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UNCERTAINTY INEQUALITY AND APPROXIMATE INVERSION FORMULAS FOR r-WEIGHTED FOCK SPACES

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We introduce r-weighted Fock space $\mathscr{F}_{r,\beta}$ which generalizes some previously known Hilbert spaces, and study the multiplication operator M_r and its adjoint. A general uncertainty inequality of Heisenberg type is obtained. We also consider the extremal functions for the r-difference operator D_r on the space and obtain approximate inversion formulas.

1. Introduction

Fock space $\mathscr{F}$ (see [1]) is the Hilbert space of analytic functions f on $\mathbb{C}$ such that

$$\|f\|_{\mathscr{F}}^2 := \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \mathrm{d}x\mathrm{d}y < \infty, \quad z = x+iy.$$

The space $\mathscr{F}$ is called also Segal-Bargmann space [2] and it was applied in many works [4, 20, 21, 30]. Precisely, Chen and Zhu [4] proved an uncertainty principle of Heisenberg type for the Fock space $\mathscr{F}$; and recently the author of the paper [20, 21] studied the extremal functions for the difference and primitive operators on the Fock space $\mathscr{F}$. In this paper we are going to prove a generalized uncertainty principle and to examine the theory of extremal functions in the context of r-weighted Fock spaces.

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We introduce the weighted Fock space $\mathscr{F}_{r,\beta}$, which is the set of all analytic functions f in $\mathbb{C}$, with $f(z)=\sum_{n=0}^{\infty}a_n z^{rn}$, such that

$$\|f\|_{\mathscr{F}_{r,\beta}}^2 := \sum_{n=0}^{\infty}\beta_{n,r}|a_n|^2 < \infty,$$

where $\beta=\{\beta_{n,r}\}$ is a positive sequence so that $\limsup_{n\to\infty}(\beta_{n,r})^{-1/n}=\infty$.

The space $\mathscr{F}_{r,\beta}$ is a reproducing kernel Hilbert space (RKHS) that gives a generalization of some Hilbert spaces of analytic functions in the complex plane $\mathbb{C}$ like, the Bessel type Fock space $\mathscr{F}_{2,\alpha}$ (see [5, 26]), the Airy type Fock space $\mathscr{F}_{3,\nu}$ (see [14, 23]), and the hyper-Bessel type Fock space $\mathscr{F}_{r,\alpha}$ (see [22, 24]).

For $f\in\mathscr{F}_{r,\beta}$ with $f(z)=\sum_{n=0}^{\infty}a_n z^{rn}$, we define the multiplication operator

$$M_r f(z) := z^r f(z) = \sum_{n=1}^{\infty}a_{n-1}z^{rn},$$

and its adjoint operator

$$L_{\mathscr{F}_{r,\beta}} f(z) := \sum_{n=0}^{\infty}\frac{\beta_{n+1,r}}{\beta_{n,r}}a_{n+1}z^{rn}.$$

These operators satisfy the commutation rule

$$[L_{\mathscr{F}_{r,\beta}}, M_r] = \frac{\beta_{1,r}}{\beta_{0,r}}I + E_{r,\beta},$$

where I is the identity operator and $E_{r,\beta}$ is the operator given by

$$E_{r,\beta}f(z) := \sum_{n=1}^{\infty}\left[\frac{\beta_{n+1,r}}{\beta_{n,r}} - \frac{\beta_{n,r}}{\beta_{n-1,r}} - \frac{\beta_{1,r}}{\beta_{0,r}}\right]a_n z^{rn}.$$

Thanks to this commutation identity, we deduce the following uncertainty inequality for the space $\mathscr{F}_{r,\beta}$, that is

$$\|(M_r+L_{\mathscr{F}_{r,\beta}}-a)f\|_{\mathscr{F}_{r,\beta}}\|(M_r-L_{\mathscr{F}_{r,\beta}}-b)f\|_{\mathscr{F}_{r,\beta}} \geq \frac{\beta_{1,r}}{\beta_{0,r}}\|f\|_{\mathscr{F}_{r,\beta}}^2, \quad a,b\in\mathbb{C}.$$

Let $D_r:\mathscr{F}_{r,\beta}\to\mathscr{F}_{r,\beta}$ be the r-difference operator given by

$$D_r f(z) := \frac{1}{z^r}(f(z)-f(0)), \quad f\in\mathscr{F}_{r,\beta}.$$

Building on the ideas of Saitoh et al. [16–18], we find the minimizer (denoted by $F^*_{\lambda,D_r}(h)$) for the extremal problem:

$$\inf_{f\in\mathscr{F}_{r,\beta}}\left\{\lambda\|f\|_{\mathscr{F}_{r,\beta}}^2 + \|D_r f - h\|_{\mathscr{F}_{r,\beta}}^2\right\},$$

where $h \in \mathscr{F}_{r,\beta}$ and $\lambda > 0$. We prove that the extremal function $F^*_{\lambda,D_r}(h)$ is given by

$$F^*_{\lambda,D_r}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{F}_{r,\beta}},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\overline{z})^{r(n+1)} w^{rn}}{\lambda \beta_{n+1,r} + \beta_{n,r}}, \quad w \in \mathbb{C}.$$

Moreover, we establish approximate inversion formulas for the r-difference operator D_r on the r-weighted Fock space $\mathscr{F}_{r,\beta}$. A pointwise approximate inversion formulas for the operator D_r are also discussed.

