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GLOBAL SOLVABILITY OF THE LAPLACE EQUATION IN WEIGHTED SOBOLEV SPACES

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We consider a non-local boundary value problem for the Laplace equation in an unbounded strip, studying the weak and strong solvability of the problem within the framework of the weighted Sobolev space $W_\nu^{1,p}$ with a Muckenhoupt weight. Utilising tools from non-harmonic analysis, we prove that any weak solution belonging to $W_\nu^{2,p}$ is also a strong solution and satisfies the corresponding boundary conditions. It is worth noting that such problems do not fall within the scope of the general theory of elliptic equations and therefore require a specialized approach.

Keywords: Laplace equation, infinite strip, biorthonormal systems, weak and strong solutions, weighted Sobolev spaces.

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1. Introduction

The classical existence and regularity theory for linear PDEs leaves untreated many problems arising in mechanics and mathematical physics. An example of such a model problem is the following degenerate elliptic equation studied

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by Moiseev in [14]. More precisely, he considered the following degenerate equation in an infinite strip:

$$\begin{cases} y^m u_{xx} + u_{yy} = 0, & (x,y) \in (0,2\pi) \times (0,\infty), \\ u(x,0) = f(x), & x \in (0,2\pi), \\ u(0,y) = u(2\pi,y), & y \in (0,\infty), \\ u_x(0,y) = 0, & y \in (0,\infty), \end{cases} \tag{1.1}$$

with $m > -2$ and $f \in C^{2,\alpha}[0,2\pi]$, where $\alpha \in (0,1)$. This problem is non-local, and the boundary conditions are given on semi-infinite lines. Under the natural assumption of boundedness of the solution at infinity, the author obtained the existence and uniqueness of a classical solution and an explicit integral representation for it, allowing the relaxation of regularity assumptions on f.

Similar boundary problems for mixed-type equations were investigated by Frankl in his study on the transonic flow around symmetric airfoils (see [6, 7]). Further results on the existence of classical solutions of problem (1.1) were obtained in [2] for uniformly elliptic equations and in [10] for multidimensional parabolic equations.

We start our studies with the case $m = 0$, while the degenerate problem is a subject of further research. Our goal is twofold: to obtain strong/weak solvability of (1.1) and to study the regularity of the solutions in new function spaces. Our interest is focused on weighted Lebesgue spaces L^p_ν, where $p \in (1,\infty)$, and the weight function ν belongs to the Muckenhoupt class A_p. Since we study problem (1.1) in an unbounded domain with respect to y, we assume that the weight depends only on x. The Sobolev spaces are built upon functions having distributional derivatives in some weighted x-space and integrable with respect to y.

Starting with the existence of weak solutions in $W^{1,p}_\nu$, we show that under suitable boundary conditions, this weak solution is also strong.

Let us note that this problem cannot be treated with the classical methods developed for linear elliptic operators. Our technique is based on spectral theory and the approach developed in [14]. Specifically, we use biorthonormal systems and Fourier series techniques in Banach function spaces, as described by Duffin and Schaeffer [5] (see also [11]). This approach extends harmonic analysis methods beyond Hilbert spaces to apply Fourier series methods (see also [3, 15, 17]).

The present paper extends some results obtained in [2, 12, 14], transitioning from classical to generalized solutions.

In the following, we use the standard notation:

- $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;

- $\Pi = (0,2\pi) \times (0,\infty)$ is an unbounded strip in $\mathbb{R}^2$ with boundary

$$\partial\Pi = J_0 \cup \bar{J} \cup J_{2\pi},$$

where

$$J = \{x \in (0,2\pi),\ y = 0\},$$
$$J_0 = \{x = 0,\ y \in (0,\infty)\},$$
$$J_{2\pi} = \{x = 2\pi,\ y \in (0,\infty)\};$$

- $C^\infty_{2\pi}(\bar{J}) = \{\eta(x) \in C^\infty([0,2\pi]) : \eta(2\pi) = 0\}$;
- The letter C indicates a positive constant, whose value may vary from line to line.

2. Auxiliary Results

Consider the following set of *test functions*:

$$C^\infty_{J_0}(\overline{\Pi}) = \Big\{\varphi \in C^\infty(\overline{\Pi}) : \varphi|_{J \cup J_{2\pi}} = 0,\ \exists\ \xi_\varphi > 0 \text{ such that } \varphi(x,y) = 0 \quad \forall\ (x,y) \in [0,2\pi] \times [\xi_\varphi,\infty)\Big\}. \tag{2.1}$$

Let $\nu : (0,2\pi) \to [0,\infty]$ be a weight function such that $\nu \in L^1(0,2\pi)$, $\nu(0) = \nu(2\pi)$, and $|\nu^{-1}(\{0;\infty\})| = 0$. For a fixed $p \in (1,\infty)$, we consider the weighted Lebesgue space $L^p_\nu(\Pi)$, endowed with the norm:

$$\|u\|_{L^p_\nu(\Pi)} = \int_0^\infty \Big(\int_0^{2\pi} |u(x,y)|^p\, \nu(x)\, dx\Big)^{\frac{1}{p}} dy.$$

The corresponding ν-weighted Sobolev space $W^{m,p}_\nu(\Pi)$ is defined as the set of all measurable functions having distributional derivatives up to order m in $L^p_\nu(\Pi)$, for which the following norm is finite:

$$\|u\|_{W^{m,p}_\nu(\Pi)} = \sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_{L^p_\nu(\Pi)}.$$

For $p \in (1,\infty)$, we denote the Lebesgue and Sobolev spaces over the interval $(0,2\pi)$ as $L^p_\nu(0,2\pi)$ and $W^{m,p}_\nu(0,2\pi)$, respectively, endowed with the respective norms:

$$\|f\|_{L^p_\nu(0,2\pi)} = \Big(\int_0^{2\pi} |f(x)|^p \nu(x)\, dx\Big)^{\frac{1}{p}}, \qquad \|f\|_{W^{m,p}_\nu(0,2\pi)} = \sum_{k=0}^{m} \|f^{(k)}\|_{L^p_\nu(0,2\pi)}.$$

For completeness, we define the Muckenhoupt class $A_p(0,2\pi)$ of weights ν defined on $\mathbb{R}$, 2π-periodic and that satisfy the condition:

$$\sup_{I\subset\mathbb{R}}\left(\frac{1}{|I|}\int_I \nu(x)\,dx\right)\left(\frac{1}{|I|}\int_I \nu(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} = [\nu]_p < \infty, \tag{2.2}$$

where the supremum is taken over all bounded intervals $I\subset\mathbb{R}$, (for sake of periodicity, it is enough taking $I\subset(0,2\pi)$) and $[\nu]_p$ is the Muckenhoupt constant of the weight ν.

