get

$$\beta_2^{q+1}(S/I) \leq \binom{n_2}{2} + n_2 - 1 - qn_2 - q + \binom{q+1}{2} < \binom{n_2-q+1}{2} \leq \binom{n-q}{2}. \tag{3.3}$$

From (3.2) and (3.3) we get the required result. $\square$

Lemma 3.7. If $q = 2$, then $\beta_3^3(S/I) \leq \binom{n-1}{3}$.

Proof. For convenience, we denote $\alpha_j := \alpha_j(S/I)$ for all j. If $\alpha_2 = \binom{n}{2}$, since $\alpha_3 \leq \binom{n}{3}$, then

$$\beta_3^3 = \alpha_3 - \alpha_2 + \alpha_1 - 1 \leq \binom{n}{3} - \binom{n}{2} + n - 1 = \binom{n-1}{3},$$

and there is nothing to prove. Hence, we may assume $\alpha_2 < \binom{n}{2}$.

Since $\alpha_0 = 1$, from (1.1) we get

$$\beta_3^3(S/I) = \alpha_3 - \alpha_2 + \alpha_1 - 1. \tag{3.4}$$

From Lemma 3.5 it follows that $\beta_3^3(S/I) \leq \alpha_3$. Hence, if $\alpha_3 \leq \binom{n-1}{3}$, then we are done.

Suppose $\alpha_3 > \binom{n-1}{3}$. From Lemma 3.2 we get $\alpha_2 > \binom{n-1}{2}$ and therefore $\alpha_1 = n$. We have two cases to consider:

(i) $\alpha_3 = \binom{n-1}{3} + \binom{n_2}{2}$ with $2 \leq n_2 \leq n-2$. From Lemma 3.2 we get $\alpha_2 \geq \binom{n-1}{2} + n_2$. Thus, from (3.4) it follows that

$$\beta_3^3(S/I) \leq \binom{n-1}{3} + \binom{n_2}{2} - \binom{n-1}{2} - n_2 + n - 1.$$

Hence, it is enough to show that

$$\binom{n_2}{2} - n_2 = \frac{n_2(n_2-3)}{2} \leq \binom{n-1}{2} - (n-1) = \frac{(n-1)(n-4)}{2},$$

which is clear as $n_2 \leq n-2$.

(ii) $\alpha_3 = \binom{n-1}{3} + \binom{n_2}{2} + \binom{n_1}{1}$ with $1 \leq n_1 < n_2 \leq n-2$. From Lemma 3.2 it follows that $\alpha_2 \geq \binom{n-1}{2} + n_2 + 1$. Thus, from (3.4), as in the case (i), it suffices to show that

$$\binom{n_2}{2} - n_2 + n_1 - 1 = \frac{n_2(n_2-3)}{2} + n_1 - 1 \leq \frac{(n-1)(n-4)}{2}.$$

Since $n_1 \leq n-3$, in order to prove the above condition, it is enough to show that

$$\frac{n_2(n_2-3)}{2} \leq \frac{(n-3)(n-4)}{2},$$

which is true, since $n_2 \leq n-2$ and $(n-2)(n-5) \leq (n-3)(n-4)$ for all $n \geq 3$.

Thus, the proof is complete. □

Lemma 3.8. If $q = 3$, then

$$\beta_3^4(S/I) \leq \binom{n-2}{3} \text{ and } \beta_4^4(S/I) \leq \binom{n-1}{4}.$$

Proof. For convenience, we denote

$$\beta_k^d = \beta_k^d(S/I) \text{ and } \alpha_k = \alpha_k(S/I), \text{ for all } 0 \leq k \leq d \leq n.$$

From hypothesis we have $\alpha_0 = 1$, $\alpha_1 = n$ and $n \geq q+2 = 5$; see Remark 2.5. It follows that

$$\beta_2^3 = \alpha_2 - 2n + 3 \geq 0, \ \beta_3^3 = \alpha_3 - \alpha_2 + n - 1 \geq 0.$$

Therefore, we have that

$$\alpha_2 \geq 2n - 3 \text{ and } \alpha_3 \geq \alpha_2 - n + 1. \tag{3.5}$$

Also, at least one of β_k^4, with $1 \leq k \leq 4$, is negative. Henceforth, we consider the cases:

(i) If $\beta_1^4 = \alpha_1 - 4 < 0$, then $n = \alpha_1 \leq 3$, a contradiction with the fact that $n \geq 5$.

(ii) If $\beta_2^4 = \alpha_2 - 3\alpha_1 + 6 < 0$, then, from (3.5), it follows that

$$3n - 7 \geq \alpha_2 \geq 2n - 3. \tag{3.6}$$

Let n_2 such that

$$\binom{n_2 - 1}{2} < 3n - 7 \leq \binom{n_2}{2}. \tag{3.7}$$

Since $\alpha_2 \leq 3n - 7$, from Lemma 3.2 it follows that

$$\alpha_3 \leq \binom{n_2}{3} \text{ and } \alpha_4 \leq \binom{n_2}{4}. \tag{3.8}$$

From (3.6) and (3.8) and the choice of n_2 we get

$$\beta_3^4 = \alpha_3 - 2\alpha_2 + 3\alpha_1 - 4 \leq \binom{n_2}{3} - n + 2. \tag{3.9}$$

Hence, in order to show that $\beta_3^4 \leq \binom{n-2}{3}$, it suffices to prove that

$$\binom{n_2}{3} \leq \binom{n-2}{3} + (n-2). \tag{3.10}$$

Note that $n_2 \leq n$, since $3n - 7 < \binom{n}{2}$. If $n_2 \leq n - 2$, then (3.10) obviously holds and, moreover, from Lemma 3.5 and (3.8) it follows that

$$\beta_4^4 \leq \alpha_4 \leq \binom{n_2}{4} \leq \binom{n-2}{4},$$

as required. Therefore, we can assume that $n_2 \in \{n-1, n\}$. We consider two subcases:

(i.a) $n_2 = n-1$. From (3.8) and Lemma 3.3 we have

$$\beta_4^4 \leq \alpha_4 \leq \binom{n-1}{4},$$

as required. From (3.7) it follows that

$$n^2-5n+8 \leq 6n-14 \leq n^2-3n+2.$$

An easy calculation shows that these inequalities hold only for $n \in \{7,8\}$.

- If $n=7$, then $n_2=6$. Also, from (3.6) we have $11 \leq \alpha_2 \leq 14$. Using Lemma 3.2 we deduce Table 1:

Table 1:

α_2	11	12	13	14
$\max(\alpha_3)$	10	11	13	16

From the table, we deduce that $\max(\alpha_3-2\alpha_2)=-12$. It follows that

$$\beta_3^4 \leq -12+21-4=5 \leq \binom{5}{3}.$$

- If $n=8$, then $n_2=7$. Also, from (3.6) we have $13 \leq \alpha_2 \leq 17$. Using Lemma 3.2 we deduce Table 2:

Table 2:

α_2	13	14	15	16	17
$\max(\alpha_3)$	13	16	20	20	21

From the table, we deduce that $\max(\alpha_3-2\alpha_2)=-10$. It follows that

$$\beta_3^4 \leq -10+24-4=10 \leq \binom{6}{3}.$$

(i.b) $n_2 = n$. From (3.7) it follows that

$$n^2-3n \leq 6n-14 \leq n^2-n.$$

These conditions hold if and only if $n \in \{2,3,4,5,6,7\}$. Since $n \geq 5$, we have in fact $n \in \{5,6,7\}$.

- $n=5$. From (3.6) it follows that $7 \leq \alpha_2 \leq 8$. Lemma 3.2 implies that
$$\alpha_2 = 7 \Rightarrow \alpha_3 \leq 4 \text{ and } \alpha_2 = 8 \Rightarrow \alpha_3 \leq 5.$$
It follows immediately that
$$\beta_3^4 = \alpha_3 - 2\alpha_2 + 3\alpha_1 - 4 = \alpha_3 - 2\alpha_2 + 11 \leq 1 = \binom{5-2}{3},$$
as required. On the other hand, from Lemma 3.5 we have $\beta_4^4 \leq \alpha_4$. If $\alpha_4 \geq 2$, then $\alpha_3 \geq 7$, a contradiction as $\alpha_3 \leq 5$. Thus $\beta_4^4 \leq 1 = \binom{5-1}{4}$, as required.
- $n=6$. From (3.6) it follows that $9 \leq \alpha_2 \leq 11$. As in the subcase (i.a), we deduce that $\max\{\alpha_3 - 2\alpha_2\} = -10$. Since $\alpha_1 = 6$, we have
$$\beta_3^4 \leq -10 + 18 - 4 = 4 = \binom{6-2}{3},$$
as required. On the other hand, since $\alpha_3 \leq 10 = \binom{5}{3}$, from Lemma 3.2 it follows that $\alpha_4 \leq \binom{5}{4} = 5$. Thus, from Lemma 3.5 it follows that
$$\beta_4^4 \leq \alpha_4 \leq 5 = \binom{6-1}{4}.$$
- $n=7$. From (3.6) it follows that $11 \leq \alpha_2 \leq 14$. As in the subcase (i.a), we deduce that $\max\{\alpha_3 - 2\alpha_2\} = -12$. Since $\alpha_1 = 7$, we have
$$\beta_3^4 \leq -12 + 21 - 4 = 5 \leq \binom{7-2}{3},$$
as required. On the other hand, since $\alpha_3 \leq 16 = \binom{5}{3} + \binom{4}{2}$, Lemma 3.2 implies $\alpha_4 \leq \binom{5}{4} + \binom{4}{3} = 9$. Thus, from Lemma 3.5 it follows that
$$\beta_4^4 \leq \alpha_4 \leq 9 < 15 = \binom{7-1}{4}.$$