The paper is organized as follows. In Section 2 we introduce the r-weighted Fock space $\mathscr{F}_{r,\beta}$. In Section 3 we establish a generalized uncertainty inequality of Heisenberg type for the space $\mathscr{F}_{r,\beta}$. In Section 4 we examine the extremal functions for the r-difference operator D_r. Finally, in Section 5, we establish approximate inversion formulas for the operator D_r on the r-weighted Fock space $\mathscr{F}_{r,\beta}$.

2. The r-Weighted Fock space

In this work r is a positive integer ($r \geq 2$)and $\alpha = (\alpha_1, \dots, \alpha_{r-1})$ a vector having $(r-1)$ real components with $|\alpha| = \alpha_1 + \dots + \alpha_{r-1}$.

We begin by recalling some results about trigonometric functions of r-order and Bessel functions of vector index [13].

Let ω_k, $k = 1, \dots, r$, the r-th roots of unity

$$\omega_k = e^{2i\pi(k-1)/r}.$$

Let $z \in \mathbb{C}$. A function $f(z)$ is called r-even if

$$f(\omega_k z) = f(z), \quad k = 1, \dots, r.$$

For example, the r- hyperbolic cosine [13] given by

$$\cosh_r(z) = \sum_{n=0}^{\infty} \frac{z^{rn}}{(rn)!},$$

is r-even and satisfies $|\cosh_r(z)| \leq e^{|z|}$.

We suppose now that the components of the vector α satisfy

$$\alpha_k \geq -1 + \frac{k}{r}, \quad k = 1, \dots, r-1.$$

The r-modified Bessel function given by

$$I_{r,\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{rn}}{c_n(r,\alpha)}, \tag{1}$$

where

$$c_n(r,\alpha)=r^{rn}n!\prod_{i=1}^{r-1}\frac{\Gamma(\alpha_i+n+1)}{\Gamma(\alpha_i+1)}, \tag{2}$$

is r-even and satisfies

$$|I_{r,\alpha}(z)|\le e^{|z|}\prod_{i=1}^{r-1}\frac{\Gamma(\alpha_i-i/r+1)}{\Gamma(\alpha_i+i/r)}.$$

We consider a sequence $\beta=\{\beta_{n,r}\}$, with $\beta_{n,r}>0$, such that

$$\limsup_{n\to\infty}(\beta_{n,r})^{-1/n}=\infty.$$

The r-weighted Fock space $\mathscr{F}_{r,\beta}$ is the set of all analytic functions f in $\mathbb{C}$, with $f(z)=\sum_{n=0}^{\infty}a_nz^{rn}$, such that

$$\|f\|^2_{\mathscr{F}_{r,\beta}}:=\sum_{n=0}^{\infty}\beta_{n,r}|a_n|^2<\infty.$$

It is a Hilbert space when equipped with the inner product

$$\langle f,g\rangle_{\mathscr{F}_{r,\beta}}=\sum_{n=0}^{\infty}\beta_{n,r}a_n\overline{b_n},$$

where $f,g\in\mathscr{F}_{r,\beta}$ with $f(z)=\sum_{n=0}^{\infty}a_nz^{rn}$ and $g(z)=\sum_{n=0}^{\infty}b_nz^{rn}$.
The set $\left\{\frac{z^{rn}}{\sqrt{\beta_{n,r}}}\right\}_{n=0}^{\infty}$ forms a Hilbert's basis for the space $\mathscr{F}_{r,\beta}$. The function $K_{\mathscr{F}_{r,\beta},z}$, $z\in\mathbb{C}$, given by

$$K_{\mathscr{F}_{r,\beta},z}(w):=\sum_{n=0}^{\infty}\frac{(w\bar{z})^{rn}}{\beta_{n,r}},\quad w\in\mathbb{C},$$

is a reproducing kernel for the r-weighted Fock space $\mathscr{F}_{r,\beta}$.

If $\beta_{n,r}=(rn)!$, the r-weighted Fock space denoted by $\mathscr{F}_r$ is the set of all analytic functions f in $\mathbb{C}$, with $f(z)=\sum_{n=0}^{\infty}a_nz^{rn}$, such that

$$\|f\|^2_{\mathscr{F}_r}:=\sum_{n=0}^{\infty}(rn)!|a_n|^2<\infty.$$

This Hilbert space has the reproducing kernel

$$K_{\mathscr{F}_{r},z}(w) = \cosh_r(w\bar{z}), \quad w,z \in \mathbb{C}.$$

If $\beta_{n,r} = c_n(r,\alpha) = r^{rn} n! \prod_{i=1}^{r-1} \frac{\Gamma(\alpha_i+n+1)}{\Gamma(\alpha_i+1)}$, the corresponding r-weighted Fock space is the hyper-Bessel type Fock space $\mathscr{F}_{r,\alpha}$ introduced in [22, 24]. This Hilbert space has the reproducing kernel

$$K_{\mathscr{F}_{r,\alpha},z}(w) = I_{r,\alpha}(w\bar{z}), \quad w,z \in \mathbb{C},$$

where $I_{r,\alpha}$ is the r-modified Bessel function given by (1).

We note that, the space $\mathscr{F}_{2,\alpha}$ is introduced by Cholewinski in [5], and the space $\mathscr{F}_{3,\nu}$ is intoduced by Nemri et al. in [14], and by Soltani in [23].

