LE MATEMATICHE
Vol. LXXIX (2024) – Issue II, pp. 535–554
doi: 10.4418/2024.79.2.16

EXTREMAL FUNCTIONS AND UNCERTAINTY PRINCIPLES FOR FOURIER MULTIPLIERS ON THE LAGUERRE HYPERGROUP

A. CHANA - A. AKHIIDJ - S. ARHILAS

The main purpose of this paper is to introduce the Fourier multipliers operators on the Laguerre hypergroup and to give some new results related to these operators as Parseval's, Plancherel's, Calderón's reproducing formulas and Heisenberg's, Donoho-Stark's uncertainty principles. Next, using the theory of reproducing kernels we give best estimates and an integral representation of the extremal functions related to these operators.

1. Introduction

Let $\mathbb{H}^d := \mathbb{C}^d \times \mathbb{R}$ be the $(2d+1)$-dimensional Heisenberg group with multiplication law

$$(z,t)\left(z',t'\right) = \left(z+z',t+t'-\operatorname{Im}\left(zz'\right)\right),$$

where $zz' = \sum_{k=0}^{d} z_k \overline{z_k'}$, is the usual positive definite Hermitian form on $\mathbb{C}^d$. If we put $T = \frac{\partial}{\partial t}$ and

$$Z_j = \frac{\partial}{\partial z_j} - i\bar{z}_j\frac{\partial}{\partial t}, \quad \overline{Z_j} = \frac{\partial}{\partial \bar{z}_j} + i\bar{z}_j\frac{\partial}{\partial t}, \quad j = 1,\ldots,d,$$

Received on 0, 0
AMS 2010 Subject Classification: 42B10, 47G30, 47B10.
Keywords: Fourier transform, Multiplier Operators, Laguerre hypergroup, Calderón's reproducing formulas, Heisenberg's uncertainty principle, Donoho-Stark's uncertainty principle .

Then the system $T, Z_j, \overline{Z_j}$ forms a basis of the left invariant vector fields of h_d^c, the complexification of the Lie algebra h_d of $\mathbb{H}_d$, where

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial \bar{z}_j} = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}.$$

Set

$$X_j = \frac{\partial}{\partial x_j} - iy_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + ix_j\frac{\partial}{\partial t}, \quad j = 1,\dots,d.$$

Thus $X_1,\dots,X_d,Y_1,\dots,Y_d,T$ is a basis of h_d. A function f on $\mathbb{H}_d$ is said to be radial if it is invariant under the action of the unitary group $U(d)$. Let

$$L^p_{\text{rad}}\left(\mathbb{H}^d\right) := \left\{f \in L^p\left(\mathbb{H}_d\right) : f(vz,t) = f(z,t) \text{ for all } v \in U(d)\right\}.$$

The theory of harmonic analysis on $L^p_{rad}(\mathbb{H}_d)$ was developed by many authors (one can consult [5, 9, 13, 14, 17, 18]). When one considers the problems of radial functions on the Heisenberg group $\mathbb{H}^d$, the underlying manifold can be regarded as the Laguerre hypergroup $\mathbb{K} := [0,\infty) \times \mathbb{R}$. In [17, 18] the authors introduced a generalized translation operator on $\mathbb{K}$ and established the theory of harmonic analysis on $L^2(\mathbb{K}, d\mu_\alpha)$, where the weighted Lebesgue measure μ_α on $\mathbb{K}$ is given by

$$d\mu_\alpha(x,t) := \frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}, \quad \alpha \geq 0, \tag{1}$$

and Γ is the Gamma function.
In their seminal papers [6, 12], Hörmander and, respectively, Mikhlin initiated the study of boundedness of the translation invariant operators on $\mathbb{R}^d$. The translation invariant operators on $\mathbb{R}^d$ characterized using the classical Euclidean Fourier transform $\mathcal{F}(f)$ therefore they also known as Fourier multipliers.
Let $1 < p < \infty$ and given a measurable function

$$m : \mathbb{R}^d \longrightarrow \mathbb{C}$$

its Fourier multiplier is the linear map $\mathcal{T}_m$ given for all $\lambda \in \mathbb{R}^d$ by the relation

$$\mathcal{F}(\mathcal{T}_m(f))(\lambda) = m(\lambda)\mathcal{F}(f)(\lambda) \tag{2}$$

The Hörmander-Mikhlin fundamental condition gives a criterion for boundedness of Fourier multiplier $\mathcal{T}_m$ in terms of derivatives of the symbol m, more precisely if

$$\left|\partial_\lambda^\gamma m(\lambda)\right| \lesssim |\lambda|^{-|\gamma|} \quad for \quad 0 \leq |\gamma| \leq \left[\frac{d}{2}\right] + 1 \tag{3}$$

Then, $\mathcal{T}_m$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^d)$ into itself . The condition (3) imposes m to be a bounded function, smooth over $\mathbf{R}^d\backslash\{0\}$ satisfying certain local and asymptotic behavior. Locally, m admits a singularity at 0 with a mild control of derivatives around it up to order $\left[\frac{d}{2}\right]+1$. This singularity links to deep concepts in harmonic analysis and justifies the key role of Hörmander-Mikhlin theorem in Fourier multiplier L_p-theory, this condition defines a large class of Fourier multipliers including Riesz transforms and Littelwood-Paley partitions of unity which are crucial in Fourier summability or Pseudo-differential operator.The boundedness of Fourier multipliers is useful to solve problems in the area of mathematical analysis as Probability theory see [11], Stochastic processus see [3],and the study of nonlinear partial differential equations see [8].

The general theory of reproducing kernels is stared with Aronszajn's in [1] in 1950, next the authors in [10, 15, 16] applied this theory to study Tikhonov regularization problem and they obtained approximate solutions for bounded linear operator equations on Hilbert spaces with the viewpoint of numerical solutions by computers. This theory has gained considerable interest in various field of mathematical sciences especially in Engineering and numerical experiments by using computers see [10, 16] for more information.

For its importance the theory of Fourier multipliers has been generalized in different sets for example in the Dunkl set [19–22], in the Sturm-Liouville hypergroup [23], this paper focuses on the generalized Fourier transform on the Laguerre hypergroup $\mathbb{K} := [0,\infty)\times\mathbb{R}$, more precisely we consider a system of partial differential operator Δ_1 and Δ_2 defined on $\mathbb{K} := [0,+\infty)\times[0,+\infty)$, by

$$\begin{cases} \Delta_1 := \frac{\partial}{\partial t}; \quad \alpha \geq 0, \quad t>0 \\ \Delta_2 := \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}, \quad x>0 \end{cases}$$

Where α is a nonnegative number and for $\alpha = d-1$ the operators Δ_2 is the radial part of sublaplacian on the Heisenberg group $\mathbb{H}^d$. The Fourier-Laguerre transform $\mathcal{F}_\alpha$ generalizing the usual Fourier transform $\mathcal{F}$ and it is defined on $L^1_\alpha(\mathbb{K})$ by

$$\mathcal{F}_\alpha(f)(\lambda,m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t)f(x,t)d\mu_\alpha(x,t), \quad \text{for } (\lambda,m)\in\hat{\mathbb{K}}$$

where $\hat{\mathbb{K}}$ is the dual space of $\mathbb{K}$, μ_α is the measure on $\mathbb{K}$ given by the relation (1) and $\varphi_{\lambda,m}$ is the character of the hypergroup $\mathbb{K}$ given later. Let σ be a function in $L^2_\alpha(\mathbb{K})$ and $\beta>0$ be a positive real number, the Laguerre L^2_α-multiplier operators is defined for smooth function on $\mathbb{K}$ as

$$\mathcal{T}_{\sigma,\beta}(f)(x,t) := \mathcal{F}_\alpha^{-1}\left(\sigma_\beta\mathcal{F}_\alpha(f)\right)(x,t) \tag{4}$$

where the function σ_β is given by

$$\sigma_\beta(\lambda, m) := \sigma(\beta\lambda, m) \tag{5}$$

The operators (4) are a generalization of the classical Fourier multiplier operators given by the relation (2). The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis on the Laguerre hypergroup, in section 3, we introduce the Laguerre L^2_α-multiplier operators $\mathcal{T}_{\sigma,\beta}$ and we give for them a Plancherel's, point- wise reproducing formulas and Heisenberg's, Donoho-Stark's uncertainty principles. The last section of this paper is devoted to give an application of the general theory of reproducing kernels to Fourier multiplier theory and to give best estimates and an integral representation of the extremal functions related to the Laguerre L^2_α multiplier operators on weighted Sobolev spaces.