(iii) If $\beta_3^4 < 0$, then, in particular, $\beta_3^4 \leq \binom{n-2}{3}$. If $\alpha_4 \leq \binom{n-1}{4}$, then, from Lemma 3.5, it follows that $\beta_4^4 \leq \binom{n-1}{4}$ and there is nothing to prove. Hence, we may assume that $\alpha_4 \geq \binom{n-1}{4} + \binom{3}{3}$. From Lemma 3.2 it follows that:
$$\alpha_3 \geq \binom{n-1}{3} + \binom{3}{2} \text{ and } \alpha_2 \geq \binom{n-1}{2} + \binom{3}{1}. \tag{3.11}$$

Since $\beta_2^4, \beta_3^3 \geq 0$, it follows that

$$\alpha_2 \geq 3n-6 \text{ and } \alpha_2 - n + 1 \leq \alpha_3 \leq 2\alpha_2 - 3n + 3. \tag{3.12}$$

From the fact that $n \geq 5$, (3.11) and (3.12) it follows that

$$\binom{n-1}{3} + 3 \leq \alpha_3 \leq 2\alpha_2 - 3n + 3 \leq 2\binom{n}{2} - 3n + 3 = (n-1)(n-3), \tag{3.13}$$

from which we deduce that $5 \leq n \leq 7$. We have the subcases:

(iii.a) $n = 5$. From (3.13) it follows that $7 \leq \alpha_3 \leq 8$. Since $\alpha_3 \leq 8 = \binom{4}{3} + \binom{3}{2} + \binom{1}{1}$, from Lemma 3.2 it follows that $\alpha_4 \leq 2$. Hence

$$\beta_4^4 = \alpha_4 - \alpha_3 + \alpha_2 - 4 \leq 2 - 7 + 10 - 4 = 1 = \binom{5-1}{4}.$$

(iii.b) $n = 6$. From (3.13) it follows that $13 \leq \alpha_3 \leq 15$. From Lemma 3.2, we deduce the following:

Table 3:

α_3	13	14	15
$\max(\alpha_4)$	6	6	7

From the above table, and the fact that $\alpha_2 \leq \binom{6}{2} = 15$, it follows that

$$\beta_4^4 = \alpha_4 - \alpha_3 + \alpha_2 - 5 \leq -7 + 15 - 5 = 3 < 5 = \binom{6-1}{4}.$$

(iii.c) $n = 7$. From (3.13) it follows that $23 \leq \alpha_3 \leq 24$. From Lemma 3.2, we deduce the following:

Table 4:

α_3	23	24
$\max(\alpha_4)$	16	17

From the above table, and the fact that $\alpha_2 \leq \binom{7}{2} = 21$, it follows that

$$\beta_4^4 = \alpha_4 - \alpha_3 + \alpha_2 - 6 \leq -7 + 21 - 6 = 8 \leq 15 = \binom{7-1}{4}.$$

(iv) If $\beta_4^4 < 0$, then, in particular, $\beta_4^4 \leq \binom{n-1}{4}$. Also, since $\beta_4^4 = \alpha_4 - \beta_3^3$, we have

$$1 \leq \alpha_4 + 1 \leq \beta_3^3 \leq \alpha_3 \leq \binom{n}{3} \text{ and thus } \alpha_3 \geq \alpha_2 - n + 2. \tag{3.14}$$

In order to avoid the previous cases, we can assume that $\beta_2^4, \beta_3^4 \geq 0$ and therefore

$$\alpha_2 \geq 3n-6,\ \alpha_3 \geq 2\alpha_2 - 3n+4. \tag{3.15}$$

From (3.15) it follows that

$$\beta_3^4 = \alpha_3 - 2\alpha_2 + 3n - 4 \leq \alpha_3 - 2(3n-6) + 3n - 4 = \alpha_3 - 3n + 8. \tag{3.16}$$

If $\alpha_3 \leq \binom{n-2}{3} + 3n - 8$, then there is nothing to prove. Assume that

$$\alpha_3 \geq \binom{n-2}{3} + 3n - 7. \tag{3.17}$$

If $n = 5$, then (3.17) implies $\alpha_3 \geq 9 = \binom{4}{3} + \binom{3}{2} + \binom{2}{1}$ and, therefore, by Lemma 3.2, we have $\alpha_2 \geq 10$. It follows that

$$\beta_3^4 = \alpha_3 - 2\alpha_2 + 3\cdot 5 - 4 \leq \alpha_3 - 9 \leq \binom{5}{3} - 9 = 1 = \binom{5-2}{3},$$

as required. Therefore, we may assume $n \geq 6$.

In order to prove that $\beta_3^4 \leq \binom{n-2}{3}$ it is enough to prove that

$$\alpha_3 - 2\alpha_2 \leq \binom{n-2}{3} - 3n + 4 = \frac{n(n-1)(n-8)}{6}. \tag{3.18}$$

We consider two subcases:

(iv.a) $\alpha_3 > \binom{n-1}{3}$. From Lemma 3.2 it follows that $\alpha_2 > \binom{n-1}{2}$. If $\alpha_3 = \binom{n}{3}$, then $\alpha_2 = \binom{n}{2}$ and therefore $\alpha_3 - 2\alpha_2 = \frac{n(n-1)(n-8)}{6}$, hence (3.18) holds.

If $\alpha_2 = \binom{n-1}{2} + n_1$ for some $1 \leq n_1 \leq n-2$, then $\alpha_3 \leq \binom{n-1}{3} + \binom{n_1}{2}$. Thus

$$\begin{aligned}\alpha_3 - 2\alpha_2 &\leq \binom{n-1}{3} - 2\binom{n-1}{2} + \binom{n_1}{2} - 2n_1 = \\ &= \frac{(n-1)(n-2)(n-9)}{6} + \frac{n_1(n_1-5)}{2}.\end{aligned} \tag{3.19}$$

Since $n \geq n_1 + 2$, it is clear that $\frac{n_1(n_1-5)}{2} \leq \frac{(n-2)(n-7)}{2}$. Thus, (3.19) implies

$$\begin{aligned}\alpha_3 - 2\alpha_2 &\leq \frac{(n-1)(n-2)(n-9)}{6} + \frac{3(n-2)(n-7)}{6} = \\ &= \frac{(n-2)(n^2-7n-12)}{6}.\end{aligned} \tag{3.20}$$

Since $n(n-1)(n-8) - (n-2)(n^2-7n-12) = 6(n-4) > 0$, from (3.20) it follows that (3.18) is satisfied.

(iv.b) $\alpha_3 \leq \binom{n-1}{3}$. From (3.17) it follows that

$$\binom{n-1}{3} - \binom{n-2}{3} - 3n + 7 \geq 0 \Leftrightarrow n^2 - 11n + 20 \geq 0.$$

Since $n \geq 5$, the above inequality implies $n \geq 9$.
From (3.17) we deduce that

$$\alpha_3 \geq \binom{n-2}{3} + 3 \cdot 9 - 7 = \binom{n-2}{3} + \binom{6}{2} + \binom{5}{1}.$$

From (3.2) it follows that $\alpha_2 \geq \binom{n-2}{2} + 7$. Therefore, we get

$$\alpha_3 - 2\alpha_2 \leq \binom{n-1}{3} - (n-2)(n-3) - 14.$$

It is easy to check that

$$\binom{n-1}{3} < (n-2)(n-3) + 14 + \frac{n(n-1)(n-8)}{6}.$$

Thus (3.18) holds.

□

Theorem 3.9. Let $(0) \neq I \subset S$ be a proper squarefree monomial ideal with $q = \operatorname{hdepth}(S/I) \leq 3$. Then

$$\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I) + 1.$$

Proof. If I is principal, then, according to Theorem 2.3, there is nothing to prove. Also, using the argument from Remark 2.5, we can assume that $I \subset \mathfrak{m}^2$. (We need this assumption in order to apply several of the previous lemmas.)

If $q = 0$, then there is nothing to prove, hence we may assume $q \geq 1$. From Lemma 3.5, it is enough to show that

$$\beta_{k+1}^{q+1}(S/I) \leq \binom{n-q+k-1}{k+1} \text{ for all } 0 \leq k \leq q. \tag{3.21}$$

Since $\alpha_0(S/I) = 1$, we have that:

$$\beta_1^{q+1}(S/I) = \alpha_1(S/I) - (q+1)\alpha_0(S/I) \leq n - q - 1 = \binom{n-q+0-1}{1},$$

and thus (3.21) holds for $k = 0$. Also, from Lemma 3.6 it follows that (3.21) holds for $k = 1$. In particular, the case $q = 1$ is proved.