An immediate consequence of the definition (2.2) is that if $\nu\in A_p(0,2\pi)$, then $\nu,\nu^{-1/(p-1)}\in L^1_{\text{loc}}(\mathbb{R})$. Moreover, the following properties hold (cf. [8]):

Lemma 2.1. *Let $\nu\in A_p(0,2\pi)$, $p\in(1,\infty)$:*

- Inclusion property: *There exists $q\in(1,p)$ such that $\nu\in A_q(0,2\pi)$.*
- Reverse Hölder inequality: *There exists $\delta>0$ depending only on p and $[\nu]_p$ such that*

$$\left(\frac{1}{|I|}\int_I \nu(x)^{1+\delta}\,dx\right)^{\frac{1}{1+\delta}} \le \frac{C_\delta}{|I|}\int_I \nu(x)\,dx$$

for each interval I, where the constant C_δ does not depend on I.

For our purpose, we additionally assume that the weight satisfies the condition $\nu(0)=\nu(2\pi)$.

The next Lemma gives a characterization of the weighted spaces that we are going to use, and the proof follows easily from [1, Lemma 2.6].

Lemma 2.2. *Let $\nu\in A_p(0,2\pi)$, with $1<p<\infty$. Then:*

1. *$L^p_\nu(0,2\pi)\subset L^1(0,2\pi)$ is continuously embedded.*
2. *$\overline{C_0^\infty(0,2\pi)} = L^p_\nu(0,2\pi)$, where the closure is taken with respect to the norm in $L^p_\nu(0,2\pi)$.*

In what follows, we need the Young-Hausdorff inequality related to the classical system of functions $\{1,\cos(nx),\sin(nx)\}_{n\in\mathbb{N}}$.

Theorem 2.3 (Young-Hausdorff [13])**.** *Let $f\in L^p(0,2\pi)$, $1<p\le 2$. Consider, for $n\in\mathbb{N}_0$, the integrals*

$$f_n^c = \int_0^{2\pi} f(x)\cos(nx)\,dx, \qquad f_n^s = \int_0^{2\pi} f(x)\sin(nx)\,dx. \tag{2.3}$$

Then $\{f_n^c, f_n^s\}_{n\in\mathbb{N}_0} \subseteq \ell_{p'}$ *where* $p' = p/(p-1)$, *and there exists a constant* C, *depending only on* p, *such that*

$$\Big(|f_0^c|^{p'} + \sum_{n=1}^{\infty}\big(|f_n^c|^{p'} + |f_n^s|^{p'}\big)\Big)^{\frac{1}{p'}} \leq C\|f\|_{L^p(0,2\pi)}. \tag{2.4}$$

Conversely, if $\{f_n^c, f_n^s\}_{n\in\mathbb{N}_0} \subseteq \ell_p$, $1 < p \leq 2$, *then* $f \in L^{p'}(0,2\pi)$, *and there exists a constant* C', *depending only on p, such that*

$$\|f\|_{L^{p'}(0,2\pi)} \leq C'\Big(|f_0^c|^p + \sum_{n=1}^{\infty}\big(|f_n^c|^p + |f_n^s|^p\big)\Big)^{\frac{1}{p}}. \tag{2.5}$$

Definition 2.4 ([4, 16]). Let $(X, \|\cdot\|)$ be a Banach space, $\{\vartheta_n\}_{n\in\mathbb{N}}$ be a vector sequence in X and $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in the dual space X^*, the pair $(y_n;\vartheta_n)$ is a biorthonormal system if and only if

$$(y_m;\vartheta_n) = \delta_{mn}, \quad \forall m,n \in \mathbb{N},$$

where $(\cdot,\cdot)$ denotes the duality pairing and δ_{mn} is the Kronecker delta:

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

If $(X, (\cdot,\cdot))$ is an Hilbert space, the definition of biorthonormal system is the same, just replacing the duality pairing with his inner product.

Neither the sequence $\{y_n\}$ nor $\{\vartheta_n\}$ is required to be orthonormal: it is easy to exhibit an example of biorthonormal system that is not orthonormal (see $\{y_n^s\}_{n\in\mathbb{N}}$ in the next). The sequences $\{y_n\}$ and $\{\vartheta_n\}$ are called biorthogonal bases if they are complete in the corresponding space. In this case, each sequence $\{y_n\}$ and $\{\vartheta_n\}$ spans the space (or a dense subspace if the space is infinite-dimensional).

We introduce the following systems of functions:

$$y_0^c = 1, \quad y_n^c(x) = \cos(nx), \quad y_n^s(x) = x\sin(nx), \quad n \geq 1, \tag{2.6}$$

$$\vartheta_0^c(x) = \frac{2\pi - x}{2\pi^2},\ \vartheta_n^c(x) = \frac{2\pi - x}{\pi^2}\cos(nx),\ \vartheta_n^s(x) = \frac{1}{\pi^2}\sin(nx), \quad n \geq 1. \tag{2.7}$$

Then the sequences

$$\{y_n\}_{n\in\mathbb{N}_0} := \{y_n^c, y_n^s\}_{n\in\mathbb{N}_0}, \qquad \{\vartheta_n\}_{n\in\mathbb{N}_0} := \{\vartheta_0^c, \vartheta_n^c, \vartheta_n^s\}_{n\in\mathbb{N}}, \tag{2.8}$$

indeed form a *biorthonormal system* in $L^2(0,2\pi)$. More precisely, for any function $f \in L^2(0,2\pi)$ the linear continuous operator $(f;\vartheta)$ acts on $L^2(0,2\pi)$ as

$$(f;\vartheta) = \int_0^{2\pi} f(x)\vartheta(x)\,dx$$

providing the pairing necessary to verify biorthonormality conditions. Direct calculations show that the sequences (2.9) are biorthonormal, that is:

$$\begin{aligned} (y_k^c;\vartheta_n^c) &= \delta_{kn}, \qquad (y_k^s;\vartheta_n^s) = \delta_{kn},\\ (y_k^s;\vartheta_n^c) &= 0, \qquad (y_k^c;\vartheta_n^s) = 0 \end{aligned} \tag{2.9}$$

forming a *biorthonormal system* in $L^2(0,2\pi)$ according to Definition 2.4. Moreover, (2.6) forms a *Riesz basis* in $L^2(0,2\pi)$ (cf. [14]), meaning we can expand any function $f \in L^2(0,2\pi)$ in a *biorthonormal series* of the form

$$f(x) = (f;\vartheta_0^c)y_0^c(x) + \sum_{n=1}^{\infty}\Big((f;\vartheta_n^c)y_n^c(x) + (f;\vartheta_n^s)y_n^s(x)\Big). \tag{2.10}$$

Our goal is to extend this theory to weighted Lebesgue spaces.