2. Harmonic Analysis on the Laguerre Hypergroup

In this section we set some notations and we recall some results in harmonic analysis on the Laguerre hypergroup, for more details we refer the reader to [5, 13, 14].

In the following we denote by

• $L^p_\alpha(\mathbb{K}), 1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{K}$, satisfying

$$\|f\|_{p,\mu_\alpha} := \begin{cases} \left(\int_{\mathbb{K}} |f(x,t)|^p d\mu_\alpha(x,t)\right)^{1/p} < \infty, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{(x,t)\in\mathbb{K}} |f(x,t)| < \infty, & p = \infty. \end{cases}$$

• $\mathcal{C}_*(\mathbb{K})$ the space of continuos function on $\mathbb{R}^2$, even with respect to the first variable.

• $\mathcal{C}_{*,c}(\mathbb{K})$ the subspace of $\mathcal{C}_*(\mathbb{K})$ formed by functions with compact support.

• $\mathcal{L}^{(\alpha)}_m(x)$ is the Laguerre function defined on $[0,+\infty[$ by

$$\mathcal{L}^{(\alpha)}_m(x) := \frac{e^{-\frac{x}{2}} L^{(\alpha)}_m(x)}{L^{(\alpha)}_m(0)},$$

where $L^{(\alpha)}_m$ is the Laguerre polynomial of degree m and order α given by

$$L^{(\alpha)}_m(x) = \sum_{k=0}^{m} \frac{(-1)^k\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{x^k}{k!(m-k)!}.$$

• $\hat{\mathbb{K}} := [0,+\infty) \times \mathbb{N}$ equipped with weighted Lebesgue measure γ_α given by

$$\int_{\hat{\mathbb{K}}} g(\lambda,m)d\gamma_\alpha(\lambda,m) = \sum_{m=0}^{\infty} L^{(\alpha)}_m(0) \int_{\mathbb{R}} g(\lambda,m)|\lambda|^{\alpha+1}d\lambda.$$

where $d\lambda$ is the classical Lebesgue measure in $\mathbb{R}$.
• $L^p_\alpha(\hat{\mathbb{K}})$ with $p \in [1,+\infty]$ the space of measurable functions on $\hat{\mathbb{K}}$ satisfying

$$\|g\|_{p,\gamma_\alpha} := \begin{cases} \left(\int_{\hat{\mathbb{K}}} |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{p}} < \infty, & 1 \le p < \infty, \\ \text{ess } \sup_{(\lambda,m)\in\hat{\mathbb{K}}} |g(\lambda,m)| < \infty, & p = \infty. \end{cases}$$

2.1. The Eigenfunctions of the Partial Differential Operators Δ_1 and Δ_2

For $(\lambda,m) \in \hat{\mathbb{K}}$ we consider the following Cauchy problem

$$(S) : \begin{cases} \Delta_1(u) = i\lambda u, \\ \Delta_2(u) = -4|\lambda|(m+\frac{\alpha+1}{2})u \\ u(0,0) = 1; \frac{\partial u}{\partial x}(0,0) = \frac{\partial u}{\partial t}(0,0) = 0. \end{cases}$$

From [14], the Cauchy problem (S) admits a unique solution given by

$$\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}_m^{(\alpha)}\left(|\lambda|x^2\right) \quad \text{for } (x,t) \in \mathbb{K} \quad \text{and } (\lambda,m) \in \hat{\mathbb{K}},$$

The function $\varphi_{\lambda,m}$ is infinitely differentiable on $\mathbb{R}^2$, even with respect to the first variable and we have the following important result

$$\sup_{(x,t)\in\mathbb{K}} \left|\varphi_{\lambda,m}(x,t)\right| = 1. \tag{6}$$

2.2. Fourier Transform on the Laguerre Hypergroup

Definition 2.1. The Fourier-Laguerre transform $\mathcal{F}_\alpha$ defined on $L^1_\alpha(\mathbb{K})$ by

$$\mathcal{F}_\alpha(f)(\lambda,m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t)f(x,t)d\mu_\alpha(x,t), \quad for(\lambda,m) \in \hat{\mathbb{K}}.$$

Some basic properties of this transform are as follows, for the proofs, we refer the reader to [4, 15].

Proposition 2.2.
(1) For every $f \in L^1_\alpha(\mathbb{K})$ we have

$$\|\mathcal{F}_\alpha(f)\|_{\infty,\gamma_\alpha} \le \|f\|_{1,\mu_\alpha}. \tag{7}$$

(2)(Inversion formula) For $f \in \left(L^1_\alpha \cap L^2_\alpha\right)(\mathbb{K})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha(\hat{\mathbb{K}})$ we have

$$f(x,t) = \int_{\hat{\mathbb{K}}} \varphi_{\lambda,m}(x,t)\mathcal{F}_\alpha(f)(\lambda,m)d\gamma_\alpha(\lambda,m), \quad a.e \quad (x,t) \in \mathbb{K}. \tag{8}$$

(3) (Parseval formula) For all $f,g \in L^2_\alpha(\mathbb{K})$ we have

$$\int_{\mathbb{K}} f(x,t)\overline{g(x,t)}d\mu_\alpha(x,t) = \int_{\hat{\mathbb{K}}} \mathcal{F}_\alpha(f)(\lambda,m)\overline{\mathcal{F}_\alpha(g)(\lambda,m)}d\gamma_\alpha(\lambda,m) \qquad (9)$$

In particular we have

$$\|f\|_{2,\mu\alpha} = \|\mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}. \qquad (10)$$

(4) (Plancherel theorem) The Laguerre-Bessel transform $\mathcal{F}_\alpha$ can be extended to an isometric isomorphism from $L^2_\alpha(\mathbb{K})$ into $L^2_\alpha(\hat{\mathbb{K}})$.

2.3. The Translation Operators on the Laguerre Hypergroup

Definition 2.3. Let $f \in C_{*,c}(\mathbb{K})$. For all (x,t) and (y,s) in $\mathbb{K}$, for $\alpha > 0$ we put

$$\tau_\alpha^{(x,t)} f(y,s) = \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 f\left(\langle x,y\rangle_{r,\theta}, s+t+xyr\sin\theta\right) r\left(1-r^2\right)^{\alpha-1} drd\theta,$$

where $\langle x,y\rangle_{r,\theta} := \sqrt{x^2+y^2+2xyr\cos\theta}$. The operators $\tau_\alpha^{(x,t)}$, are called generalized translation operators on $\mathbb{K}$.

The following proposition summarizes some properties of the translation operators see [14, 18].