Similarly, since (3.21) holds for $k \in \{0, 1\}$, the case $q = 2$ follows from Lemma 3.7 and the case $q = 3$ follows from Lemma 3.8. □

Corollary 3.1. *Let $I \subset S = K[x_1,\dots,x_n]$ be a squarefree monomial ideal. If $n \leq 5$, then*

$$\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)+1.$$

Proof. Let $q = \operatorname{hdepth}(S/I)$. If $n \leq 4$, then $q \leq 3$ and the conclusion follows from Theorem 3.9. If $n = 5$, then $q \leq 4$. If $q = 4$, then, according to Theorem 2.3, I is principal and, moreover, $\operatorname{hdepth}(I) = 5$. Also, if $q \leq 3$, then we are done by Theorem 3.9. $\square$

Let $I \subset S$ be a squarefree monomial ideal with $\operatorname{hdepth}(S/I) \geq 4$. If S/I is not Cohen-Macaulay, then the inequality $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)+1$ does not necessarily hold, as the following example shows:

Example 3.10. We consider the ideal

$$I = (x_1x_2, x_1x_3, x_1x_4, x_1x_5x_6) \subset S = K[x_1,x_2,\dots,x_6].$$

By straightforward computations, we get

$$\alpha(S/I) = (1,6,12,10,5,1,0) \text{ and } \alpha(I) = (0,0,3,10,10,5,1).$$

Also, from (1.1), we get

$$\beta^4(S/I) = (1,2,0,0,2),\ \beta_2^5(S/I) = -2 < 0,\ \beta^4(I) = (0,0,3,4,3) \text{ and } \beta_4^5(I) = -1.$$

Hence, $\operatorname{hdepth}(S/I) = \operatorname{hdepth}(I) = 4$.

Similarly to Lemma 3.4, we have the following:

Lemma 3.11. Let $I \subset S$ be a proper squarefree monomial ideal with $\operatorname{hdepth}(S/I) = q$. The following are equivalent:

(1) $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)$.

(2) $\beta_k^q(S/I) \leq \binom{n-q+k-1}{k}$, for all $1 \leq k \leq q$.

Proposition 3.2. *For any squarefree monomial ideal $I \subset S$, condition (2) from Lemma 3.11 holds for $k \in \{1,2\}$.*

Proof. If $q \geq 1$, then $\beta_1^q(S/I) = \alpha_1(S/I) - q \leq n-q$, as required. If $q \geq 2$, then, according to Lemma 3.6, applied for $q-1$, it follows that $\beta_2^q(S/I) \leq \binom{n-q+1}{2}$, as required. $\square$

As Example 3.13 shows, condition (2) of Lemma 3.11 does not hold in general for $k = 3$.

Theorem 3.12. If $I \subset S$ is a squarefree monomial with $\operatorname{hdepth}(S/I) = 4$, then

$$\operatorname{hdepth}(I) \geq 4.$$

Proof. An in the proof of Theorem 3.9, we can assume that I is not principal and $I \subseteq \mathfrak{m}^2$. In particular, $n \geq 4+2 = 6$. For convenience, we denote $\alpha_k = \alpha_k(S/I)$ and $\beta_k^s = \beta_k^s(S/I)$ for all $0 \leq k \leq s \leq n$.

Since in the proof of the Case 4 of Lemma 3.8 we have $\beta_1^4, \beta_2^4, \beta_3^4 \geq 0$ and we don't use any assumption on α_4 and on β_4^4, we can apply the same arguments in order to conclude that

$$\beta_3^4 \leq \binom{n-2}{3}.$$

Hence, in order to complete the proof, we have to show that $\beta_4^4 \leq \binom{n-1}{4}$.We will use similar methods as in the proof of Case 3 of Lemma 3.8 with the difference that, in that case, we had $\beta_3^4 < 0$. From Proposition 3.2 and the fact that $\operatorname{hdepth}(S/I) = 4$, we have

$$0 \leq \beta_1^4 = \alpha_1 - 4 = n-4 \text{ and } 0 \leq \beta_2^4 = \alpha_2 - 3\alpha_1 + 6 \leq \binom{n-3}{2}. \tag{3.22}$$

Also, we have that

$$0 \leq \beta_3^4 = \alpha_3 - 2\alpha_2 + 3n - 4 \leq \binom{n-2}{3} \text{ and } \beta_4^4 = \alpha_4 - \alpha_3 + \alpha_2 - n + 1 \geq 0 \tag{3.23}$$

On the other hand, since $\beta_4^4 = \alpha_4 - \beta_3^3$ and $\beta_3^3 \geq 0$, we can assume that

$$\alpha_4 \geq \binom{n-1}{4} + 1 = \binom{n-1}{4} + \binom{3}{3},$$

otherwise there is nothing to prove. Therefore, from Lemma 3.2, we have $\alpha_3 \geq \binom{n-1}{3} + 3$. Since $\binom{n}{4} - \binom{n}{3} + \binom{n}{2} - n + 1 = \binom{n-1}{4}$ and $\alpha_2 \leq \binom{n}{2}$, in order to complete the proof it suffices to show that

$$\alpha_4 - \alpha_3 \leq \binom{n}{4} - \binom{n}{3} = \frac{1}{24}n(n-1)(n-2)(n-7). \tag{3.24}$$

We consider two cases:

(i) $\alpha_3 = \binom{n-1}{3} + \binom{n_2}{2}$ with $n-1 > n_2 \geq 3$. From Lemma 3.2, we have $\alpha_4 \leq \binom{n-1}{4} + \binom{n_2}{3}$ and thus

$$\alpha_4 - \alpha_3 \leq \binom{n-1}{4} - \binom{n-1}{3} + \binom{n_2}{3} - \binom{n_2}{2} =$$

$$= \frac{1}{24}(n-1)(n-2)(n-3)(n-8) + \frac{1}{6}n_2(n_2-1)(n_2-5). \tag{3.25}$$

Since $n \geq 6$, if $n_2 \leq 5$, then (3.25) implies (3.24) and we are done. Now, assume that $n_2 \geq 6$ and thus $n \geq 8$. Hence, from (3.25) it follows that

$$\alpha_4 - \alpha_3 \leq \frac{1}{24}(n-1)(n-2)(n-3)(n-8) + \frac{1}{6}(n-2)(n-3)(n-7) =$$
$$= \frac{1}{24}(n-2)(n-3)(n^2-5n-20) = \frac{1}{24}(n-2)(n^3-8n^2-5n+60) \leq$$
$$\leq \frac{1}{24}(n-2)(n^3-8n^2+7n) = \binom{n}{4} - \binom{n}{3},$$

and thus we are done.

(ii) $\alpha_3 = \binom{n-1}{3} + \binom{n_2}{2} + \binom{n_1}{1}$ with $n-1 > n_2 \geq 3$ and $n_2 > n_1 \geq 1$. From Lemma 3.2, we have $\alpha_4 \leq \binom{n-1}{4} + \binom{n_2}{3} + \binom{n_1}{2}$ and thus

$$\alpha_4 - \alpha_3 \leq \binom{n-1}{4} - \binom{n-1}{3} + \binom{n_2}{3} - \binom{n_2}{2} + \binom{n_1}{2} - n_1 =$$

$$= \frac{1}{24}(n-1)(n-2)(n-3)(n-8) + \frac{1}{6}n_2(n_2-1)(n_2-5) + \frac{1}{2}n_1(n_1-3). \tag{3.26}$$

If $n_2 \leq 4$, then $n_1 \leq 3$ and thus, since $n \geq 6$, (3.26) implies (3.24). If $n_2 = 5$, then $n_1 \leq 4$ and $n \geq 7$ and, again, (3.26) implies (3.24).
If $n_2 \geq 6$, then $n \geq 8$ and from (3.26) we get

$$\binom{n}{4} - \binom{n}{3} - (\alpha_4 - \alpha_3) \geq \binom{n}{4} - \binom{n}{3} - \binom{n-1}{4} + \binom{n-1}{3} - \binom{n-2}{3}$$
$$+ \binom{n-2}{2} - \binom{n-3}{2} + \binom{n-3}{1} = n-4,$$

and thus the conclusion follows from (3.24).

$\square$

Corollary 3.3. *Let $I \subset S = K[x_1,\dots,x_6]$ be a squarefree monomial ideal. Then*

$$\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I).$$

Proof. Let $q = \operatorname{hdepth}(S/I)$. If $q = 5$, then, according to Theorem 2.3, I is principal and we have $\operatorname{hdepth}(I) = 6$. If $q \leq 4$, then the conclusion follows from Theorems 3.9 and 3.12. $\square$

We may ask if the inequality $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)$ holds in general. The answer is negative:

Example 3.13. Consider the ideal

$$I = (x_1) \cap (x_2, x_3, \dots, x_{13}) \subset K[x_1, \dots, x_{13}].$$

It is easy to check that $\alpha_0(S/I) = 1$, $\alpha_1(S/I) = 13$ and $\alpha_k(S/I) = \binom{12}{k}$ for all $2 \leq k \leq 13$. By straightforward computations, we get $\operatorname{hdepth}(S/I) = 8$. On the other hand, we have $\alpha_0(I) = \alpha_1(I) = 0$, $\alpha_2(I) = 12$ and $\alpha_3(I) = 66$. Since $\beta_3^8(I) = 66 - 6 \cdot 12 < 0$ it follows that $\operatorname{hdepth}(I) < 8$. In fact, we have $\operatorname{hdepth}(I) = 7$.