Theorem 2.5. *Let $\nu \in A_p(0,2\pi)$ with $p \in (1,\infty)$ and $\nu(0) = \nu(2\pi)$. Then the system* (2.6) *forms a basis in $L^p_\nu(0,2\pi)$.*

Proof. First, we need to show that (2.6) is a *minimal system.* To achieve this, it is sufficient to prove that (2.7) is a biorthonormal system to (2.6) in $L^p_\nu(0,2\pi)$.

Let us observe that the functional $b_{\vartheta_n^c}(\cdot) = (\vartheta_n^c;\cdot)$, $n \in \mathbb{N}_0$ is uniformly bounded in $L^p_\nu(0,2\pi)$. Indeed, for any $f \in L^p_\nu(0,2\pi)$, using the uniform boundedness of the system $\{\vartheta_n\}$ and Hölder's inequality, we obtain:

$$\begin{aligned} |b_{\vartheta_n^c}(f)| &\le \frac{2}{\pi}\int_0^{2\pi} |f(x)|\nu(x)^{\frac{1}{p}}\nu(x)^{-\frac{1}{p}}\,dx\\ &\le \frac{2}{\pi}\Big(\int_0^{2\pi}|f(x)|^p\nu(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_J \nu(x)^{-\frac{p'}{p}}\,dx\Big)^{\frac{1}{p'}}\\ &\le 4\|f\|_{L^p_\nu(0,2\pi)}[\nu]_p^{\frac{1}{p}}\|\nu\|_{L^1(0,2\pi)}^{-\frac{1}{p}} \le C\|f\|_{L^p_\nu(0,2\pi)}. \end{aligned} \tag{2.11}$$

This implies that $b_{\vartheta_n^c}(\cdot) \in \big(L^p_\nu(0,2\pi)\big)^*$ for all $n \in \mathbb{N}_0$.

Furthermore, using (2.9), we have:

$$b_{\vartheta_n^c}(y_k^c) = \delta_{kn}, \quad b_{\vartheta_n^c}(y_k^s) = 0 \quad \forall\, k,n \in \mathbb{N}.$$

Similarly, it can be shown that $b_{\vartheta_n^s}(\cdot) \in (L^p_\nu(0,2\pi))^*$ and satisfies

$$b_{\vartheta_n^s}(y_k^s) = \delta_{kn}, \quad b_{\vartheta_n^s}(y_k^c) = 0, \quad \forall k,n \in \mathbb{N}.$$

This proves the biorthogonality of (2.7) and (2.6) in $L^p_\nu(0,2\pi)$, thereby establishing the minimality of (2.6) in $L^p_\nu(0,2\pi)$.

Next, we need to demonstrate that (2.6) is also *complete* in $L^p_\nu(0,2\pi)$. By density arguments and Lemma 2.2, it suffices to show that any function in $C_0^\infty(0,2\pi)$ can be approximated by linear combinations of functions from the system (2.6) in the space $L^p_\nu(0,2\pi)$.

Let $f \in C_0^\infty(0,2\pi)$, we define

$$g(x) = \frac{2\pi - x}{\pi^2} f(x), \qquad g \in C_0^\infty(0,2\pi).$$

For all $n \in \mathbb{N}$, we take the biorthogonal coefficients of f :

$$\begin{aligned}
(f;\vartheta_0^c) &= \frac{1}{2\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\,dx = \frac{1}{2}(g;1) = \frac{1}{2}g_0,\\
(f;\vartheta_n^c) &= \frac{1}{\pi^2}\int_0^{2\pi} f(x)\,(2\pi - x)\cos(nx)\,dx = (g;\cos(nx)) = g_n^c,\\
(f;\vartheta_n^s) &= \frac{1}{\pi^2}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi^2}(f;\sin(nx)) = \frac{1}{\pi^2}f_n^s.
\end{aligned} \tag{2.12}$$

Integrating by parts twice using the regularity of f, we obtain the bounds:

$$\begin{aligned}
|(f;\vartheta_n^c)| &\le \frac{1}{n^2}\int_0^{2\pi} \left|g''(x)\cos(nx)\right| dx \le \frac{C}{n^2},\\
|(f;\vartheta_n^s)| &\le \frac{1}{n^2\pi^2}\int_0^{2\pi} \left|f''(x)\sin(nx)\right| dx \le \frac{C}{n^2}.
\end{aligned} \tag{2.13}$$

These estimates guarantee the total convergence of the biorthonormal series:

$$F(x) = (f;\vartheta_0^c) + \sum_{n=1}^{\infty}\Big((f;\vartheta_n^c)\cos(nx) + (f;\vartheta_n^s)x\cos(nx)\Big), \tag{2.14}$$

and also uniform convergence by the Weierstrass theorem.

According to the results of [14], the system (2.9) forms a basis in $L^2(0,2\pi)$, and hence $F = f$. By Lemma 2.2, it also follows that (2.14) converges to f in $L^p_\nu(0,2\pi)$, and therefore, (2.9) is a basis in $L^p_\nu(0,2\pi)$. To confirm this, consider the projectors:

$$\begin{aligned}
S_{n,m}(f)(x) &= \sum_{k=0}^{n}(f;\vartheta_k^c)y_k^c(x) + \sum_{k=1}^{m}(f;\vartheta_n^s)y_k^s(x)\\
&= \frac{1}{2}g_0 + \sum_{k=1}^{n} g_k^c\cos(kx) + \frac{x}{\pi^2}\sum_{k=1}^{m} f_k^s\sin(kx).
\end{aligned} \tag{2.15}$$

Due to the orthogonality of the trigonometric system, the $L_\nu^p(0,2\pi)$-norm of (2.15) can be estimated as:

$$\|S_{n,m}(f)\|_{L_\nu^p(0,2\pi)} \leq \Big\| \frac{1}{2} g_0 + \sum_{k=1}^{n} g_k^c \cos(kx) \Big\|_{L_\nu^p(0,2\pi)} + \frac{2}{\pi} \Big\| \sum_{k=1}^{m} f_k^s \sin(kx) \Big\|_{L_\nu^p(0,2\pi)}.$$