Proposition 2.4. *For all $(x,t),(y,s) \in \mathbb{K}$, $\quad f \in C_*(\mathbb{K})$ we have:*

(1)

$$\tau_\alpha^{(x,t)}(f)(y,s) = \tau_\alpha^{(y,s)}(f)(x,t). \qquad (11)$$

(2)

$$\int_{\mathbb{K}} \tau_\alpha^{(x,t)}(f)(y,s)d\mu_\alpha(y,s) = \int_{\mathbb{K}} f(y,s)d\mu_\alpha(y,s). \qquad (12)$$

(3) for $f \in L^p_\alpha(\mathbb{K})$ with $p \in [1;+\infty]$ $\quad \tau_\alpha^{(x,t)}(f) \in L^p_\alpha(\mathbb{K})$ and we have

$$\left\|\tau_\alpha^{(x,t)}(f)\right\|_{p,\mu_\alpha} \le \|f\|_{p,\mu_\alpha}. \qquad (13)$$

(4) For $f \in L^1_\alpha(\mathbb{K})$, $\quad \tau_\alpha^{(x,t)}(f) \in L^1_\alpha(\mathbb{K})$ and we have

$$\mathcal{F}_\alpha\left(\tau_\alpha^{(x,t)}(f)\right)(\lambda,m) = \varphi_{\lambda,m}(x,t)\mathcal{F}_\alpha(f)(\lambda,m), \quad \forall(\lambda,m) \in \hat{\mathbb{K}}. \qquad (14)$$

By using the generalized translation, we define the generalized convolution product of $f,g \in \mathcal{S}_*(\mathbb{K})$ by

$$(f *_\alpha g)(x,t) = \int_{\mathbb{K}} \tau_\alpha^{(x,t)}(\widetilde{f})(y,s)g(y,s)d\mu_\alpha(y,s),$$

with $\widetilde{f}(y,s) = f(y,-s)$. This convolution is commutative, associative and its satisfies the following properties see [13, 14].

Proposition 2.5.

(1)(Young's inequality) for all $p,q,r \in [1;+\infty]$ such that: $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$ and for all $f \in L^p_\alpha(\mathbb{K}), g \in L^q_\alpha(\mathbb{K})$ the function $f *_\alpha g$ belongs to the space $L^r_\alpha(\mathbb{K})$ and we have

$$\|f *_\alpha g\|_{r,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha}\|g\|_{q,\mu_\alpha} \tag{15}$$

(2) For $f,g \in L^2_\alpha(\mathbb{K})$ the function $f *_\alpha g$ belongs to $L^2_\alpha(\mathbb{K})$ if and only if the function $\mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g)$ belongs to $L^2_\alpha(\hat{\mathbb{K}})$ and in this case we have

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g). \tag{16}$$

(3) For $f,g \in L^2_\alpha(\mathbb{K})$ then we have

$$\int_{\mathbb{K}} |f *_\alpha g(x,t)|^2 d\mu_\alpha(x,t) = \int_{\hat{\mathbb{K}}} |\mathcal{F}_\alpha(f)(\lambda,m)|^2 |\mathcal{F}_\alpha(g)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m), \tag{17}$$

where both integrals are simultaneously finite or infinite.

3. Fourier Multipliers on the Laguerre Hypergroup

The main purpose of this section is to introduce the Laguerre L^2_α-multiplier operators on $\mathbb{K}$ and to establish for them Calderon's reproducing formulas and some uncertainty principles.

3.1. Calderon's Reproducing Formulas for the Laguerre Multiplier operators

Definition 3.1. Let $\sigma \in L^2_\alpha(\hat{\mathbb{K}})$ and $\beta > 0$, the Laguerre L^2_α-multiplier operators are defined for smooth functions on $\mathbb{K}$ as

$$\mathcal{T}_{\sigma,\beta}(f)(x,t) := \mathcal{F}^{-1}_\alpha\left(\sigma_\beta \mathcal{F}_\alpha(f)\right)(x,t), \tag{18}$$

where the function σ_β is given by

$$\sigma_\beta(\lambda,m) := \sigma(\beta\lambda,m),$$

for all $(\lambda,m) \in \hat{\mathbb{K}}$.
By a simple change of variable we find that for all $\beta > 0, \sigma_\beta \in L^2_\alpha(\hat{\mathbb{K}})$ and

$$\left\|\sigma_\beta\right\|_{2,\gamma_\alpha} = \frac{1}{\beta^{\frac{\alpha+2}{2}}}\|\sigma\|_{2,\gamma_\alpha}. \tag{19}$$

Remark 3.2. According to the relation (16) we find that

$$\mathcal{T}_{\sigma,\beta}(f)(x,t) = \left(\mathcal{F}_{\alpha}^{-1}\left(\sigma_{\beta}\right) *_{\alpha} f\right)(x,t), \tag{20}$$

where

$$\mathcal{F}_{\alpha}^{-1}\left(\sigma_{\beta}\right)(x,t) = \frac{1}{\beta^{\alpha+2}}\mathcal{F}_{\alpha}^{-1}(\sigma)\left(\frac{x}{\sqrt{\beta}}, \frac{t}{\beta}\right). \tag{21}$$

We give some properties of the Laguerre L^2_{α}-multiplier operators.

Proposition 3.3. *(i) For every $\sigma \in L^2_{\alpha}(\hat{\mathbb{K}})$, and $f \in L^1_{\alpha}(\mathbb{K})$, the function $\mathcal{T}_{\sigma,\beta}(f)$ belongs to $L^2_{\alpha}(\mathbb{K})$, and we have*

$$\left\|\mathcal{T}_{\sigma,\beta}(f)\right\|_{2,\mu_{\alpha}} \leq \frac{1}{\beta^{\frac{\alpha+2}{2}}}\|\sigma\|_{2,\gamma_{\alpha}}\|f\|_{1,\mu_{\alpha}}.$$

(ii) For every $\sigma \in L^{\infty}_{\alpha}(\hat{\mathbb{K}})$, and for every $f \in L^2_{\alpha}(\mathbb{K})$, the function $\mathcal{T}_{\sigma,\beta}(f)$ belongs to $L^2_{\alpha}(\mathbb{K})$, and we have

$$\left\|\mathcal{T}_{\sigma,\beta}(f)\right\|_{2,\mu_{\alpha}} \leq \|\sigma\|_{\infty,\gamma_{\alpha}}\|f\|_{2,\mu_{\alpha}} \tag{22}$$

(iii) For every $\sigma \in L^2_{\alpha}(\hat{\mathbb{K}})$, and for every $f \in L^2_{\alpha}(\mathbb{K})$, then $\mathcal{T}_{\sigma,\beta}(f) \in L^{\infty}_{\alpha}(\mathbb{K})$, and we have

$$\mathcal{T}_{\sigma,\beta}(f)(x,t) = \int_{\hat{\mathbb{K}}} \sigma(\beta\lambda, m)\varphi_{\lambda,m}(x,t)\mathcal{F}_{\alpha}(f)(\lambda,m)d\gamma_{\alpha}(\lambda,m) \tag{23}$$

and

$$\left\|\mathcal{T}_{\sigma,\beta}(f)\right\|_{\infty,\mu_{\alpha}} \leq \frac{1}{\beta^{\frac{\alpha+2}{2}}}\|\sigma\|_{2,\gamma_{\alpha}}\|f\|_{2,\mu_{\alpha}}.$$

Proof. (i) By the relations (15),(20) we find that

$$\left\|\mathcal{T}_{\sigma,\beta}(f)\right\|^2_{2,\mu_{\alpha}} = \left\|\mathcal{F}_{\alpha}^{-1}\left(\sigma_{\beta}\right) *_{\alpha} f\right\|^2_{2,\mu_{\alpha}} \leq \|f\|^2_{1,\mu_{\alpha}}\left\|\mathcal{F}_{\alpha}^{-1}\left(\sigma_{\beta}\right)\right\|^2_{1,\mu_{\alpha}}$$

Plancherel's formula (10) and the relation (19) gives the desired result.
(ii) Is a consequence of Plancherel's formula (10).
(iii) By using the relations (10), (15),(19) and (3.3) we find that

$$\left\|\mathcal{T}_{\sigma,\beta}(f)\right\|_{\infty,\mu_{\alpha}} \leq \frac{1}{\beta^{\frac{\alpha+2}{2}}}\|\sigma\|_{2,\gamma_{\alpha}}\|f\|_{2,\mu_{\alpha}}$$

on the other hand the relation (22) follows from inversion formula (8). □

In the following result, we give Plancherel's and pointwise reproducing inversion formula for the Laguerre L^2_{α}-multiplier operators.