For $f \in \mathscr{F}_{r,\beta}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$ we define the operators M_r and $L_{\mathscr{F}_{r,\beta}}$ on $\mathscr{F}_{r,\beta}$ by

$$M_r f(z) := z^r f(z) = \sum_{n=1}^{\infty} a_{n-1} z^{rn}, \tag{3}$$

and

$$L_{\mathscr{F}_{r,\beta}} f(z) := \sum_{n=0}^{\infty} \frac{\beta_{n+1,r}}{\beta_{n,r}} a_{n+1} z^{rn}. \tag{4}$$

The operators $L_{\mathscr{F}_{r,\beta}}$ and M_r satisfy the commutation rule

$$[L_{\mathscr{F}_{r,\beta}}, M_r] = \frac{\beta_{1,r}}{\beta_{0,r}} I + E_{r,\beta}, \tag{5}$$

where I is the identity operator and $E_{r,\beta}$ is the operator given by

$$E_{r,\beta} f(z) := \sum_{n=1}^{\infty} \left[\frac{\beta_{n+1,r}}{\beta_{n,r}} - \frac{\beta_{n,r}}{\beta_{n-1,r}} - \frac{\beta_{1,r}}{\beta_{0,r}} \right] a_n z^{rn}.$$

If $\beta_{n,r} = (rn)!$, then $L_{\mathscr{F}_r} f(z) = \Delta_r = \frac{\mathrm{d}^r}{\mathrm{d}z^r}$ and $[\Delta_r, M_r] = r! I + E_{r,\beta}$.

If $\beta_{n,r} = c_n(r,\alpha) = r^{rn} n! \prod_{i=1}^{r-1} \frac{\Gamma(\alpha_i+n+1)}{\Gamma(\alpha_i+1)}$, then $L_{\mathscr{F}_{r,\alpha}}$ is the hyper-Bessel operator [7, 12, 13] given by

$$B_{r,\alpha} = \frac{\mathrm{d}^r}{\mathrm{d}z^r} + \frac{a_1}{z} \frac{\mathrm{d}^{r-1}}{\mathrm{d}z^{r-1}} + \ldots + \frac{a_{r-1}}{z^{r-1}} \frac{\mathrm{d}}{\mathrm{d}z}, \quad \alpha = (\alpha_1, \ldots, \alpha_{r-1}),$$

where

$$a_{r-k} = \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(j-1)!(k-j)!} \prod_{i=1}^{r-1} (r\alpha_i + j), \quad k = 1, \ldots, r-1,$$

and

$$[B_{r,\alpha}, M_r] = r^r \prod_{i=1}^{r-1} (\alpha_i + 1) I + E_{r,\beta}.$$

We note that, when $r = 2$, we obtain the classical Bessel operator [5, 26]

$$B_{2,\alpha} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2\alpha+1}{z}\frac{\mathrm{d}}{\mathrm{d}z}, \quad \alpha > -1/2,$$

and we have

$$[B_{2,\alpha}, M_2] = 4(\alpha+1)I + 4z\frac{\mathrm{d}}{\mathrm{d}z}.$$

When $r = 3$, $\alpha_1 = -2/3$ and $\alpha_2 = \nu - 1/3$, we obtain the generalized Airy operator [8, 14]

$$B_{3,\nu} = \frac{\mathrm{d}^3}{\mathrm{d}z^3} + \frac{3\nu}{z}\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{3\nu}{z^2}\frac{\mathrm{d}}{\mathrm{d}z}, \quad \nu > 0,$$

and we have

$$[B_{3,\nu}, M_3] = 3(3\nu+2)I + 18(\nu+1)z\frac{\mathrm{d}}{\mathrm{d}z} + 9z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}.$$

In the next of this paper, we suppose that the sequence $\{\beta_{n,r}\}$ satisfies the condition

$$\frac{\beta_{n+1,r}}{\beta_{n,r}} - \frac{\beta_{n,r}}{\beta_{n-1,r}} \geq \frac{\beta_{1,r}}{\beta_{0,r}}, \quad n \geq 1. \tag{6}$$

The condition (6) is verified in the precedent three cases: in the Bessel type Fock space $\mathscr{F}_{2,\alpha}$, in the Airy type Fock space $\mathscr{F}_{3,\nu}$ and in the hyper-Bessel type Fock space $\mathscr{F}_{r,\alpha}$.

3. The generalized uncertainty principle

Heisenberg [10] demonstrated that the position and momentum of a particle can not be determined simultaneously with arbitrary precision.This principle has been formulated by the following inequality

$$\sigma_x \sigma_p \geq \frac{h}{4\pi},$$

where h represents Planck's constant and σ_x, σ_p signify the errors of the position and the momentum of the particle respectively. There exist many similar uncertainty principles, in quantum physics and in mathematics [3, 4, 6, 11, 25–27]. In this section we are going to prove a generalized uncertainty principle for the r-weighted Fock space $\mathscr{F}_{r,\beta}$.

We define the Hilbert space $\mathscr{U}_{r,\beta}^{(1)}$ as the space of all $f \in \mathscr{F}_{r,\beta}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$ such that

$$\|f\|^2_{\mathscr{U}_{r,\beta}^{(1)}} := \sum_{n=0}^{\infty} \beta_{n+1,r} |a_n|^2 < \infty.$$

We define the Hilbert space $\mathscr{U}_{r,\beta}^{(2)}$ as the space of all $f \in \mathscr{F}_{r,\beta}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$ such that

$$\|f\|^2_{\mathscr{U}_{r,\beta}^{(2)}} := \beta_{1,r}|a_0|^2 + \sum_{n=1}^{\infty} \frac{(\beta_{n,r})^2}{\beta_{n-1,r}} |a_n|^2 < \infty.$$

By condition (6) we obtain the inequality

$$\|f\|_{\mathscr{U}_{r,\beta}^{(2)}} \leq \|f\|_{\mathscr{U}_{r,\beta}^{(1)}}.$$

Therefore, we have the continuous inclusion $\mathscr{U}_{r,\beta}^{(1)} \subseteq \mathscr{U}_{r,\beta}^{(2)}$.