In the first term, we have a partial sum of the Fourier series for $g \in L_\nu^p(0,2\pi)$, while in the second term, we have the corresponding partial sum for $f \in L_\nu^p(0,2\pi)$. Since the trigonometric system forms a basis in $L_\nu^p(0,2\pi)$ if and only if $\nu \in A_p(0,2\pi)$ (cf. [9]), we obtain:

$$\|S_{n,m}(f)\|_{L_\nu^p(0,2\pi)} \leq C\Big(\|g\|_{L_\nu^p(0,2\pi)} + \|f\|_{L_\nu^p(0,2\pi)} \Big) \leq c\|f\|_{L_\nu^p(0,2\pi)}$$

where we have used that $|g(x)| \leq C|f(x)|$ for $x \in (0,2\pi)$. The last estimate holds for all $n,m \in \mathbb{N}$, with a constant independent of f. This implies that the projectors $\{S_{n,m}\}$ are uniformly bounded in $L_\nu^p(0,2\pi)$ and hence, the system (2.9) forms a basis in $L_\nu^p(0,2\pi)$. □

3. Solvability results

Let us consider the following non-local problem for the Laplace equation, written in formal way:

$$\begin{cases} \Delta u(x,y) = 0 & \text{for a.e. } (x,y) \in \Pi \\ u(0,y) = u(2\pi,y) & \text{for a.e. } y \in (0,\infty) \\ u(x,0) = f(x) & \text{for a.e. } x \in (0,2\pi) \\ u_x(0,y) = h(y) & \text{for a.e. } y \in (0,\infty) \end{cases} \tag{3.1}$$

where we initially suppose that $f \in L^1(0,2\pi)$ and $h \in L^1(\mathbb{R}_+)$.

By *weighted strong solution* of (3.1), we mean a function $u \in W_\nu^{2,p}(\overline{\Pi})$ with $\nu \in L^1(0,2\pi)$, verifying the partial differential equation in (3.1) and the boundary conditions almost everywhere.

Let $\varphi \in C_{J_0}^\infty(\overline{\Pi})$ be a test function from the class in (2.1), with support in the bounded rectangle $\Pi_{\xi_\varphi} = (0,2\pi) \times (0,\xi_\varphi)$ for some $\xi_\varphi > 0$. Multiplying the equation in (3.1) with φ, integrating over Π, and applying the Gauss-Ostrogradsky theorem, we obtain:

$$\begin{aligned} 0 &= \iint_{\Pi} \Delta u(x,y)\varphi(x,y)\,dxdy = \iint_{\Pi_{\xi_\varphi}} \Delta u(x,y)\varphi(x,y)\,dxdy \\ &= -\iint_{\Pi_{\xi_\varphi}} \nabla u(x,y)\nabla\varphi(x,y)\,dxdy + \int_{\partial\Pi_{\xi_\varphi}} \varphi(x,y)\,\frac{\partial u(x,y)}{\partial \boldsymbol{n}}\,dl \end{aligned}$$

where $\boldsymbol{n}$ is the outer normal to $\partial\Pi_{\xi_\varphi}$ and ξ_φ depends on the support of φ.

Applying the boundary conditions of (3.1) and taking into account that on J_0 we have $\boldsymbol{n} = (-1,0)$, we obtain:

$$\iint\limits_{\Pi} \nabla u \nabla \varphi \, dxdy = -\int_0^\infty \varphi(0,y)h(y)\, dy, \qquad \forall \varphi \in C^\infty_{J_0}(\overline{\Pi}).$$

This permits us to give the following notion of a weak solution.

Definition 3.1. A function $u \in W^{1,p}_\nu(\Pi)$, $p \in (1,\infty)$, is a *weak solution* of (3.1) if it is differentiable in the distributional sense and satisfies:

$$\begin{cases} \iint\limits_{\Pi} \nabla u \nabla \varphi \, dxdy + \int_0^\infty \varphi(0,y)h(y)\, dy = 0, & \\ u(0,y) = u(2\pi,y), & y \in (0,\infty), \\ u(x,0) = f(x), & x \in (0,2\pi), \end{cases} \tag{3.2}$$

for each $\varphi \in C^\infty_{J_0}(\overline{\Pi})$.

Theorem 3.2. *Let $\nu \in A_p(0,2\pi)$, $1 < p < \infty$ and let $f \in W^{1,p}_\nu(0,2\pi)$ such that $f(0) = f(2\pi) = 0$ and $h \in L^1(\mathbb{R}_+)$. If problem* (3.2) *has a weak solution $u \in W^{1,p}_\nu(\Pi)$ then it is unique.*

Proof. To prove the uniqueness of the weak solution of (3.2), it suffices to show that the homogeneous problem:

$$\begin{cases} \iint\limits_{\Pi} \nabla u \nabla \varphi \, dxdy = 0, & \varphi \in C^\infty_{J_0}(\overline{\Pi}), \\ u(0,y) = u(2\pi,y), & y \in (0,\infty), \\ u(x,0) = 0, & x \in (0,2\pi), \end{cases} \tag{3.3}$$

has only the trivial solution.

For any bounded domain Π_ξ, we have (as in (2.11))

$$\|u\|_{W^{1,1}(\Pi_\xi)} \le C\|u\|_{W^{1,p}_\nu(\Pi_\xi)} < \infty,$$

with a constant $C = C(p, \|\nu\|_{L^1(0,2\pi)}, [\nu]_p)$.

Thus, u has a trace u^ξ on the upper boundary $J_\xi = \{(x,\xi) : \ x \in (0,2\pi)\}$, and by the absolute continuity on lines of the Sobolev functions $W^{1,1}$, we have:

$$\begin{aligned} & u^\xi(x) := u(x,\xi) = \int_0^\xi \frac{\partial u(x,y)}{\partial y}\, dy, \\ & u^0(x) = 0, \qquad \text{for a.e. } x \in (0,2\pi). \end{aligned} \tag{3.4}$$

It follows that $u^{\xi} \in L^p_\nu(0,2\pi)$, and moreover, the estimate:

$$\|u^{\xi}\|_{L^p_\nu(0,2\pi)} \leq c_{\xi}\|u\|_{W^{1,p}_\nu(\Pi)}, \tag{3.5}$$

is valid, with a constant c_ξ depending only on ξ and p.