Note that, according to Lemma 3.3, $\beta_3^8(I) < 0$ is equivalent to $\beta_3^8(S/I) > \binom{13-8+3-1}{3} = \binom{7}{3}$. Hence, the condition (2) from Lemma 3.11 does not hold.

Example 3.14. Consider the ideal

$$I = (x_1) \cap (x_i x_j \ : \ 2 \leq i < j \leq 14) \subset K[x_1, \dots, x_{14}].$$

It is easy to check that $\alpha_0(S/I) = 1$, $\alpha_{14}(S/I) = 0$, $\alpha_1(S/I) = 14$, $\alpha_2(S/I) = \binom{14}{2}$ and $\alpha_k(S/I) = \binom{13}{k}$ for all $3 \leq k \leq 13$.

By straightforward computations, we get $\operatorname{hdepth}(S/I) = 7$. On the other hand, we have $\alpha_0(I) = \alpha_1(I) = \alpha_2(I) = 0$, $\alpha_3(I) = \binom{13}{2}$ and $\alpha_4(I) = \binom{13}{3}$. Since $\beta_4^7(I) = \alpha_4(I) - 4\alpha_3(I) < 0$ it follows that $\operatorname{hdepth}(I) < 7$. In fact, it is easy to check that $\operatorname{hdepth}(I) = 6$. Again, $\beta_4^7(I) < 0$ is equivalent to $\beta_4^7(S/I) > \binom{14-7+4-1}{4} = \binom{10}{4}$.

Example 3.15. Let $n = 10$ and $\alpha_j = \binom{10}{j}$ for $0 \leq j \leq 4$. We also let $\alpha_j = \binom{9}{j} + \binom{8}{j-1} + \binom{6}{j-2}$ for $5 \leq j \leq 7$. Hence $\alpha = (1, 10, 45, 120, 197, 216, 155, 70)$. From Kruskal-Katona Theorem there exists a squarefree monomial $I \subset S = K[x_1, \dots, x_{10}]$ such that $\alpha_j = \alpha_j(S/I)$ for all $0 \leq j \leq 7$. By straightforward computations, we have $\operatorname{hdepth}(S/I) = 7$ and, moreover,

$$\beta_5^7(S/I) = 24 > \binom{10-7+5-1}{5} = \binom{7}{5} = 21.$$

This implies that $\operatorname{hdepth}(I) \leq 6$. In fact, we have $\operatorname{hdepth}(I) = 6$.

The above examples yield us to propose the following conjectures:

Conjecture 3.4. If $I \subset S$ is a squarefree monomial ideal with $\operatorname{hdepth}(S/I) \leq 6$, then

$$\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I).$$

Conjecture 3.5. Let $I \subset S = K[x_1, \dots, x_n]$ be a squarefree monomial. If $n \leq 9$, then

$$\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I).$$

In the next section we will give a partial answer to these conjectures.

4. The case $\operatorname{hdepth}(S/I) = 5$.

Let $(0) \neq I \subset S$ be a proper squarefree monomial ideal with $q := \operatorname{hdepth}(S/I) \geq 5$. We will also assume that I is not principal and $I \subset \mathfrak{m}^2$. In particular, we have $n \geq q+2$.

Lemma 4.1. Suppose that $q \in \{5,6,7\}$. We have that:

$$\beta_3^q(S/I) \leq \binom{n-q+2}{3}.$$

Proof. We denote $\alpha_j := \alpha_j(S/I)$ for all $j \geq 0$, and $\beta_k^q = \beta_k^q(S/I)$ for all $0 \leq k \leq q$. From (1.1) we have that

$$\beta_3^q = \alpha_3 - (q-2)\alpha_2 + \binom{q-1}{2}\alpha_1 - \binom{q}{3}. \tag{4.1}$$

Since, from hypothesis, we have that

$$\alpha_0 = 1,\ \alpha_1 = n \text{ and } \binom{n}{3} - (q-2)\binom{n}{2} + \binom{q-1}{2}n - \binom{q}{3} = \binom{n-q+2}{3},$$

in order to complete the proof it is enough to show that

$$\alpha_3 - (q-2)\alpha_2 \leq \binom{n}{3} - (q-2)\binom{n}{2} = \frac{n(n-1)(n-3q+4)}{6}. \tag{4.2}$$

Since $q = \operatorname{hdepth}(S/I)$, it follows that

$$\beta_2^q = \alpha_2 - (q-1)\alpha_1 + \binom{q}{2} = \alpha_2 - n(q-1) + \binom{q}{2} \geq 0,\ \alpha_2 \geq \frac{(2n-q)(q-1)}{2}. \tag{4.3}$$

We consider the function

$$f(x) = \binom{x}{3} - (q-2)\binom{x}{2} = \frac{x(x-1)(x-3q+4)}{6}, \text{ where } x \geq 0. \tag{4.4}$$

The derivative of $f(x)$ is the quadratic function

$$f'(x) = \frac{1}{2}x^2 - (q-1)x + \frac{q}{2} - \frac{2}{3}.$$

Since the discriminant of $f'(x)$ is $\Delta = q^2 - 3q + \frac{7}{3}$, it follows that

$$f'(x) \leq 0 \text{ for } x \in \left[q-1-\sqrt{q^2-3q+\frac{7}{3}}, q-1+\sqrt{q^2-3q+\frac{7}{3}}\right].$$

Therefore

$$f(0) = f(1) = 0,\ f(2q-3) = f(2q-2),$$
$$f(x) \searrow \text{ on } [1,2q-3] \text{ and } f(x) \nearrow \text{ on } [2q-2,\infty). \tag{4.5}$$

Assume that $\alpha_2 = \binom{n_2}{2}$ for some integer $n_2 \geq 2$. From Lemma 3.2 it follows that $\alpha_3 \leq \binom{n_2}{3}$ and, therefore

$$\alpha_3 - (q-2)\alpha_2 \leq \binom{n_2}{3} - (q-2)\binom{n_2}{2} = f(n_2). \tag{4.6}$$

If $n_2 = n$, then from (4.6) it follows that equation (4.1) is satisfied and thus there is nothing to prove. Hence, we may assume that $n_2 \leq n-1$. Since $\alpha_2 = \binom{n_2}{2}$, from (4.3) it follows that

$$n_2(n_2-1) \geq (2n-q)(q-1). \tag{4.7}$$

If $n_2 \geq 2q-3$ and, henceforth $n \geq 2q-2$, then $f(n_2) \leq f(n)$ and, again, we are done from (4.6). Thus, we can assume that $n_2 \leq 2q-4$. Also, if $n \geq 3q-4$, then from (4.4) it follows that $f(n_2) \leq f(n)$ and there is nothing to prove. Thus $n \leq 3q-5$. We consider three cases:

(i) $q = 5$. From all the above, it follows that $n \leq 10$ and $n_2 \leq 6$. Also, $n \geq q+2 = 7$. Therefore we have $n_2(n_2-1) \leq 30$ and $(2n-q)(q-1) \geq (14-5)\cdot 4 = 36$. Thus (4.7) is impossible.

(ii) $q = 6$. Similarly, we have $8 \leq n \leq 13$ and $n_2 \leq 8$. If $n_2 \leq 7$, then $n_2(n_2-1) \leq 42$, while $(2n-q)(q-1) \geq 50$, thus (4.7) leads to a contradiction again. Also, if $n_2 = 8$, then $n \geq 9$ and thus

$$(2n-q)(q-1) \geq 60 > 56 = n_2(n_2-1).$$

So, again, (4.7) does not hold.

(iii) $q = 7$. We have $9 \leq n \leq 16$ and $n_2 \leq 10$. If $n_2 \leq 8$, then $n_2(n_2-1) \leq 56$, while $(2n-q)(q-1) \geq 66$. If $n_2 = 9$, then $n \geq 10$ and

$$(2n-q)(q-1) \geq 78 \geq 72 = n_2(n_2-1).$$

On the other hand, if $n_2 = 10$, then $n \geq 11$ and $(2n-q)(q-1) \geq 90 = n_2(n_2-1)$. Hence, (4.7) is satisfied for $n = 11$ and $n_2 = 10$. Since $\operatorname{hdepth}(S/I) = 7$ it follows that

$$\beta_4^7(S/I) = \alpha_4 - 4\alpha_3 + \binom{5}{2}\alpha_2 - \binom{6}{3}n + \binom{7}{3} \geq 0,$$

and therefore

$$\alpha_4 \geq 4\alpha_3 - 450 + 220 - 35 = 4\alpha_3 - 265. \tag{4.8}$$

Since $\alpha_2 = \binom{10}{2}$, from Lemma 3.2 it follows that $\alpha_4 \leq 210$. Hence, from (4.8) we get that $4\alpha_3 \leq 475$ and thus $\alpha_3 \leq 118$. However, as $118 = \binom{9}{3} + \binom{8}{2} + \binom{6}{1}$ this implies $\alpha_4 \leq \binom{9}{4} + \binom{8}{3} + \binom{6}{2} = 197$, which from (4.8) implies again that $4\alpha_3 \leq 265 + 197 = 462$. Thus $\alpha_3 \leq 115$. It follows that

$$\alpha_3 - (q-2)\alpha_2 = 115 - 5 \cdot \binom{10}{2} = -110 = \binom{11}{3} - 5 \cdot \binom{11}{2},$$

hence (4.2) is satisfied.