In this section we establish an uncertainty inequality of Heisenberg type for the space $\mathscr{F}_{r,\beta}$. We will use the following three lemmas.

Lemma 3.1. *The operators M_r and $L_{\mathscr{F}_{r,\beta}}$ satisfy the following properties.*

(i) $Dom(M_r) = \mathscr{U}_{r,\beta}^{(1)}$ and $Dom(L_{\mathscr{F}_{r,\beta}}) = \mathscr{U}_{r,\beta}^{(2)}$.

(ii) For $f \in \mathscr{U}_{r,\beta}^{(1)}$ and $g \in \mathscr{U}_{r,\beta}^{(2)}$, we have $\langle M_r f, g\rangle_{\mathscr{F}_{r,\beta}} = \langle f, L_{\mathscr{F}_{r,\beta}} g\rangle_{\mathscr{F}_{r,\beta}}$.

Proof. Let $f \in \mathscr{F}_{r,\beta}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$. From (3) and (4) we have

$$\|M_r f\|^2_{\mathscr{F}_{r,\beta}} = \sum_{n=0}^{\infty} \beta_{n+1,r}|a_n|^2 = \|f\|^2_{\mathscr{U}_{r,\beta}^{(1)}},$$

and

$$\|L_{\mathscr{F}_{r,\beta}} f\|^2_{\mathscr{F}_{r,\beta}} = \sum_{n=1}^{\infty} \frac{(\beta_{n,r})^2}{\beta_{n-1,r}} |a_n|^2 = \|f\|^2_{\mathscr{U}_{r,\beta}^{(2)}} - \beta_{1,r}|f(0)|^2.$$

Consequently $\mathrm{Dom}(M_r) = \mathscr{U}_{r,\beta}^{(1)}$ and $\mathrm{Dom}(L_{\mathscr{F}_{r,\beta}}) = \mathscr{U}_{r,\beta}^{(2)}$.

On the other hand for $f \in \mathscr{U}_{r,\beta}^{(1)}$ and $g \in \mathscr{U}_{r,\beta}^{(2)}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$ and $g(z) = \sum_{n=0}^{\infty} b_n z^{rn}$, we have

$$\langle M_r f, g\rangle_{\mathscr{F}_{r,\beta}} = \sum_{n=1}^{\infty} \beta_{n,r} a_{n-1} \overline{b_n} = \sum_{n=0}^{\infty} \beta_{n+1,r} a_n \overline{b_{n+1}} = \langle f, L_{\mathscr{F}_{r,\beta}} g\rangle_{\mathscr{F}_{r,\beta}}.$$

The lemma is proved. □

We define the Hilbert space $\mathscr{S}^{(1)}_{r,\beta}$ as the space of all $f \in \mathscr{F}_{r,\beta}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$ such that

$$\|f\|^2_{\mathscr{S}^{(1)}_{r,\beta}} := \sum_{n=0}^{\infty} \frac{(\beta_{n+1,r})^2}{\beta_{n,r}} |a_n|^2 < \infty.$$

We define the Hilbert space $\mathscr{S}^{(2)}_{r,\beta}$ as the space of all $f \in \mathscr{F}_{r,\beta}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$ such that

$$\|f\|^2_{\mathscr{S}^{(2)}_{r,\beta}} := \frac{(\beta_{1,r})^2}{\beta_{0,r}} |a_0|^2 + \sum_{n=1}^{\infty} \frac{(\beta_{n,r})^3}{(\beta_{n-1,r})^2} |a_n|^2 < \infty.$$

By condition (6) we obtain the inequalities

$$\|f\|_{\mathscr{S}^{(2)}_{r,\beta}} \leq \|f\|_{\mathscr{S}^{(1)}_{r,\beta}}, \quad \|f\|_{\mathscr{U}^{(1)}_{r,\beta}} \leq \sqrt{\frac{\beta_{0,r}}{\beta_{1,r}}} \|f\|_{\mathscr{S}^{(1)}_{r,\beta}}.$$

Therefore, we have the continuous inclusions $\mathscr{S}^{(1)}_{r,\beta} \subseteq \mathscr{S}^{(2)}_{r,\beta}$ and $\mathscr{S}^{(1)}_{r,\beta} \subseteq \mathscr{U}^{(1)}_{r,\beta}$.

Lemma 3.2. *We have* $Dom(L_{\mathscr{F}_{r,\beta}} M_r) = \mathscr{S}^{(1)}_{r,\beta}$ *and* $Dom(M_r L_{\mathscr{F}_{r,\beta}}) = \mathscr{S}^{(2)}_{r,\beta}$.