To write the developing in series of the solution of (3.3), we calculate the biorthonormal coefficients of $u(x,y)$ as in (2.12). These coefficients are given by:

$$\begin{aligned} u^c_0(y) &= (u^y;\vartheta^c_0) = \frac{1}{2\pi^2}\int_0^{2\pi} u^y(x)(2\pi-x)\,dx,\\ u^c_n(y) &= (u^y;\vartheta^c_n) = \frac{1}{\pi^2}\int_0^{2\pi} u^y(x)(2\pi-x)\cos(nx)\,dx,\\ u^s_n(y) &= (u^y;\vartheta^s_n) = \frac{1}{\pi^2}\int_0^{2\pi} u^y(x)\sin(nx)\,dx. \end{aligned} \tag{3.6}$$

Thus, the biorthonormal series rapresentation of u is:

$$u(x,y) = u^c_0(y) + \sum_{n=1}^{\infty}\Big(u^c_n(y)\cos(nx) + u^s_n(y)x\sin(nx)\Big). \tag{3.7}$$

It follows directly from (3.5) and the initial condition in (3.3) that $\|u^y\|_{L^p_\nu(0,2\pi)}$ vanishes as $y \to 0^+$. Consequently, we have $u^c_0(0) = u^c_n(0) = u^s_n(0) = 0$.

For any $\psi(y) \in C^\infty_0(\mathbb{R}_+)$, the functions $\varphi_n(x,y) = \psi(y)\sin(nx)$ belong to $C^\infty_{J_0}(\overline{\Pi})$. Moreover, $\frac{\partial u}{\partial y} \in L^1(\Pi_\xi)$ (see [13]), the functions $\{u^c_n, u^s_n\}_{n\in\mathbb{N}_0}$ are differentiable, and

$$\frac{du^s_n(y)}{dy} = \frac{1}{\pi^2}\int_0^{2\pi}\frac{\partial u(x,y)}{\partial y}\sin(nx)\,dx.$$

Multiplying both sides by $\psi'(y)$, integrating in y over $\mathbb{R}_+$, using the fact that u solves problem (3.3) and taking into account that $\psi(0)=0$, we obtain

$$\begin{aligned} \int_0^\infty \frac{du^s_n(y)}{dy}\psi'(y)\,dy &= \frac{1}{\pi^2}\iint_\Pi \frac{\partial u(x,y)}{\partial y}\psi'(y)\sin(nx)\,dxdy\\ &= \frac{1}{\pi^2}\iint_\Pi \frac{\partial u}{\partial y}\frac{\partial \varphi_n}{\partial y}\,dxdy = -\frac{1}{\pi^2}\iint_\Pi \frac{\partial u}{\partial x}\frac{\partial \varphi_n}{\partial x}\,dxdy\\ &= -\frac{n}{\pi^2}\iint_\Pi \frac{\partial u(x,y)}{\partial x}\psi(y)\cos(nx)\,dxdy\\ &= -\frac{n^2}{\pi^2}\iint_\Pi u(x,y)\psi(y)\sin(nx)\,dxdy\\ &= -n^2\int_0^\infty u^s_n(y)\psi(y)\,dy. \end{aligned} \tag{3.8}$$

To estimate the L^p-norms of the coefficients, we calculate:

$$
\begin{aligned}
|u_n^s(y)|^p &\le \frac{1}{\pi^{2p}}\Big(\int_0^{2\pi}|u(x,y)|\,dx\Big)^p \\
&= \frac{1}{\pi^{2p}}\Big(\int_0^{2\pi}|u(x,y)|\nu(x)^{\frac{1}{p}}\nu(x)^{-\frac{1}{p}}\,dx\Big)^p \\
&\le \frac{1}{\pi^{2p}}\Big(\int_0^{2\pi}\nu(x)^{-\frac{p'}{p}}\,dx\Big)^{\frac{p}{p'}}\int_0^{2\pi}|u(x,y)|^p\nu(x)\,dx,
\end{aligned}
$$

and hence,

$$\|u_n^s\|_{L^p(\mathbb{R}_+)} \le C\|u\|_{L^p_\nu(\Pi)},$$

with a constant $= C(p,\|\nu\|_{L^1(0,2\pi)},[\nu]_p)$.

In a similar way, we establish that $\frac{du_n^s}{dy} \in L^p(\mathbb{R}_+)$. Moreover, direct calculations show that the second derivative of u_n^s exists in the distributional sense. Precisely,

$$
\begin{aligned}
\int_0^\infty \frac{d^2u_n^s(y)}{dy^2}\psi(y)\,dy &= \frac{1}{\pi^2}\iint\limits_{\Pi}\frac{\partial^2 u(x,y)}{\partial y^2}\psi(y)\sin(nx)\,dxdy \\
&= -\frac{1}{\pi^2}\iint\limits_{\Pi}\frac{\partial u(x,y)}{\partial y}\psi'(y)\sin(nx)\,dxdy \\
&= n^2\int_0^\infty u_n^s(y)\psi(y)\,dy,
\end{aligned}
$$

which implies $\frac{d^2u_n^s}{dy^2} = n^2u_n^s$ for almost each $y\in\mathbb{R}_+$, and $u_n^s \in W^{2,p}(\mathbb{R}_+)$.

Hence, for all $n\in\mathbb{N}$, the functions u_n^s solve:

$$
\begin{cases}
\dfrac{d^2u_n^s(y)}{dy^2} = n^2u_n^s(y) & \text{for a.e. } y\in\mathbb{R}_+ \\
u_n^s(0) = 0 & \lim\limits_{y\to\infty}|u(x,y)| = 0.
\end{cases}
\tag{3.9}
$$

Since we are looking for bounded solutions, we consider general solution of the form $u_n^s(y) = ae^{-ny}$. It is easy to see that the initial condition in (3.9) gives $a = 0$, which means (3.9) has only trivial solution. Hence $u_n^s(y) = 0$.

In order to calculate the coefficients u_n^c we use similar reasoning, defining the functions

$$\phi_0(x,y) = \frac{1}{2\pi^2}\,\psi(y)(2\pi - x), \quad \phi_n(x,y) = \frac{1}{\pi^2}\,\psi(y)(2\pi - x)\cos(nx)$$

for any choice of $\psi\in C_0^\infty(\mathbb{R}_+)$.

Calculations similar to those above, and using the boundary conditions of (3.3), give

$$\begin{aligned}\int_0^\infty \frac{du_0^c(y)}{dy}\psi'(y)\,dy &= \frac{1}{2\pi^2}\iint\limits_{\Pi} \frac{\partial u(x,y)}{\partial y}\psi'(y)(2\pi - x)\,dxdy \\ &= \iint\limits_{\Pi} \frac{\partial u}{\partial y}\frac{\partial \phi_0}{\partial y}\,dxdy = -\iint\limits_{\Pi} \frac{\partial u}{\partial x}\frac{\partial \phi_0}{\partial x}\,dxdy \\ &= \frac{1}{2\pi^2}\int_0^\infty \Big(\int_0^{2\pi} \frac{\partial u(x,y)}{\partial x}\,dx\Big)\psi(y)\,dy = 0.\end{aligned}$$

From the *Fundamental Lemma of Calculus of Variations,* it follows:

$$\frac{d^2u_0^c}{dy^2} = 0 \quad \Longrightarrow \quad u_0^c(y) = ay + b.$$

Since $u_0^c \in L^p(\mathbb{R}_+)$, it is necessary that $a = b = 0$.