Now, assume that $\alpha_2 = \binom{n_2}{2} + \binom{n_1}{1}$, where $n_2 > n_1 \geq 1$ and $n_2 \leq n-1$. From Lemma 3.2 it follows that $\alpha_3 \leq \binom{n_2}{3} + \binom{n_1}{2}$. Therefore, we have that:

$$\alpha_3 - (q-2)\alpha_2 \leq \binom{n_2}{3} - (q-2)\binom{n_2}{2} + \binom{n_1}{2} - (q-2)\binom{n_1}{1}. \tag{4.9}$$

On the other hand, from (4.3) it follows that

$$(n_2+2)(n_2-1) \geq n_2(n_2-1) + 2n_1 = 2\alpha_2 \geq (2n-q)(q-1). \tag{4.10}$$

We consider the function

$$g(y) = \binom{y}{2} - (q-2)y, \text{ where } y \geq 0.$$

Since $g'(y) = y - (q - \frac{3}{2})$ it follows that

$$g(0) = 0, g(y) \searrow \text{ on } [0, q-2],\ g(q-2) = g(q-1) = -\binom{q-1}{2},$$

$$g(y) \nearrow \text{ on } [q-1, \infty) \text{ and } g(2q-3) = 0. \tag{4.11}$$

From (4.2) and (4.9), in order to complete the proof we have to show that

$$f(n_2) + g(n_1) \leq f(n) = \frac{n(n-1)(n-3q+4)}{6}. \tag{4.12}$$

Assume that $n_2 \geq 2q-3$. We have $n \geq 2q-2$. If $n_1 \leq 2q-3$, then from (4.5) and (4.11) it follows that

$$f(n_2) + g(n_1) \leq f(n_2) \leq f(n),$$

and thus we are done. Now, assume that $n_1 \geq 2q-2$. We have $n_2 \geq 2q-1$ and $n \geq 2q$. From (4.5) and (4.11), by straightforward computations, it follows that

$$f(n)-(f(n_2)+g(n_1)) \geq f(n_2+1)-f(n_2)-g(n_1) \geq$$
$$\geq f(n_2+1)-f(n_2)-g(n_2-1) = n_2-q+1 > 0,$$

and thus there is nothing to prove.

Now, we consider the case $n_2 \leq 2q-4$. It follows that $1 \leq n_1 \leq 2q-5$ and $g(n_1) < 0$. If $n \geq 3q-4$, from (4.4) we get

$$f(n) \geq f(n_2) > f(n_2)+g(n_1),$$

and thus we are done. Hence, we may assume that $n \leq 3q-5$. We consider three cases:

(i) $q=5$. We have $7 \leq n \leq 10$. Also $n_2 \leq 6$ and $n_1 \leq 5$. From (4.10) it follows that $n=7$, $n_2=6$ and $n_1 \in \{3,4,5\}$. Since $q-2=3$, we have $g(n_1) \leq g(5) = -5$ and therefore

$$f(n_2)+g(n_1) \leq f(6)+g(5) = -25-5 = -30 \leq -28 = f(7) = f(n),$$

and thus the required conclusion follows from (4.12).

(ii) $q=6$. We have $8 \leq n \leq 13$. Also $n_2 \leq \min\{n-1,8\}$. If $n \geq 11$, then (4.10) leads to contradiction. If $n=10$, then (4.10) implies $n_2=8$ and $n_1=7$. Hence

$$f(n_2)+g(n_1) = f(8)+g(7) = -63 < f(10) = -60,$$

and thus are done by (4.12).

If $n=9$, then (4.10) implies $n_2=8$ and $2 \leq n_1 \leq 7$. Since $g(n_1) \leq -7$ for all $2 \leq n_1 \leq 7$ it follows that

$$f(n_2)+g(n_1) = f(8)+g(n_1) \leq -56-7 = -64 < f(9) = -60,$$

and thus are done by (4.12).

If $n=8$, then (4.10) implies $n_2=7$ and $4 \leq n_1 \leq 6$. Note that $g(4) = g(5) = -10$ and $g(6) = -9$. It follows that

$$f(7)+g(n_1) \leq -49-9 = -58 < -56 = f(8),$$

and thus we are done by (4.12).

(iii) $q = 7$. We have $9 \leq n \leq 16$. Also $n_2 \leq \min\{n-1, 10\}$. If $n \geq 13$, then (4.10) leads to contradiction.

If $n = 12$, then (4.10) implies $n_2 = 10$ and $6 \leq n_1 \leq 9$. It follows that

$$f(n_2) + g(n_1) \leq f(10) + g(9) = -114 < -110 = f(12),$$

and we are done by (4.12).

If $n = 11$, then (4.10) implies $n_2 = 10$. Also, $1 \leq n_1 \leq 9$. It follows that

$$f(n_2) + g(n_1) \leq f(10) + g(1) = -110 = f(11),$$

and we are done by (4.12).

If $n = 10$, then (4.10) implies $n_2 = 9$ and $3 \leq n_1 \leq 8$. It follows that

$$f(n_2) + g(n_1) \leq f(9) + g(3) = -108 < -105 = f(10),$$

and we are done by (4.12).

Finally, if $n = 9$, then (4.10) implies $n_2 = 8$ and $5 \leq n_1 \leq 7$. It follows that

$$f(n_2) + g(n_1) \leq f(8) + g(7) = -98 < -96 = f(9),$$

and we are done by (4.12).

□

Note that, if $q \geq 8$, then the conclusion of Lemma 4.1 does not hold in general; see Example 3.14. In the following, we will assume that $q = \operatorname{hdepth}(S/I) = 5$.

Lemma 4.2. With the above notations, we have $\beta_4^5(S/I) \leq \binom{n-2}{4}$.

Proof. We denote $\alpha_j := \alpha_j(S/I)$ for all $j \geq 0$, and $\beta_k^q = \beta_k^q(S/I)$ for all $0 \leq k \leq q$. Since $\operatorname{hdepth}(S/I) \geq 5$ it follows that $n \geq 7$ and

$$\beta_2^5 = \alpha_2 - 4n + 10 \geq 0 \text{ and } \beta_3^5 = \alpha_3 - 3\alpha_2 + 6n - 10 \geq 0. \tag{4.13}$$

From (4.13) it follows that

$$\alpha_2 \geq 4n - 10,\ 3\alpha_2 \leq \alpha_3 + 6n - 10 \text{ and } \alpha_3 \geq 3\alpha_2 - 6n + 10 \geq 6n - 20. \tag{4.14}$$

On the other hand, the conclusion is equivalent to

$$\beta_4^5 = \alpha_4 - 2\alpha_3 + 3\alpha_2 - 4n + 5 \leq \binom{n-2}{4}. \tag{4.15}$$

x	1	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	0	0	-2	-7	-15	-25	-35	-42	-42	-30	0	55

We consider the functions

$$f(x) = \binom{x}{4} - 2\cdot\binom{x}{3},\ g(x) = \binom{x}{3} - 2\cdot\binom{x}{2} \text{ and } h(x) = \binom{x}{2} - 2\cdot\binom{x}{1}.$$

Note that $f(x) = \frac{1}{24}x(x-1)(x-2)(x-11)$ and we have:

Also, f is decreasing on $[2,8]$ and is increasing on $[9,\infty)$.

We have that $g(x) = \frac{1}{6}x(x-1)(x-8)$ and:

x	1	2	3	4	5	6	7	8	9
$g(x)$	0	-2	-5	-8	-10	-10	-7	0	12

Also, g is decreasing on $[1,5]$ and is increasing on $[6,\infty)$.

We have that $h(x) = \frac{1}{2}x(x-5)$ and:

x	1	2	3	4	5	6	7	8
$h(x)$	-2	-3	-3	-2	0	3	7	12

Also, h is increasing on $[3,\infty)$.

We consider several cases:

(a) $\alpha_3 = \binom{n_3}{3}$. From Lemma 3.2 it follows that $\alpha_4 \leq \binom{n_3}{4}$ and $\alpha_2 \geq \binom{n_3}{2}$. If $n_3 = n$, then $\alpha_2 = \binom{n}{2}$ and therefore

$$\beta_4^5 \leq \binom{n}{4} - 2\binom{n}{3} + 3\binom{n}{2} - 4n + 5 = \binom{n-2}{4},$$

as required. Hence, we can assume that $n_3 < n$.

Since $\alpha_2 \leq \binom{n}{2}$ and $\alpha_4 \leq \binom{n_3}{4}$, a sufficient condition to have $\beta_4^5 \leq \binom{n-2}{4}$ is $f(n_3) \leq f(n)$. This is clearly satisfied for $n \geq 11$; see the table with values of $f(x)$. Thus, we may assume that $n \leq 10$. We consider the subcases:

(i) $n = 10$. Since $f(10) = -30$, the inequality $f(n_3) \leq f(10)$ is satisfied for $n_3 \in \{7,8,9\}$ and there is nothing to prove. Thus $n_3 \leq 6$ and $\alpha_3 \leq \binom{6}{3} = 20$. From (4.14) it follows that

$$\alpha_2 \geq 4n - 10 = 30 \text{ and } 3\alpha_2 \leq \alpha_3 + 6n - 10 \leq 70,$$

which is impossible.