Proof. From (3) and (4) we have

$$L_{\mathscr{F}_{r,\beta}} M_r f(z) = \sum_{n=0}^{\infty} \frac{\beta_{n+1,r}}{\beta_{n,r}} a_n z^{rn}, \quad M_r L_{\mathscr{F}_{r,\beta}} f(z) = \sum_{n=1}^{\infty} \frac{\beta_{n,r}}{\beta_{n-1,r}} a_n z^{rn}.$$

Therefore

$$\|L_{\mathscr{F}_{r,\beta}} M_r f\|^2_{\mathscr{F}_{r,\beta}} = \sum_{n=0}^{\infty} \frac{(\beta_{n+1,r})^2}{\beta_{n,r}} |a_n|^2 = \|f\|^2_{\mathscr{S}^{(1)}_{r,\beta}},$$

and

$$\|M_r L_{\mathscr{F}_{r,\beta}} f\|^2_{\mathscr{F}_{r,\beta}} = \sum_{n=1}^{\infty} \frac{(\beta_{n,r})^3}{(\beta_{n-1,r})^2} |a_n|^2 = \|f\|^2_{\mathscr{S}^{(2)}_{r,\beta}} - \frac{(\beta_{1,r})^2}{\beta_{0,r}} |f(0)|^2.$$

Consequently $\mathrm{Dom}(L_{\mathscr{F}_{r,\beta}} M_r) = \mathscr{S}^{(1)}_{r,\beta}$ and $\mathrm{Dom}(M_r L_{\mathscr{F}_{r,\beta}}) = \mathscr{S}^{(2)}_{r,\beta}$. □

Lemma 3.3. *[See [9], Proposition 2.1]. Let A and B be self-adjoint operators on a Hilbert space* $\mathscr{H}$*, then*

$$\|(A-a)f\|_{\mathscr{H}} \|(B-b)f\|_{\mathscr{H}} \geq \frac{1}{2} |\langle [A,B]f, f\rangle_{\mathscr{H}}|,$$

for all $f \in Dom([A,B])$ *and all* $a, b \in \mathbb{C}$.

Theorem 3.4. *Let $f \in \mathscr{F}_{r,\beta}$. For all $a, b \in \mathbb{C}$, we have*

$$\|(M_r + L_{\mathscr{F}_{r,\beta}} - a)f\|_{\mathscr{F}_{r,\beta}} \|(M_r - L_{\mathscr{F}_{r,\beta}} - b)f\|_{\mathscr{F}_{r,\beta}} \geq \frac{\beta_{1,r}}{\beta_{0,r}} \|f\|^2_{\mathscr{F}_{r,\beta}}. \tag{7}$$

Proof. Let $f \in \mathscr{F}_{r,\beta}$. First the inequality (7) is true for $f \notin \mathscr{S}^{(1)}_{r,\beta}$. Now, let A and B be the operators defined for $f \in \mathscr{S}^{(1)}_{r,\beta}$ by

$$A := (M_r + L_{\mathscr{F}_{r,\beta}})f, \quad B := i(M_r - L_{\mathscr{F}_{r,\beta}})f.$$

By (5), Lemma 3.1 and Lemma 3.2, the operators A and B possess the following properties.

(i) $A^* = A$ and $B^* = B$,

(ii) $[A, B] = -2i[M_r, L_{\mathscr{F}_{r,\beta}}] = 2i(\frac{\beta_{1,r}}{\beta_{0,r}} I + E_{r,\beta})$,

(iii) $\mathrm{Dom}([A, B]) = \mathscr{S}^{(1)}_{r,\beta}$.

Thus, the inequality (7) follows from Lemma 3.3 and the fact that

$$\langle E_{r,\beta} f, f\rangle_{\mathscr{F}_{r,\beta}} \geq 0.$$

This completes the proof of the theorem. □

In particular cases we obtain.

Remark 3.5. If $f \in \mathscr{F}_r$ and $a, b \in \mathbb{C}$, we have

$$\|(M_r + \Delta_r - a)f\|_{\mathscr{F}_r} \|(M_r - \Delta_r - b)f\|_{\mathscr{F}_r} \geq r! \|f\|^2_{\mathscr{F}_r}.$$

And, if $f \in \mathscr{F}_{r,\alpha}$ and $a, b \in \mathbb{C}$, we have

$$\|(M_r + B_{r,\alpha} - a)f\|_{\mathscr{F}_{r,\alpha}} \|(M_r - B_{r,\alpha} - b)f\|_{\mathscr{F}_{r,\alpha}} \geq r^r \prod_{i=1}^{r-1} (\alpha_i + 1) \|f\|^2_{\mathscr{F}_{r,\alpha}}.$$

4. The r-difference operator

Tikhonov regularization in statistics is the method of ridge regression. In general, this method related to the Levenberg-Marquardt algorithm for solving nonlinear least squares problems. Tikhonov regularization has been invented independently in many different contexts. It became widely known from its application to integral equations [28, 29].

Let $\mathscr{H}$ be a Hilbert space, and let $T:\mathscr{F}_{r,\beta}\to\mathscr{H}$ be a bounded linear operator from $\mathscr{F}_{r,\beta}$ into $\mathscr{H}$. Let $\lambda>0$. We denote by $\langle .,.\rangle_{\lambda,\mathscr{F}_{r,\beta}}$ the inner product defined on the space $\mathscr{F}_{r,\beta}$ by

$$\langle f,g\rangle_{\lambda,\mathscr{F}_{r,\beta}}:=\lambda\langle f,g\rangle_{\mathscr{F}_{r,\beta}}+\langle Tf,Tg\rangle_{\mathscr{H}}.$$

The two norms $\|.\|_{\mathscr{F}_{r,\beta}}$ and $\|.\|_{\lambda,\mathscr{F}_{r,\beta}}$ are equivalent. In particular, we have

$$|f(z)|\leq\|f\|_{\lambda,\mathscr{F}_{r,\beta}}\left[\frac{K_{\mathscr{F}_{r,\beta},z}(z)}{\lambda}\right]^{1/2},\quad f\in\mathscr{F}_{r,\beta},\ z\in\mathbb{C}.$$

Then the space $\mathscr{F}_{r,\beta}$, equipped with the norm $\|.\|_{\lambda,\mathscr{F}_{r,\beta}}$ has a reproducing kernel $K_{\lambda,\mathscr{F}_{r,\beta},z}$. Therefore, we have the functional equation

$$(\lambda I+T^*T)K_{\lambda,\mathscr{F}_{r,\beta},z}=K_{\mathscr{F}_{r,\beta},z},\quad z\in\mathbb{C},\tag{8}$$

where I is the unit operator and $T^*:\mathscr{H}\longrightarrow\mathscr{F}_{r,\beta}$ is the adjoint of T.