On the other hand, for all $n \in \mathbb{N}$, we have:

$$\begin{aligned}\int_0^\infty \frac{du_n^c(y)}{dy}\psi'(y)\,dy &= \frac{1}{\pi^2}\iint\limits_{\Pi} \frac{\partial u}{\partial y}(2\pi - x)\cos(nx)\,\psi'(y)\,dxdy \\ &= \iint\limits_{\Pi} \frac{\partial u}{\partial y}\frac{\partial \phi_n}{\partial y}\,dxdy = -\iint\limits_{\Pi} \frac{\partial u}{\partial x}\frac{\partial \phi_n}{\partial x}\,dxdy \\ &= \frac{1}{\pi^2}\iint\limits_{\Pi} \frac{\partial u}{\partial x}\Big(\cos(nx) + (2\pi - x)n\sin(nx)\Big)\psi(y)\,dxdy.\end{aligned} \tag{3.10}$$

From the boundary conditions of (3.3), the definition of ϕ_n, it follows:

$$\begin{aligned}&\frac{1}{\pi^2}\int_0^{2\pi} \frac{\partial u(x,y)}{\partial x}\cos(nx)\,dx = nu_n^s(y) \\ &\frac{1}{\pi^2}\int_0^{2\pi} \frac{\partial u(x,y)}{\partial x}(2\pi - x)n\sin(nx)\,dx = nu_n^s(y) - n^2u_n^c(y).\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), keeping in mind that $u_n^s(y) = 0$ for $y \in \mathbb{R}_+$, and arguing as above, we obtain:

$$\int_0^\infty \left(\frac{du_n^c(y)}{dy}\psi'(y) + \big(n^2u_n^c(y) - nu_n^s(y)\big)\psi(y)\right)dy = 0,$$

which implies (after integrating by parts the first term):

$$\int_0^\infty \Big(-\frac{d^2u_n^c(y)}{dy^2} + n^2u_n^c(y) - nu_n^s(y)\Big)\psi(y)\,dy = 0.$$

From the *Fundamental Lemma of Calculus of Variations*, it follows that u_n^c solves the following problem

$$\begin{cases} \dfrac{d^2 u_n^c(y)}{dy^2} = n^2 u_n^c(y) \quad \text{for a.e. } y \in \mathbb{R}_+ \\ u_n^c(0) = 0 \end{cases} \tag{3.12}$$

where we used the result that $u_n^s = 0$. Hence, this problem has only the trivial solution in $W^{2,p}(\mathbb{R}_+)$, that is, $u_n^c(y) = 0$ for all $y \in \mathbb{R}_+$ and any $n \in \mathbb{N}$.

Consequently, all biorthonormal coefficients of $u^y(x)$ are equal to zero for almost each $x \in (0, 2\pi)$, and hence $u(x,y) = 0$ for almost each $(x,y) \in \Pi$. □

Concerning the question of existence of weak solutions of (3.2), we study the particular case when $h = 0$.

Theorem 3.3. *Let $\nu \in A_p(0,2\pi)$ with $1 < p < \infty$ and $f \in W_\nu^{1,p}(0,2\pi)$ such that $f(0) = f(2\pi) = 0$. Then the problem*

$$\begin{cases} \displaystyle\iint_\Pi \nabla u \nabla \varphi \, dxdy = 0 & \forall \varphi \in C_{J_0}^\infty(\overline{\Pi}) \\ u(0,y) = u(2\pi, y) & y \in (0,\infty) \\ u(x,0) = f(x) & x \in (0,2\pi) \end{cases} \tag{3.13}$$

has a unique solution satisfying the estimate:

$$\|u\|_{W_\nu^{1,p}(\Pi)} \le c \|f\|_{W_\nu^{1,p}(0,2\pi)}.$$

Proof. Let $u \in W_\nu^{1,p}(\Pi)$ and $\{u_n^c(y), u_n^s(y)\}_{n \in \mathbb{N}_0}$ be the biorthonormal coefficients of $u(x,y)$ with respect to the system (2.6). Consider the biorthonormal series (3.7) related to u. Since problem (3.13) is not homogeneous, we obtain:

$$u^y(x) = \int_0^y \frac{\partial u(x,\tau)}{d\tau} \, d\tau = u(x,y) - f(x) \quad \text{for a.e. } x \in (0,2\pi).$$

It is easy to see that $u^y \in L_\nu^p(0,2\pi)$ for a.e. $y \in \mathbb{R}_+$. Moreover

$$\|u^y\|_{L_\nu^p(0,2\pi)} = \|u(\cdot,y) - f(\cdot)\|_{L_\nu^p(0,2\pi)} \to 0 \quad \text{as } y \to 0^+.$$

Our goal is to prove that the series

$$u(x,y) = u_0^c(y) + \sum_{n=1}^\infty \Big(u_n^c(y)\cos(nx) + u_n^s(y) x \sin(nx) \Big) \tag{3.14}$$

with

$$u_n^c(y) = (u; \vartheta_n^c), \quad u_n^s(y) = (u; \vartheta_n^s),$$

is a solution of (3.3).

First of all, note that the total convergence in $L^p_\nu(\Pi)$ follows analogously to the convergence of (2.14).

By formally deriving (3.14) and arguing as above, we obtain the following problems:

$$\begin{cases} \dfrac{d^2 u_n^s(y)}{dy^2} = n^2 u_n^s(y) & y \in \mathbb{R}^+ \\ u_n^s(0) = (f; \vartheta_n^s) & n = 1, 2, \dots . \end{cases} \tag{3.15}$$

Due to the boundedness of u_n^s as $y \to \infty$, the unique solution of problem (3.15) is:

$$u_n^s(y) = (f; \vartheta_n^s) e^{-ny}.$$

By similar arguments applied to $\{u_n^c\}$, we obtain:

$$\begin{cases} \dfrac{d^2 u_n^c(y)}{dy^2} = n^2 u_n^c(y) - n u_n^s(y) & \text{for a.e. } y \in \mathbb{R}_+ \\ u_n^c(0) = (f; \vartheta_n^c) & n = 0, 1, \dots \end{cases} \tag{3.16}$$