Theorem 3.4. *Let $\sigma \in L^2_\alpha(\hat{\mathbb{K}})$ satisfying the admissibility condition:*

$$\int_0^\infty \left|\sigma_\beta(\lambda,m)\right|^2 \frac{d\beta}{\beta} = 1, \quad (\lambda,m) \in \hat{\mathbb{K}}. \tag{24}$$

(i) (Plancherel formula) For all f in $L^2_\alpha(\mathbb{K})$, we have

$$\int_{\mathbb{K}} |f(x,t)|^2 d\mu_\alpha(x,t) = \int_0^\infty \left\|\mathcal{T}_{\sigma,\beta}(f)\right\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta}. \tag{25}$$

(ii) (First calderón's formula) Let $f \in L^1_\alpha(\mathbb{K})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha(\hat{\mathbb{K}})$ then we have

$$f(x,t) = \int_0^\infty \left(\mathcal{T}_{\sigma,\beta}(f) *_\alpha \overline{\mathcal{F}_\alpha^{-1}(\sigma_\beta)}\right)(x,-t)\frac{d\beta}{\beta}, \quad a.e.\ (x,t) \in \mathbb{K}.$$

Proof. (i) By using Fubini's theorem and the relations (17) and (20) we find that

$$\begin{aligned}\int_0^\infty \left\|\mathcal{T}_{\sigma,\beta}(f)\right\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta} &= \int_0^\infty \left[\int_{\mathbb{K}} \left|\mathcal{T}_{\sigma,\beta}(f)(x,t)\right|^2 d\mu_\alpha(x,t)\right] \frac{d\beta}{\beta} \\ &= \int_0^\infty \left[\int_{\mathbb{K}} \left|\mathcal{F}_\alpha^{-1}\left(\sigma_\beta\right) *_\alpha f(x,t)\right|^2 d\mu_\alpha\right] \frac{d\beta}{\beta} \\ &= \int_0^\infty \left[\int_{\hat{\mathbb{K}}} |\mathcal{F}_\alpha(f)(\lambda,m)|^2 d\gamma_\alpha\right] \left|\sigma_\beta(\lambda,m)\right|^2 \frac{d\beta}{\beta}\end{aligned}$$

the admissibility condition (24) and Plancherel's formula (10) gives the desired result.
(ii) Let $f \in L^1_\alpha(\mathbb{K})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha(\hat{\mathbb{K}})$, by Fubini's theorem and the relations (8),(9),(14) and the admissibility condition (24) we find the result □

To establish the second Calderon's reproducing formula for the Laguerre L^2_α-multiplier operators, we need the following technical result.

Proposition 3.5. *Let $\sigma \in L^2_\alpha(\hat{\mathbb{K}}) \cap L^\infty_\alpha(\hat{\mathbb{K}})$ satisfy the admissibility condition (24) then the function defined by*

$$\Phi_{\gamma,\delta}(\lambda,m) = \int_\gamma^\delta \left|\sigma_\beta(\lambda,m)\right|^2 \frac{d\beta}{\beta}$$

belongs to $L^2_\alpha(\hat{\mathbb{K}}) \cap L^\infty_\alpha(\hat{\mathbb{K}})$ for all $0 < \gamma < \delta < \infty$.

Proof. Using Hölder's inequality for the measure $\frac{d\beta}{\beta}$, we get

$$\left|\Phi_{\gamma,\delta}(\lambda,m)\right|^2 \leq \log(\delta/\gamma) \int_\gamma^\delta \left|\sigma_\beta(\lambda,m)\right|^4 \frac{d\beta}{\beta}, \quad (\lambda,m) \in \hat{\mathbb{K}}.$$

Then using Fubini's theorem, we obtain

$$\left\|\Phi_{\gamma,\delta}\right\|_{2,\gamma_\alpha}^2 \leq \log(\delta/\gamma)\|\sigma\|_{\infty,\gamma_\alpha}^2 \int_\gamma^\delta \|\sigma\|_{2,\gamma_\alpha}^2 \frac{d\beta}{\beta}$$

by using the relation (19) we find that

$$\left\|\Phi_{\gamma,\delta}\right\|_{2,\gamma_\alpha}^2 \leq \log(\delta/\gamma)\|\sigma\|_{\infty,\gamma_\alpha}^2 \|\sigma\|_{2,\gamma_\alpha}^2 \int_\gamma^\delta \frac{d\beta}{\beta^{\alpha+3}} < \infty$$

So $\Phi_{\gamma,\delta}$ belongs to $L^2_\alpha(\hat{\mathbb{K}})$, furthermore by the relation (24) we get $\left\|\Phi_{\gamma,\delta}\right\|_{\infty,\gamma_\alpha} \leq 1$ therefore $\Phi_{\gamma,\delta}$ belongs to $L^2_\alpha(\hat{\mathbb{K}}) \cap L^\infty_\alpha(\hat{\mathbb{K}})$. □

Theorem 3.6. *(Second Calderón's formula). Let $f \in L^2_\alpha(\mathbb{K}), \sigma \in L^2_\alpha(\hat{\mathbb{K}}) \cap L^\infty_\alpha(\hat{\mathbb{K}})$ satisfy the admissibility condition (3.7) and $0 < \gamma < \delta < \infty$. Then the function*

$$f_{\gamma,\delta}(x,t) = \int_\gamma^\delta \left(\mathcal{T}_{\sigma,\beta}(f) *_\alpha \overline{\mathcal{F}_\alpha^{-1}(\sigma_\beta)}\right)(x,-t)\frac{d\beta}{\beta}, \quad (x,t) \in \mathbb{K}$$

belongs to $L^2_\alpha(\mathbb{K})$ and satisfies

$$\lim_{(\gamma,\delta)\to(0,\infty)} \left\|f_{\gamma,\delta} - f\right\|_{2,\mu_\alpha} = 0 \tag{26}$$

Proof. By a simple computation we find that

$$f_{\gamma,\delta}(x,t) = \int_{\hat{\mathbb{K}}} \Phi_{\gamma,\delta}(\lambda,m)\varphi_{\lambda,m}(x,t)\mathcal{F}_\alpha(f)(\lambda,m)d\gamma_\alpha(\lambda,m),$$

by proposition (3.5) we have $\Phi_{\gamma,\delta} \in L^\infty_\alpha(\hat{\mathbb{K}})$ then we have $f_{\gamma,\delta} \in L^2_\alpha(\mathbb{K})$ and

$$\mathcal{F}_\alpha\left(f_{\gamma,\delta}\right)(\lambda,m) = \Phi_{\gamma,\delta}(\lambda,m)\mathcal{F}_\alpha(f)(\lambda,m)$$

on the other hand by Plancherel's formula (10) we find that

$$\lim_{(\gamma,\delta)\to(0,\infty)} \left\|f_{\gamma,\delta} - f\right\|_{2,\mu_\alpha}^2 = \lim_{(\gamma,\delta)\to(0,\infty)} \int_{\hat{\mathbb{K}}} |\mathcal{F}_\alpha(f)(\lambda,m)|^2 \left(1-\Phi_{\gamma,\delta}(\lambda,m)\right)^2 d\gamma_\alpha$$

by using the admissibility condition (24), the relation (26) follows from the dominated convergence theorem. □

3.2. Uncerainty Principles for the Laguerre L^2_α-Multiplier operators

The main purpose of this subsection is to establish Heisenberg's and Donoho-Stark's uncertainty principles for the the Laguerre L^2_α-multiplier operators $\mathcal{T}_{\sigma,\beta}$.