(ii) $n = 9$. Since $f(8) = f(9)$ we can assume that $n_3 \leq 7$ and, therefore, $\alpha_3 \leq \binom{7}{3} = 35$ and $\alpha_4 \leq \binom{7}{4} = 35$. From (4.14) it follows that

$$\alpha_2 \geq 4n-10 = 26 \text{ and } 3\alpha_2 \leq \alpha_3 + 6n - 10 \leq 79.$$

Thus $\alpha_2 = 26$. Also, from (4.14) it follows that $\alpha_3 \geq 6n-20 = 34$ and thus $\alpha_3 = 35$. We get

$$\beta_4^5 \leq 35 - 2\cdot 35 + 3\cdot 26 - 36 + 5 = 12 \leq \binom{9-2}{4} = 35,$$

as required.

(iii) $n = 8$. We have $n_3 \leq 7$ and thus $\alpha_3 \leq \binom{7}{3} = 35$ and $\alpha_4 \leq \binom{7}{4} = 35$. From (4.14) it follows that

$$3\alpha_2 \leq \alpha_3 + 38 \leq 73 \text{ and } \alpha_3 \geq 28 > \binom{6}{3}.$$

Therefore $\alpha_3 = 35$ and $\alpha_2 \leq 24$. We get:

$$\beta_4^5 \leq 35 - 2\cdot 35 + 3\cdot 24 - 32 + 5 = 10 \leq \binom{8-2}{4} = 15,$$

as required.

(iv) $n = 7$. We have $n_3 \leq 6$ and thus $\alpha_3 \leq \binom{6}{3} = 20$. From (4.14) it follows that

$$\alpha_2 \geq 4\cdot 7 - 10 = 18 \text{ and } 3\alpha_2 \leq \alpha_3 + 6\cdot 7 - 10 \leq 52,$$

which is impossible.

(b) $\alpha_3 = \binom{n_3}{3} + \binom{n_2}{2}$ with $n > n_3 > n_2 \geq 2$. From Lemma 3.2 it follows that $\alpha_4 \leq \binom{n_3}{4} + \binom{n_2}{3}$ and $\alpha_2 \geq \binom{n_3}{2} + \binom{n_2}{1}$. Similar to the case (a), if $f(n_3) + g(n_2) \leq f(n)$, then $\beta_4^5 \leq \binom{n-2}{4}$ and there is nothing to prove. If $n \geq 11$ and $n_2 \leq 8$, then, from the tables of values of $f(x)$ and $g(x)$ we have

$$f(n_3) + g(n_2) \leq f(n_3) \leq f(n),$$

and there is nothing to prove. If $n \geq 11$ and $n_2 \geq 9$, then

$$f(n) - (f(n_3) + g(n_2)) \geq f(n) - (f(n-1) + g(n-2)) = \frac{1}{2}(n^2 - 9n + 14) > 0,$$

and we are done. Hence, we may assume that $n \leq 10$. We consider the subcases:

(i) $n = 10$. Similar to the case (a.i), if $n_3 \in \{7,8,9\}$, then $f(n_3) + g(n_2) \leq f(10)$ and we are done. If $n_3 \leq 6$, then $\alpha_3 \leq \binom{6}{3} + \binom{5}{2} = 30$. On the other hand, from (4.14) it follows that $\alpha_3 \geq 6 \cdot 10 - 20 = 40$, which gives a contradiction.

(ii) $n = 9$. If $n_3 = 8$, then $f(n_3) + g(n_2) \leq f(n_3) = f(8) = f(9)$ and we are done. If $n_3 = 7$ and $n_2 \in \{4,5,6\}$, then $f(n_3) + g(n_2) \leq f(7) + g(4) = -43 < -42 = f(9)$ and we are also done. If $n_3 = 7$ and $n_2 \leq 3$, then $\alpha_3 \leq \binom{7}{3} + \binom{3}{2} = 38$, $\alpha_3 \geq \binom{7}{3} + \binom{2}{2} = 36$ and $\alpha_4 \leq \binom{7}{4} + \binom{3}{3} = 36$. From (4.14) it follows that

$$3\alpha_2 \leq \alpha_3 + 6n - 10 \leq 82, \text{ hence } \alpha_2 \leq 27.$$

Therefore

$$\beta_4^5 \leq 36 - 2 \cdot 36 + 3 \cdot 27 - 36 + 5 = 14 \leq \binom{n-2}{4} = \binom{7}{4} = 35,$$

as required. Now, if $n_3 \leq 6$, then, as in the case (i), we have $\alpha_3 \leq 30$, which contradict the fact that $\alpha_3 \geq 6 \cdot 9 - 20 = 34$.

(iii) $n = 8$. As in the case (ii), if $n_3 = 7$ and $n_2 \in \{4,5,6\}$, then there is nothing to prove. Also, if $n_3 = 7$ and $n_2 \leq 3$, then $\alpha_3 \in \{36,38\}$ and $\alpha_4 \leq 36$. From (4.14) it follows that $3\alpha_2 \leq \alpha_3 + 6n - 10 \leq 76$, hence $\alpha_2 \leq 25$. Therefore

$$\beta_4^5 \leq 36 - 2 \cdot 36 + 3 \cdot 25 - 32 + 5 = 12 \leq \binom{8-2}{4} = 15,$$

and we are done. Now, if $n_3 \leq 6$, then $\alpha_3 \leq \binom{6}{3} + \binom{5}{2} = 30$ and thus $\alpha_4 \leq \binom{6}{4} + \binom{5}{3} = 25$. Also, from (4.14) we have

$$\alpha_3 \geq 6 \cdot 8 - 20 = 28 \text{ and } 3\alpha_2 \leq 30 + 38 = 68.$$

Hence $\alpha_2 \leq 22$. It follows that

$$\beta_4^5 \leq 25 - 2 \cdot 28 + 3 \cdot 22 - 32 + 5 = 8 \leq \binom{6}{4} = 15,$$

and we are done.

(iv) $n = 7$. If $n_3 = 6$ and $n_2 = 5$, then $f(n_3) + g(n_2) = -25 - 10 = -35 = f(n)$ and there is nothing to prove. Assume that $n_3 \leq 6$ and $n_2 \leq 4$. It follows that $\alpha_3 \leq \binom{6}{3} + \binom{4}{2} = 26$ and $\alpha_4 \leq \binom{6}{4} + \binom{4}{3} = 19$. From (4.14) we get

$$\alpha_2 \geq 4 \cdot 7 - 10 = 18 \text{ and } 3\alpha_2 \leq \alpha_3 + 6 \cdot 7 - 10 \leq 58.$$

Hence $\alpha_2 \in \{18,19\}$. Also, $\alpha_3 \geq 6\cdot 7-20 = 22$ and thus $\alpha_3 \geq \binom{6}{3}+\binom{3}{2} = 23$. If $\alpha_3 = 26$, then $\alpha_4 \leq 19$ and

$$\beta_4^5 \leq 19-2\cdot 26+3\cdot 19-28+5 = 1 \leq \binom{7-2}{4} = 5,$$

and we are done. If $\alpha_3 = 23$, then $\alpha_4 \leq \binom{6}{4}+\binom{3}{3} = 16$ and

$$\beta_4^5 \leq 16-2\cdot 23+3\cdot 19-28+5 = 4 \leq \binom{7-2}{4} = 5,$$

and we are done, also.

(c) $\alpha_3 = \binom{n_3}{3}+\binom{n_2}{2}+\binom{n_1}{1}$ with $n > n_3 > n_2 > n_1 \geq 1$. From Lemma 3.2 it follows that $\alpha_4 \leq \binom{n_3}{4}+\binom{n_2}{3}+\binom{n_1}{2}$ and $\alpha_2 \geq \binom{n_3}{2}+\binom{n_2}{1}$. Similar to the previous cases, if $f(n_3)+g(n_2)+h(n_1) \leq f(n)$, then $\beta_4^5 \leq \binom{n-2}{4}$ and there is nothing to prove. If $n \geq 11$, then

$$f(n)-f(n_3)-g(n_2)-h(n_1) \geq f(n)-f(n-1)-g(n-2)-h(n-3) = n-5 \geq 6,$$

and we are done. Hence, we may assume that $n \leq 10$. We consider the subcases:

(i) $n = 10$. If $n_3 \in \{7,8,9\}$, then, looking at the values of $f(x), g(x)$ and $h(x)$, it is easy to check that $f(n_3)+g(n_2)+h(n_1) \leq f(10)$, and we are done. If $n_3 \leq 6$, then $\alpha_3 \leq \binom{6}{3}+\binom{5}{2}+\binom{4}{1} = 34$. On the other hand, from (4.14) it follows that $\alpha_3 \geq 6\cdot 10-20 = 40$, which gives a contradiction.