For any $h\in\mathscr{H}$ and for any $\lambda>0$, we define the extremal function $F^*_{\lambda,T}(h)$ by

$$F^*_{\lambda,T}(h)(z)=\langle h,TK_{\lambda,\mathscr{F}_{r,\beta},z}\rangle_{\mathscr{H}},\quad z\in\mathbb{C}.$$

Then by (8) we deduce that

$$\begin{aligned}F^*_{\lambda,T}(h)(z)&=\langle T^*h,K_{\lambda,\mathscr{F}_{r,\beta},z}\rangle_{\mathscr{F}_{r,\beta}}\\&=\langle T^*h,(\lambda I+T^*T)^{-1}K_{\mathscr{F}_{r,\beta},z}\rangle_{\mathscr{F}_{r,\beta}}\\&=\langle(\lambda I+T^*T)^{-1}T^*h,K_{\mathscr{F}_{r,\beta},z}\rangle_{\mathscr{F}_{r,\beta}}.\end{aligned}$$

Hence

$$F^*_{\lambda,T}(h)(z)=(\lambda I+T^*T)^{-1}T^*h(z),\quad z\in\mathbb{C}.\tag{9}$$

The extremal function $F^*_{\lambda,T}(h)$ is the unique solution (see [16], Theorem 2.5, Section 2) of the Tikhonov regularization problem

$$\inf_{f\in\mathscr{F}_{r,\beta}}\left\{\lambda\|f\|^2_{\mathscr{F}_{r,\beta}}+\|Tf-h\|^2_{\mathscr{H}}\right\}.$$

Let D_r be the r-difference operator defined for $f\in\mathscr{F}_{r,\beta}$ by

$$D_rf(z):=\frac{1}{z^r}(f(z)-f(0)).$$

For $f\in\mathscr{F}_{r,\beta}$ with $f(z)=\sum_{n=0}^{\infty}a_nz^{rn}$ we have

$$D_rf(z):=\sum_{n=0}^{\infty}a_{n+1}z^{rn}.\tag{10}$$

From (6), the operator D_r maps continuously from $\mathscr{F}_{r,\beta}$ into $\mathscr{F}_{r,\beta}$, and

$$\|D_r f\|_{\mathscr{F}_{r,\beta}} \leq \sqrt{\frac{\beta_{0,r}}{\beta_{1,r}}}\|f\|_{\mathscr{F}_{r,\beta}}.$$

Building on the ideas of Saitoh [16–18] we examine the extremal function associated with the r-difference operator D_r.

Theorem 4.1. *(i) For $f \in \mathscr{F}_{r,\beta}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$, we have*

$$D_r^* f(z) = \sum_{n=1}^{\infty} \frac{\beta_{n-1,r}}{\beta_{n,r}} a_{n-1} z^{rn}, \quad D_r^* D_r f(z) = \sum_{n=1}^{\infty} \frac{\beta_{n-1,r}}{\beta_{n,r}} a_n z^{rn}.$$

(ii) For any $h \in \mathscr{F}_{r,\beta}$ and for any $\lambda > 0$, the problem

$$\inf_{f \in \mathscr{F}_{r,\beta}} \left\{ \lambda \|f\|^2_{\mathscr{F}_{r,\beta}} + \|D_r f - h\|^2_{\mathscr{F}_{r,\beta}} \right\}$$

has a unique extremal function given by

$$F^*_{\lambda,D_r}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{F}_{r,\beta}},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\overline{z})^{r(n+1)} w^{rn}}{\lambda \beta_{n+1,r} + \beta_{n,r}}, \quad w \in \mathbb{C}.$$

Proof. (i) If $f, g \in \mathscr{F}_{r,\beta}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$ and $g(z) = \sum_{n=0}^{\infty} b_n z^{rn}$, then

$$\langle D_r f, g \rangle_{\mathscr{F}_{r,\beta}} = \sum_{n=0}^{\infty} \beta_{n,r} a_{n+1} \overline{b_n} = \sum_{n=1}^{\infty} \beta_{n-1,r} a_n \overline{b_{n-1}} = \langle f, D_r^* g \rangle_{\mathscr{F}_{r,\beta}},$$

where

$$D_r^* g(z) = \sum_{n=1}^{\infty} \frac{\beta_{n-1,r}}{\beta_{n,r}} b_{n-1} z^{rn}.$$

And therefore

$$D_r^* D_r f(z) = \sum_{n=1}^{\infty} \frac{\beta_{n-1,r}}{\beta_{n,r}} a_n z^{rn}.$$