3.2.1. Heisenberg's uncertainty principle for $\mathcal{T}_{\sigma,\beta}$

In [2], using method based on ultracontractive properties of the semigroups generated by the differential operator L given by

$$L := -(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2})$$

the authors proved the following Heisenberg's inequality for $\mathcal{F}_\alpha$, there exist a positive constant c such that for all $f \in L^2_\alpha(\mathbb{K})$ we have

$$\|f\|^2_{2,\mu_\alpha} \leq c\,\||(x,t)|_{\mathbb{K}} f\|_{2,\mu_\alpha} \left\| |(\lambda,m)|^{\frac{1}{2}}_{\widehat{\mathbb{K}}} \mathcal{F}_\alpha(f) \right\|_{2,\gamma_\alpha} \tag{27}$$

where $|\cdot|_{\mathbb{K}}$ and $|\cdot|_{\hat{\mathbb{K}}}$ are the homogeneous norms given by $|(x,t)|_{\mathbb{K}} = \left(x^4+4t^2\right)^{\frac{1}{4}}$ and $|(\lambda,m)|_{\hat{\mathbb{K}}} = 4|\lambda|(m+\frac{\alpha+1}{2})$.The main purpose of this subsection is to generalize the inequality (3.10) for the Laguerre multipliers $\mathcal{T}_{\sigma,\beta}$.

Theorem 3.7. *There exist a positive constant c such that for all $f \in L^2_\alpha(\mathbb{K})$ we have*

$$\|f\|^2_{2,\mu_\alpha} \leq c \left\| |(\lambda,m)|^{\frac{1}{2}}_{\widehat{\mathbb{K}}} \mathcal{F}_\alpha(f) \right\|_{2,\gamma_\alpha} \left[\int_0^\infty \left\||(x,t)|_{\mathbb{K}} \mathcal{T}_{\sigma,\beta}(f)\right\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta}\right]^{\frac{1}{2}}$$

Proof. Suppose that $\left\| |(\lambda,m)|^{\frac{1}{2}}_{\widehat{\mathbb{K}}} \mathcal{F}_\alpha(f) \right\|_{2,\gamma_\alpha} + \int_0^\infty \left\||(x,t)|_{\mathbb{K}} \mathcal{T}_{\sigma,\beta}(f)\right\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta} < \infty$, by using the relation (27) we have

$$\int_{\mathbb{K}} |\mathcal{T}_{\sigma,\beta}(f)(x,t)|^2 d\mu_\alpha(x,t) \leq c\left\||(x,t)|_{\mathbb{K}} \mathcal{T}_{\sigma,\beta}(f)\right\|_{2,\mu_\alpha} \left\| |(\lambda,m)|^{\frac{1}{2}}_{\widehat{\mathbb{K}}} \sigma_\beta \mathcal{F}_\alpha(f) \right\|_{2,\gamma_\alpha}$$

, integrating over $]0,+\infty[$ with respect to measure $\frac{d\beta}{\beta}$ and we get

$$\int_0^\infty \left[\int_{\mathbb{K}} |\mathcal{T}_{\sigma,\beta}(f)(x,t)|^2 d\mu_\alpha(x,t)\right] \frac{d\beta}{\beta} \leq c\int_0^\infty \left\||(x,t)|_{\mathbb{K}} \mathcal{T}_{\sigma,\beta}(f)\right\|_{2,\mu_\alpha} \left\| |(\lambda,m)|^{\frac{1}{2}}_{\widehat{\mathbb{K}}} \sigma_\beta \mathcal{F}_\alpha(f) \right\|_{2,\gamma_\alpha} \frac{d\beta}{\beta},$$

by using Plancherel's formula for $\mathcal{T}_{\sigma,\beta}$. (25) and Schwartz's inequality and the admissibility condition (24) gives the desired result. □

3.2.2. Donoho-Stark's uncertainty principle for $\mathcal{T}_{\sigma,\beta}$

Building on the ideas of Donoho and Stark In [4], the main purpose of this subsection is to give an uncertainty inequality of concentration type in $L^2_\theta(\mathbb{K})$ where $L^2_\theta(\mathbb{K})$ is the space of measurables functions on $]0,+\infty[\times\mathbb{K}$ such that

$$\|f\|_{2,\theta_\alpha} = \left[\int_0^\infty \|f(\beta,.)\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta}\right]^{\frac{1}{2}}.$$

We denote by θ_α the measure defined on $]0,+\infty[\times\mathbb{K}$ *by*

$$d\theta_\alpha(\beta,(x,t)) = d\mu_\alpha(x,t)\otimes\frac{d\beta}{\beta},$$

Definition 3.8. [4] (i) Let E be a measurable subset of $\mathbb{K}$, we say that the function $f\in L^2_\alpha(\mathbb{K})$ is ε-concentrated on E if

$$\|f-1_E f\|_{2,\mu_\alpha}\leq\varepsilon\|f\|_{2,\mu_\alpha}, \tag{28}$$

where 1_E is the indicator function of the set E.
(ii) Let F be a measurable subset of $]0,+\infty[\times\mathbb{K}$, we say that the function $\mathcal{T}_{\sigma,\beta}(f)$ is ρ-concentrated on F if

$$\|\mathcal{T}_{\sigma,\beta}(f)-1_F\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}\leq\rho\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}. \tag{29}$$

We have the following result

Theorem 3.9. *Let $f\in L^2_\alpha(\mathbb{K})$ and $\sigma\in L^2_\alpha(\hat{\mathbb{K}})\cap L^1_\alpha(\hat{\mathbb{K}})$ satisfying the admissibility condition (24), if f is ε-concentrated on E and $\mathcal{T}_{\sigma,\beta}(f)$ is ρ-concentrated on F then we have*

$$\|\sigma\|_{1,\gamma_\alpha}(\mu(E))^{\frac{1}{2}}\left[\int_F\frac{d\theta_\alpha(\beta,(x,t))}{\beta^{4\alpha+4}}\right]^{\frac{1}{2}}\geq 1-(\varepsilon+\rho).$$

Proof. Let $f\in L^2_\alpha(\mathbb{K})$ and $\sigma\in L^2_\alpha(\hat{\mathbb{K}})\cap L^\infty_\alpha(\hat{\mathbb{K}})$ satisfying (24) and assume that $\mu_\alpha(E)<\infty$ and $\int_F\frac{d\theta_\alpha(\beta,(x,t))}{\beta^{4\alpha+4}}<\infty$.
According to the relations (28),(29) we have

$$\|\mathcal{T}_{\sigma,\beta}(f)-1_F\mathcal{T}_{\sigma,\beta}(1_E f)\|_{2,\theta_\alpha}\leq\rho\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}+\|\mathcal{T}_{\sigma,\beta}(f-1_E f)\|_{2,\theta_\alpha},$$

by Plancherel's relation (25) we get

$$\|\mathcal{T}_{\sigma,\beta}(f)-1_F\mathcal{T}_{\sigma,\beta}(1_E f)\|_{2,\theta_\alpha}\leq(\varepsilon+\rho)\|f\|_{2,\mu_\alpha}$$