(ii) $n = 9$. If $n_3 = 8$, then $n_2 \leq 7$ and $n_1 \leq 6$. In particular, $g(n_2) \leq 0$. If $n_1 \leq 5$, then $h(n_1) \leq 0$. If $n_1 = 6$, then $n_2 = 7$ and $g(n_2)+h(n_1) = -7+3 = -4 \leq 0$. It follows that

$$f(n_3)+g(n_2)+h(n_1) \leq f(n_3) = f(8) = -42 = f(9) = f(n),$$

and we are done. If $n_3 = 7$ and $n_2 \in \{4,5,6\}$, then $n_1 \leq 5$, $h(n_1) \leq 0$ and, moreover, we have

$$f(n_3)+g(n_2)+h(n_1) \leq f(n_3)+g(n_2) \leq -35-8 = -43 < -42 = f(n),$$

are we are also done. If $n_3 = 7$ and $n_2 \leq 3$, then $n_1 \leq 2$ and therefore

$$\binom{7}{3}+\binom{2}{2}+\binom{1}{1} = 37 \leq \alpha_3 \leq 40 = \binom{7}{3}+\binom{3}{2}+\binom{2}{1}.$$

From (4.14) it follows that

$$3\alpha_2 \leq \alpha_3 + 6n - 10 \leq 84, \text{ hence } \alpha_2 \leq 28.$$

Note that $\alpha_4 \leq \binom{7}{4} + \binom{3}{3} + \binom{2}{2} = 37$. Therefore

$$\beta_4^5 \leq 37 - 2\cdot 37 + 3\cdot 28 - 36 + 5 = 16 \leq \binom{n-2}{4} = \binom{7}{4} = 35,$$

as required. On the other hand, if $n_3 \leq 6$, then $n_2 \leq 5$ and $n_1 \leq 4$ and thus $\alpha_3 \leq \binom{6}{3} + \binom{5}{2} + \binom{4}{1} = 34$. Also, from (4.14) it follows that $\alpha_3 \geq 6n - 20 = 34$. Thus $\alpha_3 = 34$ and $\alpha_4 \leq \binom{6}{4} + \binom{5}{3} + \binom{4}{2} = 31$. Also, from (4.14) it follows that

$$3\alpha_2 \leq \alpha_3 + 6n - 10 \leq 78, \text{ hence } \alpha_2 \leq 26.$$

Therefore

$$\beta_4^5 \leq 31 - 2\cdot 34 + 3\cdot 26 - 36 + 5 = 10 \leq \binom{7}{4}.$$

Hence, we are done.

(iii) $n = 8$. As in the case (ii), if $n_3 = 7$ and $n_2 \in \{4,5,6\}$, then there is nothing to prove. Also, using the argument from the case (ii), if $n_2 \leq 3$, then $37 \leq \alpha_3 \leq 40$ and $\alpha_4 \leq 37$. From (4.14) it follows that

$$3\alpha_2 \leq \alpha_3 + 6n - 10 \leq 78, \text{ hence } \alpha_2 \leq 26.$$

Therefore

$$\beta_4^5 \leq 37 - 2\cdot 37 + 3\cdot 26 - 32 + 5 = 14 \leq 15 = \binom{6}{4} = \binom{n-2}{4},$$

and we are done. Now, if $n_3 \leq 6$, then $\alpha_3 \leq 34$ and thus $\alpha_4 \leq 31$. Also, from (4.14) it follows that $\alpha_3 \geq 6n - 20 = 28$ and

$$3\alpha_2 \leq \alpha_3 + 6n - 10 \leq 72, \text{ hence } \alpha_2 \leq 24.$$

If $\alpha_2 \leq 22$, then

$$\beta_4^5 \leq 31 - 2\cdot 28 + 3\cdot 22 - 32 + 5 = 14 \leq \binom{8-2}{4} = 15,$$

and we are done. Similarly, if $\alpha_3 \geq 31$, then

$$\beta_4^5 \leq 31 - 2\cdot 31 + 3\cdot 24 - 32 + 5 = 14 \leq \binom{8-2}{4} = 15,$$

and we are done. So we may assume $\alpha_3 \in \{28,29,30\}$ and $\alpha_2 \in \{23,24\}$.
If $\alpha_3 = 28 = \binom{6}{3}+\binom{4}{2}+\binom{2}{1}$, then, from Lemma 3.2, it follows that $\alpha_4 \leq \binom{6}{4}+\binom{4}{3}+\binom{2}{2} = 20$ and thus

$$\beta_4^5 \leq 20-2\cdot 28+3\cdot 24-32+5 = 9 < 15,$$

as required. If $\alpha_3 = 29 = \binom{6}{3}+\binom{4}{2}+\binom{3}{1}$, then, from Lemma 3.2, it follows that $\alpha_4 \leq \binom{6}{4}+\binom{4}{3}+\binom{3}{2} = 22$ and thus

$$\beta_4^5 \leq 22-2\cdot 29+3\cdot 24-32+5 = 9 < 15,$$

as required. If $\alpha_3 = 29 = \binom{6}{3}+\binom{5}{2}$, then, from Lemma 3.2, it follows that $\alpha_4 \leq \binom{6}{4}+\binom{5}{3} = 25$ and thus

$$\beta_4^5 \leq 25-2\cdot 30+3\cdot 24-32+5 = 10 < 15,$$

and we are done.

(iv) $n=7$. If $n_3 = 6$ and $n_2 = 5$, then $n_1 \leq 4$ and therefore

$$f(n_3)+g(n_2)+h(n_1) \leq f(n_3)+g(n_2) = -25-10 = -35 = f(n),$$

and we are done. Assume that $n_3 \leq 6$ and $n_2 \leq 4$. It follows that $\alpha_3 \leq \binom{6}{3}+\binom{4}{2}+\binom{3}{1} = 29$ and $\alpha_4 \leq \binom{6}{4}+\binom{4}{3}+\binom{3}{2} = 22$. From (4.14) we get

$$\alpha_2 \geq 4\cdot 7-10 = 18 \text{ and } 3\alpha_2 \leq \alpha_3+6\cdot 7-10 \leq 61.$$

Hence $\alpha_2 \in \{18,19,20\}$. Also, $\alpha_3 \geq 6\cdot 7-20 = 22$. If $\alpha_2 = 18 = \binom{6}{2}+\binom{3}{1}$, then, from Lemma 3.2, it follows that $\alpha_4 \leq \binom{6}{4}+\binom{3}{3} = 16$ and thus

$$\beta_4^5 \leq 16-2\cdot 22+3\cdot 18-28+5 = 3 < 5 = \binom{7-2}{4}.$$

If $\alpha_2 \geq 19$, then $\alpha_3 \geq 25$ and thus and thus $\alpha_3 \in \{25,28,29\}$. If $\alpha_3 = 25$, then $\alpha_4 \leq 17$ and thus $\alpha_4 - 2\alpha_3 \leq -33$. Also $3\alpha_2 \leq 25+32 = 57$ and thus $\alpha_2 \leq 19$. It follows that

$$\beta_4^5 \leq -33+3\cdot 19-28+5 = 1 \leq \binom{7-2}{4} = 5,$$

and we are done. If $\alpha_3 \geq 28$, then

$$\beta_4^5 \leq 22-2\cdot 28+3\cdot 20-28+5 = 3 \leq \binom{7-2}{4} = 5,$$

and we are also done.

Hence, the proof is complete. □

Lemma 4.3. With the above notations, we have that $\beta_5^5 \leq \binom{n-1}{5}$.

Proof. Since $\beta_5^5 = \alpha_5 - \beta_4^4$ and $\beta_4^4 \geq 0$, the conclusion follows immediately if $\alpha_5 \leq \binom{n-1}{5}$. Hence, we may assume that

$$\alpha_5 \geq \binom{n-1}{5} + 1 = \binom{n-1}{5} + \binom{4}{4}.$$

From Lemma 3.2 it follows that $\alpha_j \geq \binom{n-1}{j} + \binom{4}{j}$ for all $j \in \{2,3,4\}$. We claim that

$$\alpha_3 - \alpha_2 \leq \binom{n}{3} - \binom{n}{2} = \frac{1}{6}n(n-1)(n-5). \tag{4.16}$$

If $\alpha_2 = \binom{n}{2}$, then there is nothing to prove. Assume that $\alpha_2 = \binom{n-1}{2} + \binom{n_1}{1}$, where $n-1 > n_1 \geq 4$. Then, from Lemma 3.2, we have $\alpha_3 \leq \binom{n-1}{3} + \binom{n_1}{2}$. It follows that

$$\begin{aligned}
\alpha_3 - \alpha_2 &\leq \binom{n-1}{3} + \binom{n_1}{2} - \binom{n-1}{2} - \binom{n_1}{1} = \\
&= \frac{1}{6}(n-1)(n-2)(n-6) + \frac{1}{2}n_1(n_1-3) \leq \\
&\leq \frac{1}{6}(n-1)(n-2)(n-6) + \frac{1}{2}(n-2)(n-5) = \\
&= \frac{1}{6}(n^3 - 6n^2 - n + 18) < \frac{1}{6}n(n-1)(n-5),
\end{aligned}$$

and thus the claim (4.16) holds. Hence, in order to complete the proof it is enough to show that

$$\alpha_5 - \alpha_4 \leq \binom{n}{5} - \binom{n}{4} = \frac{1}{120}n(n-1)(n-2)(n-3)(n-9). \tag{4.17}$$

We denote $f_k(x) = \binom{x}{k} - \binom{x}{k-1}$ for all $2 \leq k \leq 5$. We have the following table of values:

x	1	2	3	4	5	6	7	8	9
$f_5(x)$	0	0	0	-1	-4	-9	-14	-14	0
$f_4(x)$	0	0	-1	-3	-5	-5	0	14	42
$f_3(x)$	0	-1	-2	-2	0	5	14	28	48
$f_2(x)$	-1	-1	0	2	5	9	14	20	27

We consider several cases:

(a) $\alpha_4 = \binom{n-1}{4} + \binom{n_3}{3}$, where $n-1 > n_3 \geq 4$. If $n = 7$, then $n_3 \in \{4,5\}$ and therefore $\binom{n_3}{4} - \binom{n_3}{3} \in \{-3,-5\}$. Also, we have $\alpha_5 \leq \binom{6}{5} + \binom{n_3}{4}$. If $n_3 = 5$, then

$$\alpha_5 - \alpha_4 \leq \binom{6}{5} - \binom{6}{4} - 5 = -14 = \binom{7}{5} - \binom{7}{4},$$

and the condition (4.17) is fulfilled. Now, assume that $n_3 = 4$, that is $\alpha_4 = \binom{6}{4} + \binom{4}{3} = 19$. Since

$$\beta_4^4 = \alpha_4 - (\alpha_3 - \alpha_2) - 7 + 1 = 13 - (\alpha_3 - \alpha_2) \geq 0,$$

it follows that $\alpha_3 - \alpha_2 \leq 13$. Since $\alpha_5 \leq \binom{6}{5} + \binom{4}{4} = 7$, we have $\alpha_5 - \alpha_4 \leq -12$. If $\alpha_3 - \alpha_2 \leq 12$, then

$$\beta_5^5 = (\alpha_5 - \alpha_4) + (\alpha_3 - \alpha_2) + 7 - 1 \leq 6 = \binom{7-1}{5},$$

as required. So, we may assume $\alpha_3 - \alpha_2 = 13$. Note that, if $\alpha_2 = \binom{a}{2} + \binom{b}{1} \leq 20$ with $6 \geq a > b \geq 1$, then, from Lemma 3.2, we get

$$\alpha_3 - \alpha_2 \leq f_3(a) + f_2(b) \leq f_3(6) + f_2(5) = 10,$$

a contradiction. Hence $\alpha_2 = 21$ and $\alpha_3 = 34$. We get

$$\beta_4^5 = \alpha_4 - 2\alpha_3 + 3\alpha_2 - 4 \cdot 7 + 5 = 19 - 2 \cdot 34 + 3 \cdot 21 - 28 + 5 = -9,$$

a contradiction with the fact that $\operatorname{hdepth}(S/I) = 5$.

If $n = 8$, then $n_3 \in \{4,5,6\}$. It follows that

$$\alpha_5 - \alpha_4 = f_5(7) + f_4(n_3) \leq -14 - 3 = -17 < -14 = f_5(8),$$

and we are done. If $n \geq 9$, then

$$\begin{aligned} &f_5(n) - (\alpha_5 - \alpha_4) = f(n) - (f_5(n-1) + f_4(n_3)) \geq \\ &\geq f_5(n) - (f_5(n-1) + f_4(n-2)) = \frac{1}{6}(n-2)(n-3)(n-7) > 0, \end{aligned}$$

and thus we are done.

(b) $\alpha_4 = \binom{n-1}{4} + \binom{n_3}{3} + \binom{n_2}{2}$, where $n-1 > n_3 \geq 4$ and $n_3 > n_2 \geq 2$. Assume $n = 7$. Since $\alpha_5 - \alpha_4 \leq f_5(6) + f_4(n_3) + f_3(n_2)$, the only case in which $\alpha_5 - \alpha_4$ could be larger than $f_5(7) = -14$ is $n_3 = 4$ and $n_2 = 2$. This means

$$\alpha_4 = \binom{6}{4} + \binom{4}{3} + \binom{2}{2} = 15 + 4 + 1 = 20.$$

Since $\beta_3^5 \geq 0$ and $\beta_4^5 \geq 0$ it follows that

$$0 \leq \beta_3^5 + \beta_4^5 = \alpha_4 - \alpha_3 + 2n - 5 = 29 - \alpha_3,$$

thus $\alpha_3 \leq 29$. On the other hand, we have

$$\alpha_5 \leq \binom{6}{5} + \binom{4}{4} = 7 \text{ and } \alpha_2 \geq \binom{6}{2} + \binom{4}{1} + \binom{2}{0} = 20.$$

It follows that

$$\beta_5^5 = \alpha_5 - \alpha_4 + \alpha_3 - \alpha_2 + 6 \leq 7 - 20 + 29 - 20 + 6 = 2 \leq \binom{7-1}{5} = 6,$$

and we are done. If $n = 8$, then $4 \leq n_3 \leq 6$ and $2 \leq n_2 \leq 5$. In particular, $f_4(n_3) \leq -3$ and $f_3(n_2) \leq 0$. Thus

$$\alpha_5 - \alpha_4 = f_5(7) + f_4(n_3) + f_3(n_2) \leq f_5(7) - 3 = -17 < f_5(8) = -14,$$

and we are done. Now, assume $n \geq 9$. Then

$$\begin{aligned} & f_5(n) - (\alpha_5 - \alpha_4) = f_5(n) - (f_5(n-1) + f_4(n_3) + f_3(n_2)) \geq \\ & \geq f_5(n) - (f_5(n-1) + f_4(n-2) + f_3(n-3)) = \frac{(n-3)(n-6)}{2} > 0. \end{aligned}$$

Hence, we get the required result.

(c) $\alpha_4 = \binom{n-1}{4} + \binom{n_3}{3} + \binom{n_2}{2} + \binom{n_1}{1}$, where $n-1 > n_3 \geq 4$ and $n_3 > n_2 > n_1 \geq 1$. Similar to the previous cases, $\alpha_5 \leq \binom{n-1}{5} + \binom{n_3}{4} + \binom{n_2}{3} + \binom{n_1}{2}$ and thus

$$\alpha_5 - \alpha_4 \leq f_5(n-1) + f_4(n_3) + f_3(n_2) + f_2(n_1).$$

If $n = 7$, then $n_3 \geq 4$, $n_2 \geq 2$ and $n_1 \geq 1$. From the table with the values of $f_k(x)$'s, it is easy to check that

$$f_5(6) + f_4(n_3) + f_3(n_2) + f_2(n_1) \leq -9 - 3 - 1 - 1 = -14 = f_5(7).$$

If $n \geq 8$, then

$$\begin{aligned} & f_5(n) - (\alpha_5 - \alpha_4) = f_5(n) - (f_5(n-1) + f_4(n_3) + f_3(n_2) + f_2(n_1)) \geq \\ & \geq f_5(n) - (f_5(n-1) + f_4(n-2) + f_3(n-3) + f_2(n-4)) = n - 5 > 0, \end{aligned}$$

and we are done, again.

Hence, the proof is complete. $\square$

Theorem 4.4. Let $I \subset S$ be a squarefree monomial ideal with $\operatorname{hdepth}(S/I) = 5$. Then $\operatorname{hdepth}(I) \geq 5$.

Proof. The conclusion follows from Lemma 3.11, Proposition 3.2, Lemma 4.1 (the case $q = 5$), Lemma 4.2 and Lemma 4.3. □

Corollary 4.1. *Let $I \subset S = K[x_1, \dots, x_n]$ be a squarefree monomial ideal, where $n \leq 7$. Then* $\operatorname{hdepth}(I) \geq \operatorname{hdepth}(S/I)$.

Proof. If I is principal, which, by Theorem 2.3, is equivalent to $\operatorname{hdepth}(S/I) = n-1$, we have, again, by Theorem 2.3 that $\operatorname{hdepth}(I) = n$ and there is nothing to prove. Hence, we may assume that $q = \operatorname{hdepth}(S/I) \leq 5$. If $q \leq 4$ we are done by Theorem 3.9 and Theorem 3.12. For $q = 5$, the conclusion follows from Theorem 4.4. □

REFERENCES

[1] W. Bruns, C. Krattenthaler and J. Uliczka, *Stanley decompositions and Hilbert depth in the Koszul complex*, J. Commut. Algebra 2(3) (2010), 327–357.

[2] W. Bruns, C. Krattenthaler and J. Uliczka, *Hilbert depth of powers of the maximal ideal*, Commutative algebra and its connections to geometry, Contemp. Math. 555 (2011), 1–12.

[3] S. Bălănescu, M. Cimpoeaş and C. Krattenthaler, *On the Hilbert depth of monomial ideals*, arXiv:2306.09450v4, (2024).

[4] S. Bălănescu and M. Cimpoeaş, *Remarks on the Hilbert depth of squarefree monomial ideals*, arXiv:2310.12339v5, (2024).

[5] J. Herzog, *A survey on Stanley depth*, In Monomial Ideals, Computations and Applications, Springer, 2013, 3–45.

[6] R. P. Stanley, *Combinatorics and commutative algebra, 2nd edition*, Progress in mathematics 41, Birkhäuser, 2004.

[7] J. Uliczka, *Remarks on Hilbert series of graded modules over polynomial rings*, Manuscripta Math. 132 (2010), 159–168.

A. I. BORDIANU
Department of Mathematical Methods and Models
Faculty of Applied Sciences
National University of Science and Technology Politehnica Bucharest
060042 Bucharest, Romania
e-mail: andreea.bordianu@stud.fsa.upb.ro

M. CIMPOEAŞ
Department of Mathematical Methods and Models
Faculty of Applied Sciences
National University of Science and Technology Politehnica Bucharest
060042 Bucharest, Romania and
Simion Stoilow Institute of Mathematics of the Romanian Academy
014700 Bucharest, Romania
e-mail: mircea.cimpoeas@imar.ro