(ii) We put $h(z) = \sum_{n=0}^{\infty} h_n z^{rn}$ and $F^*_{\lambda,D_r}(h)(z) = \sum_{n=0}^{\infty} f_n z^{rn}$. From (9) we have $(\lambda I + D_r^* D_r) F^*_{\lambda,D_r}(h)(z) = D_r^* h(z)$. By (i) we deduce that

$$f_0 = 0, \quad f_n = \frac{\beta_{n-1,r} h_{n-1}}{\lambda \beta_{n,r} + \beta_{n-1,r}}, \quad n \geq 1.$$

Thus

$$F^*_{\lambda,D_r}(h)(z) = \sum_{n=0}^{\infty} \frac{\beta_{n,r} h_n}{\lambda \beta_{n+1,r} + \beta_{n,r}} z^{r(n+1)} = \langle h, \Psi_z \rangle_{\mathscr{F}_{r,\beta}}, \tag{11}$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\overline{z})^{r(n+1)} w^{rn}}{\lambda \beta_{n+1,r} + \beta_{n,r}}, \quad w \in \mathbb{C}.$$

The theorem is proved. □

If $\mathscr{F}_{r,\beta}$ is the r-weighted Fock space $\mathscr{F}_r$. For $f \in \mathscr{F}_r$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$ we have

$$D_r^* f(z) = \sum_{n=1}^{\infty} \frac{(rn-r)!}{(rn)!} a_{n-1} z^{rn}, \quad D_r^* D_r f(z) = \sum_{n=1}^{\infty} \frac{(rn-r)!}{(rn)!} a_n z^{rn}.$$

And for any $h \in \mathscr{F}_r$ and for any $\lambda > 0$, one has

$$F^*_{\lambda,D_r}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{F}_r},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\overline{z})^{r(n+1)} w^{rn}}{\lambda (rn+r)! + (rn)!}.$$

If $\mathscr{F}_{r,\beta}$ is the hyper-Bessel type Fock space $\mathscr{F}_{r,\alpha}$. For $f \in \mathscr{F}_{r,\alpha}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$ we have

$$D_r^* f(z) = \sum_{n=1}^{\infty} \frac{c_{n-1}(r,\alpha)}{c_n(r,\alpha)} a_{n-1} z^{rn},$$

$$D_r^* D_r f(z) = \sum_{n=1}^{\infty} \frac{c_{n-1}(r,\alpha)}{c_n(r,\alpha)} a_n z^{rn},$$

being $c_n(r,\alpha)$ the constants given by (2). And for any $h \in \mathscr{F}_{r,\alpha}$ and for any $\lambda > 0$, one has

$$F^*_{\lambda,D_r}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{F}_{r,\alpha}},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\overline{z})^{r(n+1)} w^{rn}}{\lambda c_{n+1}(r,\alpha) + c_n(r,\alpha)}.$$

5. Approximate inversion formulas

In this section we establish the estimate properties of the extremal function $F^*_{\lambda,D_r}(h)(z)$, and we deduce approximate inversion formulas for the r-difference operator D_r. These formulas are the analogous of Calderón's reproducing formulas for the Fourier type transforms [15, 19]. A pointwise approximate inversion formulas for the operator D_r are also discussed.

The extremal function $F^*_{\lambda,D_r}(h)$ given by (11) satisfies the following properties.

Lemma 5.1. *If $\lambda > 0$ and $h \in \mathscr{F}_{r,\beta}$, then*

(i) $|F^*_{\lambda,D_r}(h)(z)| \leq \frac{1}{2\sqrt{\lambda}}(K_{\mathscr{F}_{r,\beta},z}(z))^{1/2}\|h\|_{\mathscr{F}_{r,\beta}}$,

(ii) $|D_rF^*_{\lambda,D_r}(h)(z)| \leq \sqrt{\frac{\beta_{0,r}}{4\lambda\beta_{1,r}}}(K_{\mathscr{F}_{r,\beta},z}(z))^{1/2}\|h\|_{\mathscr{F}_{r,\beta}}$,

(iii) $\|F^*_{\lambda,D_r}(h)\|_{\mathscr{F}_{r,\beta}} \leq \frac{1}{2\sqrt{\lambda}}\|h\|_{\mathscr{F}_{r,\beta}}$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_{r,\beta}$ with $h(z) = \sum_{n=0}^{\infty} h_n z^{rn}$. From (11) we have

$$|F^*_{\lambda,D_r}(h)(z)| \leq \|\Psi_z\|_{\mathscr{F}_{r,\beta}}\|h\|_{\mathscr{F}_{r,\beta}}.$$

Using the fact that $(x+y)^2 \geq 4xy$ we obtain

$$\|\Psi_z\|^2_{\mathscr{F}_{r,\beta}} = \sum_{n=0}^{\infty}\beta_{n,r}\left[\frac{|z|^{r(n+1)}}{\lambda\beta_{n+1,r}+\beta_{n,r}}\right]^2 \leq \frac{1}{4\lambda}\sum_{n=0}^{\infty}\frac{|z|^{2r(n+1)}}{\beta_{n+1,r}} \leq \frac{1}{4\lambda}K_{\mathscr{F}_{r,\beta},z}(z).$$

This gives (i).