So we get

$$\begin{aligned}\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}&\leq|\mathcal{T}_{\sigma,\beta}(f)-1_F\mathcal{T}_{\sigma,\beta}(1_E f)\|_{2,\theta_\alpha}+\|1_F\mathcal{T}_{\sigma,\beta}(1_E f)\|_{2,\theta_\alpha}\\&\leq(\varepsilon+\rho)\|f\|_{2,\mu_\alpha}+\|1_F\mathcal{T}_{\sigma,\beta}(1_E f)\|_{2,\theta_\alpha},\end{aligned} \tag{30}$$

on the other hand by the relation (23) we have

$$|\mathcal{T}_{\sigma,\beta}(1_E f)|^2\leq\frac{1}{\beta^{4\alpha+4}}\|f\|^2_{2,\mu_\alpha}\|\sigma\|^2_{1,\gamma_\alpha}\mu(E),$$

so we find that

$$\|1_F\mathcal{T}_{\sigma,\beta}(1_E f)\|_{2,\theta_\alpha} \leq \|f\|_{2,\mu_\alpha}\|\sigma\|_{1,\gamma_\alpha}(\mu(E))^{\frac{1}{2}}\left[\int_F \frac{d\theta_\alpha(\beta,(x,t))}{\beta^{4\alpha+4}}\right]^{\frac{1}{2}}, \quad (31)$$

by the relations (30),(31) we deduce that

$$\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha} \leq \|f\|_{2,\mu_\alpha}\left[(\varepsilon+\rho)+\|\sigma\|_{1,\gamma_\alpha}(\mu(E))^{\frac{1}{2}}\left[\int_F \frac{d\theta_\alpha(\beta,(x,t))}{\beta^{4\alpha+4}}\right]^{\frac{1}{2}}\right]$$

Plancherel's formula (3.8) for $\mathcal{T}_{\sigma,\beta}$ gives the desired result. □

4. Extremal Functions Associated with the Laguerre L^2_α-Multiplier operators

In the following, we study the extremal function associated to the Laguerre-Bessel L^2_α-multiplier operators.

Definition 4.1. . Let ψ be a positive function on $\hat{\mathbb{K}}$ satisfying the following conditions

$$\frac{1}{\psi} \in L^1_\alpha(\hat{\mathbb{K}}) \quad (32)$$

and

$$\psi(\lambda,m) \geq 1, \quad (\lambda,m) \in \hat{\mathbb{K}}. \quad (33)$$

We define the Sobolev-type space $\mathcal{H}_\psi(\mathbb{K})$ by

$$\mathcal{H}_\psi(\mathbb{K}) = \left\{f \in L^2_\alpha(\mathbb{K}) : \sqrt{\psi}\mathcal{F}_\alpha(f) \in L^2_\alpha(\hat{\mathbb{K}})\right\}$$

provided with inner product

$$\langle f,g\rangle_\psi = \int_{\hat{\mathbb{K}}} \psi(\lambda,m)\mathcal{F}_\alpha(f)(\lambda,m)\overline{\mathcal{F}_\alpha(g)(\lambda,m)}d\gamma_\alpha(\lambda,m),$$

and the norm

$$\|f\|_\psi = \sqrt{\langle f,f\rangle_\psi}.$$

Proposition 4.2. *Let σ be a function in $L^\infty_\alpha(\hat{\mathbb{K}})$. Then the Laguerre-Bessel L^2_α multiplier operators $\mathcal{T}_{\sigma,\beta}$ are bounded and linear from $\mathcal{H}_\psi(\mathbb{K})$ into $L^2_\alpha(\mathbb{K})$ and we have for all $f \in \mathcal{H}_\psi(\mathbb{K})$*

$$\left\|\mathcal{T}_{\sigma,\beta}(f)\right\|_{2,\mu_\alpha} \leq \|\sigma\|_{\infty,\gamma_\alpha}\|f\|_\psi. \quad (34)$$

Proof. By using the relations (10),(22),(33) we get the result □

Definition 4.3. Let $\eta > 0$ and let σ be a function in $L^{\infty}_{\alpha}(\hat{\mathbb{K}})$. We denote by $\langle f, g\rangle_{\psi,\eta}$ the inner product defined on the space $\mathcal{H}_{\psi}(\mathbb{K})$ by

$$\langle f, g\rangle_{\psi,\eta} = \int_{\hat{\mathbb{K}}} \left(\eta\psi(\lambda,m) + \left|\sigma_{\beta}(\lambda,m)\right|^{2}\right) \mathcal{F}_{\alpha}(f)(\lambda,m)\overline{\mathcal{F}_{\alpha}(g)(\lambda,m)}d\gamma_{\alpha}(\lambda,m),$$

and the norm

$$\|f\|_{\psi,\eta} = \sqrt{\langle f, f\rangle_{\psi,\eta}}$$

In the following results, we show that the norm $\|\cdot\|_{\psi,\eta}$ can be expressed in function of the norm of the Hilbert space $\mathcal{H}_{\psi}(\mathbb{K})$ and the norm of Laguerre-Bessel L^2_{α}-multiplier operators. Moreover, we show the equivalence between the norms $\|\cdot\|_{\psi,\eta}$ and $\|\cdot\|_{\psi}$.

Proposition 4.4. *Let σ be a function in $L^{\infty}_{\alpha}(\hat{\mathbb{K}})$ and $f \in \mathcal{H}_{\psi}(\mathbb{K})$ then*
(i) the norm $\|\cdot\|_{\psi,\eta}$ satisfies

$$\|f\|^2_{\psi,\eta} = \|f\|^2_{\psi} + \left\|\mathcal{T}_{\sigma,\beta}(f)\right\|^2_{2,\mu_{\alpha}}.$$

(ii) The norms $\|\cdot\|_{\zeta,\eta}$ and $\|\cdot\|_{\zeta}$ are equivalent and we have

$$\sqrt{\eta}\|f\|_{\psi} \leq \|f\|_{\psi,\eta} \leq \sqrt{\eta + \|\sigma\|^2_{\infty,\gamma_{\alpha}}}\|\varphi\|_{\zeta,\eta}.$$

Proof. the results follows from Plancherel's formula (10)and the relation (34). □

Theorem 4.5. *Let $\sigma \in L^{\infty}_{\alpha}(\hat{\mathbb{K}})$ the Sobolev-type space $\left(\mathcal{H}_{\psi}(\mathbb{K})\right), \langle\cdot,\cdot\rangle_{\psi,\eta}$ is a reproducing kernel Hilbert space with kernel*

$$\mathcal{K}_{\psi,\eta}((x,t),(y,s)) = \int_{\hat{\mathbb{K}}} \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\eta\psi(\lambda,m) + \left|\sigma_{\beta}(\lambda,m)\right|^{2}} d\gamma_{\alpha}(\lambda,m),$$

that is
(i) For all $(y,s) \in \mathbb{K}$, the function $(x,t) \mapsto \mathcal{K}_{\psi,\eta}((x,t),(y,s)$ belongs to $\mathcal{H}_{\psi}(\mathbb{K})$.
(ii) For all $f \in \mathcal{H}_{\psi}(\mathbb{K})$ and $(y,s) \in \mathbb{K}$, we have the reproducing property

$$f(y,s) = \left\langle f, \mathcal{K}_{\psi,\eta}(\cdot,(y,s))\right\rangle_{\psi,\eta}.$$

