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AMPLE VECTOR BUNDLES AND INTRINSIC QUADRIC FIBRATIONS OVER IRRATIONAL CURVES

TOMMASO DE FERNEX

Let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 2$ on a smooth complex projective variety X. This work is part of the following problem: to study and classify the pair $(X, \mathcal{E})$ assuming the existence of a regular section $s \in \Gamma(X, \mathcal{E})$ whose zero locus $Z = (s)_0$ is a special subvariety of X. In [2] and [11], the case of Z quadric fibration, respectively of dimension 2 or more, over a smooth curve is discussed under the further hypothesis that the quadric fibration structure is induced on Z by an ample line bundle L on X. Here the same situation is considered, and classification is given assuming the base curve to be irrational, in the more general case that the quadric fibration structure of Z is intrinsic, i.e. not *a priori* induced by a polarization of X.

0. Introduction.

Several approaches in studying geometry of higher dimensional projective varieties rely on investigation of existence of particular subvarieties. It is well know, for instance, that, if Z is a hyperplane section or more generally, an effective ample divisor of a projective variety X, the geometric characteristics of

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The author is member of the GNSAGA of the Italian CNR.

Z will determine many properties of ambient variety X. If we wish to consider a more general case, Z having higher codimension inside X, we need a hypothesis generalizing the concept of ample divisor. We consider the following set-up:

($*$) Set-up. X is a smooth complex projective variety of dimension n and Z is a subvariety of X, of dimension $k \leq n-2$, which is defined as zero locus of a regular section s of an ample vector bundle $\mathcal{E}$ over X. In particular, by definition of regularity of the section s, Z is smooth and rk $\mathcal{E} = n - k$.

Set-up ($*$) allows us to find good relations between X and Z. First of all, due to the regularity of s, the restriction of $\mathcal{E}$ to Z is isomorphic to the normal bundle $N_{Z|X}$, leading to the adjunction formula:

$$K_Z = (K_X + \det \mathcal{E})_Z. \tag{0.1}$$

Furthermore, using also the ampleness of $\mathcal{E}$, we can use an extension of Lefschetz theorem,essentially due to Sommese [16], Proposition 1.16, case $k = 0$, which states that the restriction map

$$H^j(g) : H^j(X, \mathbb{Z}) \to H^j(Z, \mathbb{Z}), \tag{0.2}$$

induced by the embedding $g : Z \hookrightarrow X$, is isomorphism for $j < k$ and injective with torsion free cokernel for $j = k$, being $k = \dim Z$. From (0.2) we deduce the following properties (see [11], Theorem 1.1]):

(0.3) Theorem . *Let X, $\mathcal{E}$ and Z be as in ($*$), $k = \dim Z$, and let $g : Z \hookrightarrow X$ be the inclusion morphism. Then*

(1) $H^{p,q}(g) : H^q(X, \Omega^p_X) \to H^q(Z, \Omega^p_Z)$ *is isomorphism for $p + q < k$ and injective for $p + q = k$,*

(2) $\mathrm{Pic}(g) : \mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ *is isomorphism if $k > 2$ and injective with torsion-free cokernel if $k = 2$,*

(3) $\mathrm{Alb}(g) : \mathrm{Alb}(Z) \to \mathrm{Alb}(X)$ *is isomorphism if $k \geq 2$.*

Set-up ($*$) has been recently introduced by Lanteri and Maeda with the purpose of classifying the pair $(X, \mathcal{E})$, assuming that the subvariety Z is a "special" variety. In their first paper [9], Theorems A and B, they assume Z to be a projective space or a quadric hypersurface. In [10] the case of geometrically ruled surfaces over smooth curves is discussed. The case of $\mathbb{P}$-bundles of dimension ≥ 3 over smooth curves is treated in [11], Theorem B, where they add a further hypothesis: they assume the existence of an ample line bundle H

over X whose restriction to the subvariety Z gives it the structure (as polarized variety) of a scroll over the base curve.

Such kind of assumption, in which a particular structure, as polarized variety, of the subvariety Z is defined by restriction of a suitable polarization of the ambient variety X, will be called here *assumption of global polarization*. On the contrary, if Z has a particular structure, as polarized variety, which is not *a priori* induced by an ample line bundle of X, we will say the structure of Z is *intrinsically defined*, or *intrinsic*.

In [11], Theorem A, Lanteri and Maeda prove also a theorem for the $\mathbb{P}$-bundle case without the assumption of global polarization, but only in the case of irrational base curve. In the recent paper [1], Andreatta and Occhetta consider the more general case of intrinsic $\mathbb{P}$-bundles without restrictions on the dimension of the base B, but their result (ibid, Corollary 4.4) concerns just the case of B minimal in the sense of Mori, i.e. K_B numerically effective. The case of Z quadric fibration over a smooth curve is treated, respectively, in [2] if Z is a surface and [11], Theorem C, if $\dim Z \geq 3$. In both cases global polarization is required.

In view of the described considerations and results, it is natural to ask whether this extra assumption of global polarization could be avoided also in the last case above cited, i.e. when Z is a quadric fibration over a smooth curve B. the aim of this work is to answer this question when B is an irrational curve. The result splits into the following two theorems, depending on whether dim Z is 2 or more.