If $n = 0$, it follows easily that $u_0^c = (f; \vartheta_0^c)$. For $n \in \mathbb{N}$, the solutions vanishing as $y \to \infty$ are:

$$u_n^c(y) = (f; \vartheta_n^c)\, e^{-ny} + \frac{1}{2} y (f; \vartheta_n^s)\, e^{-ny}.$$

Thus, the series development of $u(x,y)$ becomes:

$$\begin{aligned} u(x,y) = (f; \vartheta_0^c) + \sum_{n=1}^{\infty} \big((f; \vartheta_n^c) + \frac{1}{2} y (f; \vartheta_n^s) \big) \cos(nx)\, e^{-ny} \\ + \sum_{n=1}^{\infty} (f; \vartheta_n^s) x \sin(nx)\, e^{-ny}. \end{aligned} \tag{3.17}$$

Calculating the derivative with respect to x, we obtain

$$\begin{aligned} u_x(x,y) &= - \sum_{n=1}^{\infty} \big((f; \vartheta_n^c) + \frac{1}{2} y (f; \vartheta_n^s) \big) n \sin(nx)\, e^{-ny} \\ &\quad + \sum_{n=1}^{\infty} (f; \vartheta_n^s) \big(\sin(nx) + xn \cos(nx) \big)\, e^{-ny} \\ &= \sum_{n=1}^{\infty} \big((f; \vartheta_n^s) - (f; \vartheta_n^c) n - \frac{1}{2} y (f; \vartheta_n^s) n \big) \sin(nx)\, e^{-ny} \\ &\quad + \sum_{n=1}^{\infty} (f; \vartheta_n^s) xn \cos(nx)\, e^{-ny} =: v(x,y) + w(x,y). \end{aligned} \tag{3.18}$$

We need to show the convergence of this series in $L^p_\nu(\Pi)$.

Let us start with $w(x,y)$; the series $v(x,y)$ can be treated in a similar manner. By (2.3) and (3.6), we have

$$w(x,y)=\sum_{n=1}^{\infty}(f;\vartheta_n^s)xn\cos(nx)\,e^{-ny}=\frac{1}{\pi^2}\sum_{n=1}^{\infty}f_n'^{c}\,x\cos(nx)\,e^{-ny}.$$

Let δ be as in Lemma 2.1, and take $\alpha=1+\delta$, so that $\alpha'=\frac{1+\delta}{\delta}$ is the conjugate of α. Applying Hölder's inequality for $f\in L^p_\nu(0,2\pi)$ and Lemma 2.1, we obtain

$$\int_0^{2\pi}|f(x)|^p\nu(x)\,dx\le C\Big(\int_0^{2\pi}|f(x)|^{p\alpha'}\,dx\Big)^{\frac{1}{\alpha'}} \tag{3.19}$$

where $C=C(p,[\nu]_p,\|\nu\|_{L^1(0,2\pi)})$. In order to estimate the norm of w, we consider the following two cases:

- $\boxed{p\ge 2:}$ Here, $p_1=p\alpha'>2$, $p_1'\in(1,2)$, and $p/p_1'>1$. Then, by (3.19), for any fixed $y\in\mathbb{R}_+$, we get

  $$\begin{aligned}\|w(\cdot,y)\|^p_{L^p_\nu(0,2\pi)}&\le C\Big(\int_0^{2\pi}|w(x,y)|^{p_1}\,dx\Big)^{\frac{p}{p_1}}\\&\le C\Big(\sum_{n=1}^{\infty}|f_n'^{c}|^{p_1'}e^{-np_1'y}\Big)^{\frac{p}{p_1'}}\le C\sum_{n=1}^{\infty}|f_n'^{c}|^{p}e^{-npy}\end{aligned}$$

  where the last inequality holds since $p/p_1'>1$. Integrating with respect to y and using Hölder's inequality for sequence, we obtain:

  $$\begin{aligned}\iint_\Pi|w(x,y)|^p\nu(x)\,dxdy&\le C\sum_{n=1}^{\infty}|f_n'^{c}|^p\int_0^{\infty}e^{-npy}\,dy\\&=C\sum_{n=1}^{\infty}\frac{|f_n'^{c}|^p}{n}\le C\Big(\sum_{n=1}^{\infty}\frac{1}{n^{\beta'}}\Big)^{\frac{1}{\beta'}}\Big(\sum_{n=1}^{\infty}|f_n'^{c}|^{p\beta}\Big)^{\frac{1}{\beta}}\\&\le C\|f'\|^p_{L^{(p\beta)'}(0,2\pi)}\end{aligned} \tag{3.20}$$

  for any $\beta>1$, where the constant C depends on ν and p.

  Let $q\in(1,p)$ be as in Lemma 2.1, and take $r=p/q$. Then $1<r<p$ and by Hölder's inequality, we have:

  $$\begin{aligned}\int_0^{2\pi}|f'|^r\,dx&=\int_0^{2\pi}|f'|^{\frac{p}{q}}\nu^{\frac{1}{q}}\nu^{-\frac{1}{q}}\,dx\\&\le\Big(\int_0^{2\pi}\nu^{-\frac{1}{q-1}}\,dx\Big)^{\frac{q-1}{q}}\Big(\int_0^{2\pi}|f'|^p\nu\,dx\Big)^{\frac{1}{q}}\\&\le C\|f'\|^r_{L^p_\nu(0,2\pi)}\end{aligned} \tag{3.21}$$

where C depends on $[\nu]_q$ and p.

Chosen $\beta > 1$ such that $\beta_1 = p\beta > p \geq 2$ and $1 < \beta_1' < r$. By the boundedness of $(0,2\pi)$, we have

$$\|f'\|_{L^{\beta_1'}(0,2\pi)} \leq C\|f'\|_{L^r(0,2\pi)}. \tag{3.22}$$

Using (3.20), (3.21), and (3.22), we obtain

$$\begin{aligned}\|w\|_{L^p_\nu(\Pi)} &\leq C\|f'\|_{L^{\beta_1'}(0,2\pi)} \leq C\|f'\|_{L^r(0,2\pi)} \\ &\leq C\|f'\|_{L^p_\nu(0,2\pi)} \leq C\|f\|_{W^{1,p}_\nu(0,2\pi)}.\end{aligned}$$

- $\boxed{1 < p < 2:}$ Taking $\alpha = 1+\delta$ with $0 < \delta < p/(2-p)$, it follows that $p_1 = p\alpha' > 2$. The arguments proceed similarly, obtaining the boundedness of the L^p_ν norms via the norm of f.

Considering all the series in the expressions for u, u_x, and u_y, we can estimate their norms, similarly. Unifying these estimates, we obtain:

$$\|u\|_{W^{1,p}_\nu(\Pi)} \leq C\|f\|_{W^{1,p}_\nu(0,2\pi)},$$

where C is independent of f. Moreover, direct calculations verify that u satisfies the differential equation in (3.13) in the weak sense.