Proof. (i) Let $(y,s) \in \mathbb{K}$, from the relations (6),(32) we have the function

$$g_{(y,s)} : (\lambda,m) \longrightarrow \frac{\varphi_{-\lambda,m}(y,s)}{\eta\psi(\lambda,m) + \left|\sigma_{\beta}(\lambda,m)\right|^{2}}$$

belongs to $L^1_\alpha(\hat{\mathbb{K}}) \cap L^2_\alpha(\hat{\mathbb{K}})$. Hence the function $\mathcal{K}_{\psi,\eta}$ is well defined and by the inversion formula (8), we obtain

$$\mathcal{K}_{\psi,\eta}((x,t),(y,s)) = \mathcal{F}_\alpha^{-1}(g_{(y,s)})(x,t)$$

by Plancherel's theorem for $\mathcal{F}_\alpha$ we find that $\mathcal{K}_{\psi,\eta}(\cdot,(y,s))$ belongs to $L^2_\alpha(\mathbb{K})$ and we have

$$\mathcal{F}_\alpha(\mathcal{K}_{\psi,\eta}(\cdot,(y,s)))(\lambda,m) = \frac{\varphi_{-\lambda,m}(y,s)}{\eta\psi(\lambda,m) + \left|\sigma_\beta(\lambda,m)\right|^2} \tag{35}$$

by the relations (6),(7),(35) we find that

$$\|\sqrt{\psi}\mathcal{F}_\alpha(\mathcal{K}_{\psi,\eta}(\cdot,(y,s)))\|_{2,\gamma_\alpha} \leq \frac{1}{\eta^2}\left\|\frac{1}{\psi}\right\|_{1,\gamma_\alpha} < \infty,$$

this prove that for every $(y,s) \in \mathbb{K}$ the function $(x,t) \mapsto \mathcal{K}_{\psi,\eta}\left((x,t),(y,s)\right.$ belongs to $\mathcal{H}_\psi(\mathbb{K})$.
(ii) By using the relation (35) we find that for all $f \in \mathcal{H}_\psi(\mathbb{K})$,

$$\langle f, \mathcal{K}_{\psi,\eta}\left(\cdot,(y,s)\right)\rangle_{\psi,\eta} == \int_{\hat{\mathbb{K}}} \varphi_{\lambda,m}(y,s)\mathcal{F}_\alpha(f)(\lambda,m)d\gamma_\alpha(\lambda,m),$$

inversion formula (8) gives the desired result. □

By taking σ a null function and $\eta = 1$ we find the following result

Corollary 4.6. *The Sobolev-type space $\left(\mathcal{H}_\psi(\mathbb{K})\right), \langle\cdot,\cdot\rangle_\psi)$ is a reproducing kernel Hilbert space with kernel*

$$\mathcal{K}_\psi((x,t),(y,s)) = \int_{\hat{\mathbb{K}}} \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\eta\psi(\lambda,m)} d\gamma_\alpha(\lambda,m).$$

The main result of this section can be stated as follows

Theorem 4.7. *Let $\sigma \in L^\infty_\alpha(\hat{\mathbb{K}})$ and $\beta > 0$, for any $h \in L^2_\alpha(\mathbb{K})$ and for any $\eta > 0$, there exist a unique function $f^*_{\eta,\beta,h}$ where the infimum*

$$\inf_{f \in \mathcal{H}_\psi(\mathbb{K})} \left\{\eta\|f\|^2_\psi + \left\|h - \mathcal{T}_{\sigma,\beta}(f)\right\|^2_{2,\mu_\alpha}\right\} \tag{36}$$

is attained. Moreover the extremal function $f^*_{\eta,\beta,h}$ is given by

$$f^*_{\eta,\beta,h}(y,s) = \int_{\mathbb{K}} h(x,t)\overline{\Theta_{\eta,\beta}((x,t),(y,s))}d\mu_\alpha(x,t),$$

where $\Theta_{\eta,\beta}$ is given by

$$\Theta_{\eta,\beta}((x,t),(y,s)) = \int_{\hat{\mathbb{K}}} \frac{\sigma_\beta(\lambda,m)\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\eta\psi(\lambda,m) + |\sigma_\beta(\lambda,m)|^2} d\gamma_\alpha(\lambda,m)$$

Proof. The existence and the unicity of the extremal function $f^*_{\eta,\beta,h}$ satisfying (36) is given in [7, 10, 15, 16], furthermore $f^*_{\eta,\beta,h}$ is given by

$$f^*_{\eta,\beta,h}(y,s) = \langle h, \mathcal{T}_{\sigma,\beta}(\mathcal{K}_{\psi,\eta}(\cdot,(y,s))\rangle_{\mu_\alpha}$$

, by inversion formula (8) and the relation (35) we get

$$\mathcal{T}_{\sigma,\beta}(\mathcal{K}_{\psi,\eta}(\cdot,(y,s)))(x,t) = \int_{\hat{\mathbb{K}}} \frac{\sigma_\beta(\lambda,m)\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s)}{\eta\psi(\lambda,m)+|\sigma_\beta(\lambda,m)|^2} d\gamma_\alpha(\lambda,m)$$

$$= \Theta_{\eta,\beta}((x,t),(y,s))$$

and the proof is complete. □

Theorem 4.8. *$\sigma \in L^\infty_\alpha(\hat{\mathbb{K}})$ and $h \in L^2_\alpha(\mathbb{K})$ then the function $f^*_{\eta,\beta,h}$ satisfies the following properties*

$$\mathcal{F}_\alpha(f^*_{\eta,\beta,h})(\lambda,m) = \frac{\overline{\sigma_\beta(\lambda,m)}}{\eta\psi(\lambda,m)+|\sigma_\beta(\lambda,m)|^2}\mathcal{F}_\alpha(h)(\lambda,m) \tag{37}$$

and

$$\|f^*_{\eta,\beta,h}\|_\psi \leq \frac{1}{\sqrt{2\eta}}\|h\|_{2,\mu_\alpha}.$$

Proof. Let $(y,s) \in \mathbb{K}$ then the function

$$k_{(y,s)} : (\lambda,m) \longrightarrow \frac{\sigma_\beta(\lambda,m)\varphi_{-\lambda,m}(y,s)}{\eta\psi(\lambda,m)+\left|\sigma_\beta(\lambda,m)\right|^2}$$

belongs to $L^2_\alpha(\hat{\mathbb{K}}) \cap L^1_\alpha(\hat{\mathbb{K}})$ and by inversion formula (8) we get

$$\Theta_{\eta,\beta}((x,t),(y,s)) = \mathcal{F}^{-1}_\alpha(k_{(y,s)})(x,t)$$

using Plancherel's theorem and Parseval's relation (9) we get $\Theta_{\eta,\beta}(\cdot,(y,s)) \in L^2_\alpha(\mathbb{K})$ and

$$f^*_{\eta,\beta,h}(y,s) = \int_{\hat{\mathbb{K}}} \frac{\overline{\sigma_\beta(\lambda,m)}}{\eta\psi(\lambda,m)+|\sigma_\beta(\lambda,m)|^2}\mathcal{F}_\alpha(h)(\lambda,m)d\gamma_\alpha(\lambda,m)$$

on the other hand the function

$$F : (\lambda,m) \longrightarrow \frac{\overline{\sigma_\beta(\lambda,m)}\mathcal{F}_\alpha(h)(\lambda,m)}{\eta\psi(\lambda,m)+\left|\sigma_\beta(\lambda,m)\right|^2}$$

belongs to $L^1_\alpha(\hat{\mathbb{K}}) \cap L^\infty_\alpha(\hat{\mathbb{K}})$, by inversion formula (8), Plancherel's theorem we find that $f^*_{\eta,\beta,h}$ belongs to $L^2_\alpha(\mathbb{K})$ and

$$\mathcal{F}_\alpha(f^*_{\eta,\beta,h})(\lambda,m) = F(\lambda,m)$$

on the other hand we have

$$|\mathcal{F}_\alpha(f^*_{\eta,\beta,h})(\lambda,m)|^2 \leq \frac{1}{2\eta\psi(\lambda,m)} |\mathcal{F}_\alpha(h)(\lambda,m)|^2$$

by Plancherel's formula (10) we find that

$$\|f^*_{\eta,\beta,h}\|_\psi \leq \frac{1}{\sqrt{2\eta}} \|h\|_{2,\mu_\alpha}.$$

□

Theorem 4.9. *(Third Calderón's formula) Let $\sigma \in L^\infty_\alpha(\hat{\mathbb{K}})$ and $f \in \mathcal{H}_\psi(\mathbb{K})$ then the extremal function given by*

$$f^*_{\eta,\beta,h}(y,s) = \int_{\mathbb{K}} \mathcal{T}_{\sigma,\beta}(f)(x,t)\overline{\Theta_{\eta,\beta}((x,t),(y,s))} d\mu_\alpha(x,t),$$

satisfies

$$\lim_{\eta\to 0^+} \left\| f^*_{\eta,\beta} - f \right\|_{2,\mu_\alpha} = 0 \tag{38}$$

moreover we have $f^*_{\eta,\beta} \longrightarrow f$ uniformly when $\eta \longrightarrow 0^+$.