On the other hand, from (10) and (11) we have

$$D_rF^*_{\lambda,D_r}(h)(z) = \sum_{n=0}^{\infty}\frac{\beta_{n,r}h_n}{\lambda\beta_{n+1,r}+\beta_{n,r}}z^{rn} = \langle h, \Phi_z\rangle_{\mathscr{F}_{r,\beta}}, \tag{12}$$

where

$$\Phi_z(w) = \sum_{n=0}^{\infty}\frac{(w\bar{z})^{rn}}{\lambda\beta_{n+1,r}+\beta_{n,r}}.$$

Then

$$|D_rF^*_{\lambda,D_r}(h)(z)| \leq \|\Phi_z\|_{\mathscr{F}_{r,\beta}}\|h\|_{\mathscr{F}_{r,\beta}}.$$

And by (6) we deduce that

$$\|\Phi_z\|^2_{\mathscr{F}_{r,\beta}} = \sum_{n=0}^{\infty}\beta_{n,r}\left[\frac{|z|^{rn}}{\lambda\beta_{n+1,r}+\beta_{n,r}}\right]^2 \leq \frac{1}{4\lambda}\sum_{n=0}^{\infty}\frac{|z|^{2rn}}{\beta_{n+1,r}} \leq \frac{\beta_{0,r}}{4\lambda\beta_{1,r}}K_{\mathscr{F}_{r,\beta},z}(z).$$

This gives (ii).

Finally, from (11) we have

$$\|F^*_{\lambda,D_r}(h)\|^2_{\mathscr{F}_{r,\beta}} = \sum_{n=1}^{\infty} \beta_{n,r} \left[\frac{\beta_{n-1,r}|h_{n-1}|}{\lambda\beta_{n,r} + \beta_{n-1,r}} \right]^2.$$

Then we obtain

$$\|F^*_{\lambda,D_r}(h)\|^2_{\mathscr{F}_{r,\beta}} \leq \frac{1}{4\lambda} \sum_{n=1}^{\infty} \beta_{n-1,r}|h_{n-1}|^2 = \frac{1}{4\lambda}\|h\|^2_{\mathscr{F}_{r,\beta}},$$

which gives (iii) and completes the proof of the lemma. □

We establish approximate inversion formulas for the operator D_r.

Theorem 5.2. *If $\lambda > 0$ and $h \in \mathscr{F}_{r,\beta}$, then*

(i) $\lim_{\lambda \to 0^+} \|D_r F^*_{\lambda,D_r}(h) - h\|_{\mathscr{F}_{r,\beta}} = 0$,

(ii) $\lim_{\lambda \to 0^+} \|F^*_{\lambda,D_r}(D_r h) - h_0\|_{\mathscr{F}_{r,\beta}} = 0$, *where* $h_0(z) = h(z) - h(0)$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_{r,\beta}$ with $h(z) = \sum_{n=0}^{\infty} h_n z^{rn}$. From (12) we have

$$D_r F^*_{\lambda,D_r}(h)(z) - h(z) = \sum_{n=0}^{\infty} \frac{-\lambda\beta_{n+1,r}h_n}{\lambda\beta_{n+1,r} + \beta_{n,r}} z^{rn}. \tag{13}$$

Therefore

$$\|D_r F^*_{\lambda,D_r}(h) - h\|^2_{\mathscr{F}_{r,\beta}} = \sum_{n=0}^{\infty} \beta_{n,r} \left[\frac{\lambda\beta_{n+1,r}|h_n|}{\lambda\beta_{n+1,r} + \beta_{n,r}} \right]^2$$

Again, by dominated convergence theorem and the fact that

$$\beta_{n,r} \left[\frac{\lambda\beta_{n+1,r}|h_n|}{\lambda\beta_{n+1,r} + \beta_{n,r}} \right]^2 \leq \beta_{n,r}|h_n|^2,$$

we deduce (i).

Finally, from (10) and (11) we have

$$F^*_{\lambda,D_r}(D_r h)(z) - h_0(z) = \sum_{n=1}^{\infty} \frac{-\lambda\beta_{n,r}h_n}{\lambda\beta_{n,r} + \beta_{n-1,r}} z^{rn}. \tag{14}$$

So, one has

$$\|F^*_{\lambda,D_r}(D_r h) - h_0\|^2_{\mathscr{F}_{r,\beta}} = \sum_{n=1}^{\infty} \beta_{n,r} \left[\frac{\lambda\beta_{n,r}|h_n|}{\lambda\beta_{n,r} + \beta_{n-1,r}} \right]^2.$$

Using the dominated convergence theorem and the fact that

$$\beta_{n,r}\left[\frac{\lambda\beta_{n,r}|h_n|}{\lambda\beta_{n,r}+\beta_{n-1,r}}\right]^2 \leq \beta_{n,r}|h_n|^2,$$

we deduce (ii). □

We deduce pointwise approximate inversion formulas for the r-difference operator D_r.

Theorem 5.3. *If* $\lambda > 0$ *and* $h \in \mathscr{F}_{r,\beta}$*, then*

(i) $\lim_{\lambda\to 0^+} D_r F^*_{\lambda,D_r}(h)(z) = h(z)$,

(ii) $\lim_{\lambda\to 0^+} F^*_{\lambda,D_r}(D_r h)(z) = h_0(z)$.

Proof. Let $h \in \mathscr{F}_{r,\beta}$ with $h(z) = \sum_{n=0}^{\infty} h_n z^{rn}$. From (13) and (14), by using the dominated convergence theorem and the fact that

$$\frac{\lambda\beta_{n+1,r}|h_n|}{\lambda\beta_{n+1,r}+\beta_{n,r}}|z|^{rn}, \frac{\lambda\beta_{n,r}|h_n|}{\lambda\beta_{n,r}+\beta_{n-1,r}}|z|^{rn} \leq |h_n||z|^{rn},$$

we obtain (i) and (ii). □

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