Finally, to verify that u satisfies the boundary conditions in (3.13), we examine the trace operators θ_0, $\theta_{2\pi}$, and θ_J that are the trace operators on $J_0, J_{2\pi}$ and J respectively.

Let us show that $\theta_J u = f$. Since $u \in W^{1,1}(\Pi)$, then $\theta_J u \in L^1(0,2\pi)$, hence we need to prove only that $\theta_J u = f$ almost everywhere on J. Consider, for all $m \in \mathbb{N}$, the partial sums

$$u^m(x,y) = u_0^c(y) + \sum_{n=1}^{m} \Big(u_n^c(y)\cos(nx) + u_n^s(y)\,x\sin(nx)\Big) \quad (x,y) \in \Pi$$

and taking the trace on J, we obtain:

$$\begin{aligned}\theta_J u^m(x) &= u^m(x,0) \\ &= (f;\vartheta_0^c) + \sum_{n=1}^{m} \Big((f;\vartheta_n^c)\cos(nx) + (f;\vartheta_n^s)x\sin(nx)\Big) = S_m(f)(x)\end{aligned}$$

where the last expression is the projector of f with respect to system (2.7). Hence, by Theorem 2.5, we have:

$$\lim_{m\to\infty} \|S_m(f) - f\|_{L^p_\nu(0,2\pi)} = \lim_{m\to+\infty} \|\theta_J u^m - f\|_{L^p_\nu(0,2\pi)} = 0.$$

The convergence also holds with respect to the norm in $L^1(0,2\pi)$.

On the other hand, by the classical theory,

$$\lim_{m\to\infty} \|\theta_J u^m - \theta_J u\|_{L^1(0,2\pi)} = 0$$

and hence $\theta_J u = f$ almost everywhere on J.

It is easy to check that $u^m(0,y) = u^m(2\pi,y)$ for all $y > 0$ and $m \in \mathbb{N}$, and arguing as above, we can obtain that $\theta_{J_0} u = \theta_{J_{2\pi}} u$. □

The following result gives a necessary condition under which the weak solution of (3.13) is also a strong one.

Theorem 3.4. *Let the conditions of Theorem 3.3 hold. Then any function that is a weak solution of* (3.13) *and belongs to* $W_\nu^{2,p}(\Pi)$ *is a strong solution of*

$$\begin{cases} \Delta u(x,y) = 0 & \text{for a.e. } (x,y) \in \Pi \\ u(y)|_{J_0} = u(y)|_{J_{2\pi}} & \text{for a.e. } y \in (0,\infty) \\ u(x,0) = f(x) & \text{for a.e. } x \in (0,2\pi) \\ u_x(0,y) = 0 & \text{for a.e. } y \in (0,\infty). \end{cases} \tag{3.23}$$

Proof. For any $\eta(x) \in C^\infty_{2\pi}(\bar{J})$ and $\psi(y) \in C^\infty_0(\mathbb{R}_+)$, we consider the test function

$$\varphi(x,y) = \psi(y)\eta(x) \in C^\infty_{J_0}(\Pi).$$

Integrating by parts, we obtain:

$$\begin{aligned} 0 &= \iint\limits_{\Pi} \nabla u \nabla \varphi \, dxdy = \int_0^\infty \int_0^{2\pi} (u_x\varphi_x + u_y\varphi_y)\, dxdy \\ &= \int_0^\infty \psi(y) \left(\int_0^{2\pi} u_x \eta'(x)\, dx \right) dy + \int_0^{2\pi} \eta(x) \left(\int_0^\infty u_y \psi'(y)\, dy \right) dx \\ &= -\eta(0) \int_0^\infty \psi(y) u_x(0,y)\, dy - \int_\Pi \varphi \Delta u \, dxdy. \end{aligned} \tag{3.24}$$

The function ψ is a test function, so $\operatorname{supp}\psi(y) \subset [0,\xi]$ for some $\xi > 0$. Then we can write (3.24) in the form

$$\iint\limits_{\Pi_\xi} \varphi(x,y) \Delta u \, dxdy = -\eta(0) \int_0^\xi u_x(0,y)\psi(y)\, dy. \tag{3.25}$$

Consider the following systems of functions:

$$\{\eta_n(x)\}_{n\in\mathbb{N}_0} = \{\eta_0^c = 1,\ \eta_n^c = \cos(nx),\ \eta_n^s = \sin(nx)\}_{n\in\mathbb{N}}, \tag{3.26}$$

$$\{\psi_n(y)\}_{n\in\mathbb{N}_0} = \left\{ \psi_0^c = 1,\ \psi_n^c = \frac{\cos(2\pi n y)}{\xi},\ \psi_n^s = \frac{\sin(2\pi n y)}{\xi} \right\}_{n\in\mathbb{N}}, \tag{3.27}$$

and the corresponding modified ones:

$$\tilde{\eta}_n(x) = x(2\pi - x)\eta_n(x) \in C_0^\infty(\bar{J}),$$
$$\tilde{\psi}_n(y) = y(\xi - y)\psi_n(y) \in C_0^\infty([0,\xi]).$$

Taking the test function in (3.24) in the form

$$\tilde{\varphi}_{mn}(x,y) = \tilde{\eta}_m(x)\tilde{\psi}_n(y) \in C_0^\infty(\Pi_\xi)$$

for any $m,n \in \mathbb{N}_0$, we obtain

$$\int_0^{2\pi}\left(\int_0^{\xi}\tilde{\psi}_n(y)\Delta u\,dy\right)x(2\pi - x)\varphi(x)\,dx = 0. \tag{3.28}$$

Since $u \in W_\nu^{2,p}(\Pi)$, it follows that $\Delta u \in L^1(\Pi)$ and

$$F(x) := \int_0^{\xi}\tilde{\psi}_n(y)\Delta u\,dy \in L^1(0,2\pi).$$

Then, by the Lebsgue theorem, (3.28) implies $F(x) = 0$ for almost each $x \in J$, and hence $\Delta u = 0$ for almost each $(x,y) \in \Pi_\xi$. Due to the arbitrariness of ξ, it follows that

$$\Delta u = 0 \quad \text{for a.e. } (x,y) \in \Pi.$$

From (3.25), we obtain

$$\int_0^{\infty} u_x(0,y)\psi(y)\,dy = 0$$

for all $\psi \in C_0^\infty([0,\infty))$, and hence

$$u_x(0,y) = 0 \quad \text{for a.e. } y > 0.$$

□

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