Proof. $f \in \mathcal{H}_\psi(\mathbb{K})$, we put $h = \mathcal{T}_{\sigma,\beta}(f)$ and $f^*_{\eta,\beta,h} = f^*_{\eta,\beta}$ in the relation (37) we find that

$$\mathcal{F}_\alpha(f^*_{\eta,\beta,h} - f)(\lambda,m) = \frac{-\eta\psi(\lambda,m)\mathcal{F}_\alpha(f)(\lambda,m)}{\eta\psi(\lambda,m) + \left|\sigma_\beta(\lambda,m)\right|^2} \tag{39}$$

therefore

$$\left\| f^*_{\eta,\beta} - f \right\|^2_\psi = \int_{\hat{\mathbb{K}}} \frac{\eta^2 (\psi(\lambda,m))^3}{\eta\psi(\lambda,m) + |\sigma_\beta(\lambda,m)|^2} |\mathcal{F}_\alpha(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)$$

On the other hand we have

$$\frac{\eta^2 (\psi(\lambda,m))^3}{\eta\psi(\lambda,m) + |\sigma_\beta(\lambda,m)|^2} |\mathcal{F}_\alpha(f)(\lambda,m)|^2 \leq \psi(\lambda,m) |\mathcal{F}_\alpha(f)(\lambda,m)|^2 \tag{40}$$

the result (38) follows from (40) and the dominated convergence theorem. Now, for all $f \in \mathcal{H}_\psi(\mathbb{K})$ we have $\mathcal{F}_\alpha(f) \in L^2_\alpha(\hat{\mathbb{K}}) \cap L^1_\alpha(\hat{\mathbb{K}})$ and by using the relations (8), (39) we find that

$$f^*_{\eta,\beta(y,s)} - f(y,s) = \int_{\hat{\mathbb{K}}} \frac{-\eta\psi(\lambda,m)\mathcal{F}_\alpha(f)(\lambda,m)}{\eta\psi(\lambda,m) + \left|\sigma_\beta(\lambda,m)\right|^2} \varphi_{\lambda,m}(y,s) d\gamma_\alpha(\lambda,m)$$

and

$$\left|\frac{-\eta\psi(\lambda,m)\mathcal{F}_\alpha(f)(\lambda,m)}{\eta\psi(\lambda,m)+\left|\sigma_\beta(\lambda,m)\right|^2}\varphi_{\lambda,m}(y,s)\right| \leq |\mathcal{F}_\alpha(f)(\lambda,m)| \tag{41}$$

By using the relation (41) and the dominated convergence theorem we deduce that

$$\lim_{\eta\to 0^+}\left|f^*_{\eta,\beta}(y,s)-f(y,s)\right| = 0$$

which completes the proof of the theorem. □

Acknowledgements

The authors are deeply indebted to the referees for providing constructive comments and help in improving the contents of this article.

REFERENCES

[1] Aronszajn, N., *Theory of reproducing kernels*, Transactions of the American mathematical society 68(3) (1950), 337–404.

[2] Atef, R., *Uncertainty Inequalities on Laguerre Hypergroup*, Mediterranean Journal of Mathematics 10(1) (2013), 333–351.

[3] Bañuelos, R., & Bogdan, K., *Lévy processes and Fourier multipliers*, Journal of Functional Analysis 250(1) (2007), 197–213. https://doi.org/10.1016/j.jfa.2007.05.013

[4] Donoho, D. L., & Stark, P. B., *Uncertainty principles and signal recovery*, SIAM Journal on Applied Mathematics 49(3) (1989), 906–931.

[5] Faraut, J., & Harzallah, K., *Deux cours d'analyse harmonique*, École d'été d'analyse harmonique de Tunis, 1984.

[6] Hörmander, L., *Estimates for translation invariant operators in L^p spaces*, 1960.

[7] Kimeldorf, G., & Wahba, G., *Some results on Tchebycheffian spline functions*, Journal of Mathematical Analysis and Applications, 33(1) (1971), 82–95.

[8] Kumar, V., & Ruzhansky, M, *$L^p - -L^q$ Boundedness of (k,a)-Fourier Multipliers with Applications to Nonlinear Equations*, International Mathematics Research Notices, 2023(2), 1073–1093.

[9] Mauceri, G., *Zonal multipliers on the Heisenberg group*, Pacific Journal of Mathematics, 95(1) (1981), 143–159.

[10] Matsuura, T., Saitoh, S., & Trong, D. D., *Approximate and analytical inversion formulas in heat conduction on multidimensional spaces*, (2005).

[11] McConnell, T. R., *On Fourier multiplier transformations of Banach-valued functions*, Transactions of the American Mathematical Society, 285(2) (1984), 739–757.

[12] Mikhlin, S.G., *On the multipliers of Fourier integrals*, Dokl. Akad. Nauk SSSR 109 (1956), 701–703.

[13] Negzaoui, S., *Lipschitz conditions in Laguerre hypergroup*, Mediterranean Journal of Mathematics 14 (2017), 1–12.

[14] Nessibi, M. M., & Trimeche, K., *Inversion of the Radon transform on the Laguerre hypergroup by using generalized wavelets*. Journal of Mathematical Analysis and Applications, 208(2) (1997), 337–363.

[15] Saitoh, S., *Hilbert spaces induced by Hilbert space valued functions*, Proceedings of the American Mathematical Society, 89(1) (1983), 74–78.

[16] Saitoh, S.,Matsuura, T. , *Analytical and numerical inversion formulas in the Gaussian convolution by using the Paley–Wiener spaces*, Applicable Analysis, 85(8) (2006), 901-915.

[17] Stempak, K., *Almost everywhere summability of Laguerre series*, Studia Mathematica, 2(100) (1991), 129–147.

[18] Stempak, K., *An algebra associated with the generalized sublaplacian*, Studia Math, 88(3) (1988), 245–256.

[19] Soltani, F., *L^p-Fourier multipliers for the Dunkl operator on the real line*, Journal of Functional Analysis, 209(1) (2004), 16–35.

[20] Soltani, F., *Tikhonov regularization for Dunkl multiplier operators*, Kodai Mathematical Journal, 39(2) (2016), 399–409.

[21] Soltani, F., *Uncertainty Principles and Extremal Functions for the Dunkl L2-Multiplier Operators*. Journal of Operators, 2014(1), 659069.

[22] Soltani, F., *Dunkl multiplier operators and applications* Integral Transforms and Special Functions, 25(11) (2014), 898–908.

[23] Soltani, F. , *Extremal functions on Sturm–Liouville hypergroups*, Complex Analysis and Operator Theory, 8 (2014), 311–325.

A. CHANA
Laboratory of Fundamental and Applied Mathematics, Department of Mathematics and Informatics, Faculty of Sciences Ain Chock, University of Hassan II, B.P 5366 Maarif, Casablanca, Morocco e-mail:
`maths.chana@gmail.com`

A. AKHIIDJ
Laboratory of Fundamental and Applied Mathematics, Department of Mathematics and Informatics, Faculty of Sciences Ain Chock, University of Hassan II, B.P 5366 Maarif, Casablanca, Morocco e-mail:
akhlidj@hotmail.fr

S. ARHILAS
Laboratory of Fundamental and Applied Mathematics, Department of Mathematics and Informatics, Faculty of Sciences Ain Chock, University of Hassan II, B.P 5366 Maarif, Casablanca, Morocco e-mail:
arhilassouhir@gmail.com *2024/08/20*