Theorem A. *Let X, $\mathcal{E}$ and Z be as in $(*)$. Assume that Z is a conic fibration over an irrational smooth curve B. Let $\pi : Z \to B$ the corresponding fibration. Then π extends to a morphism $\alpha : X \to B$. Moreover, if Z is geometrically ruled over B, then*

(0) *$\alpha : X \to B$ is a $\mathbb{P}^{n-1}$-bundle over B and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ for every fiber F.*

Otherwise the pair $(X, \mathcal{E})$ comes into one of the following cases:

(1) *$\alpha : X \to B$ is a $\mathbb{P}^{n-1}$-bundle over B and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}$ for every fiber F,*

(2) *$\alpha : X \to B$ is a quadric fibration over B and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-2)}$ for every smooth fiber F,*

(3) *$\alpha : X \to B$ factor through a $\mathbb{P}^{n-1}$-bundle fibration $\psi : X \to S$, where S is a ruled surface over B, $\mathcal{E}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ for every fiber G of ψ, and*

$\psi|_Z : Z \to S$ is a birational morphism contracting at least one (-1)-curve of Z.

Theorem B. *Let X, $\mathcal{E}$ and Z be as in $(*)$. Assume that Z is a quadric fibration of dimension ≥ 3 over an irrational smooth curve B. Let $\pi : Z \to B$ the corresponding fibration. Then π extends to a morphism $\alpha : X \to B$. Moreover, if Z is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over B with trivial monodromy, then*

(0) *$\alpha : X \to B$ factors through a $\mathbb{P}^{n-2}$-bundle fibration $\gamma : X \to S$, where S is a geometrically ruled surface over B, and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}$ for every fiber G of γ.*

Otherwise the pair $(X, \mathcal{E})$ comes into one of the following cases:

(1) *$\alpha : X \to B$ is a $\mathbb{P}^{n-1}$-bundle over B and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-k-1)}$ for every fiber F,*

(2) *$\alpha : X \to B$ is a quadric fibration over B and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-k)}$ for every smooth fiber F.*

(0.4) Remark. Case (0) of Theorem A is [10]. In cases (0), (1) and (2) of Theorem A and (1) and (2) of Theorem B, *a posteriori* there exists an ample line bundle on X inducing on Z its structure of quadric fibration over B. The effectiveness of the theorems is clear in all but case (3) of Theorem A, which looks uncertain.

(0.5) Remark. Theorems A and B hold also if $B \cong \mathbb{P}^1$, if we assume that the morphism $\pi : Z \to B$ extends to a morphism $\alpha : X \to B$.

In section 1, after giving basic notations, we review some properties of quadric fibrations. Section 2 contains a sketch of the proves of Theorems A and B and the preliminary results we will need. The proofs of the two theorems can be found in the last two sections, respectively.

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1. Quadric fibrations over smooth curves.

Notation. In this paper we work over the complex field $\mathbb{C}$. Smooth projective varieties are briefly called *manifolds*. A *polarized manifold* is a pair (X, H) consisting of a manifold X and an ample line bundle L on it. We use almost interchangeably divisors and line bundles on manifolds; consequently tensor product between line bundles is often denoted additively.

Let X be a manifold. The restriction of a vector bundle $\mathcal{E}$ (resp. of a divisor D) on X to a subvariety $Y \subset X$ is denoted with $\mathcal{E}_Y$ (resp. D_Y). The numerical class of a curve $C \subset X$ is denoted with $[C]$. If $\alpha : X \to Y$ is a morphism over an algebraic variety,

$$N_1(X/Y) := (Z_1(X/Y)/_{\equiv}) \otimes \mathbb{R}$$

is the real vector space generated by α-relative 1-cycles modulo numerical equivalence, $\rho(X/Y) := \dim N_1(X/Y)$ is the Picard number of X over Y, and $\overline{NE}(X/Y)$ is the (closed and convex) cone of α-relative curves. We write briefly $N_1(X)$, $\rho(X)$ and $\overline{NE}(X)$ if Y is a point.

If V is a closed convex cone in $\mathbb{R}^n$, a subcone $W \subset V$ is called *extremal* if it is so in the sense of convexity. A polyhedral extremal subcone is called an *extremal face*. A one dimensional subcone is called a *ray*. A ray R of $\overline{NE}(X/Y)$ is said to be *negative* if it has negative intersection with K_X. The *length* of a negative extremal ray R of $\overline{NE}(X/Y)$ is the integer number

$$l(R) := \min\{-K_X \cdot C \mid C \text{ is a rational curve, } [C] \in R\}.$$

If $\alpha : X \to Y$ is a Fano-Mori contraction, we identify $\overline{NE}(X/Y)$ with the relative extremal face of $\overline{NE}(X)$. We refer to [14] and [7] for a complete review of the properties of the cone of curves on a manifold, in particular as references for Mori's Cone Theorem and Kawamata's Contraction Theorem.

(1.1) Quadric fibrations. Even though the definition of quadric fibration usually given in literature concerns polarized manifolds, we prefer to repeat here the same definition just for manifolds, leaving the choice of polarization to be taken freely. We say that a manifold X is a *quadric fibration* over an algebraic variety Y if there exist a surjection $\pi : X \to Y$ and an ample line bundle L on X such that any fiber F of π is isomorphic to a quadric hypersurface $Q \subset \mathbb{P}^{k+1}$, where $k = \dim X - \dim Y$, and L induces $\mathcal{O}_Q(1)$ on it. A quadric fibration whose fibers are 1-dimensional is called *conic fibration*.

Proposition (1.1.1). *Let $\alpha : X \to Y$ be a quadric fibration. Then there exists a very ample line bundle L inducing the quadric fibration structure on X.*

Proof. Let L_0 be any ample line bundle defining the quadric fibration structure on X, and take the push-out $\mathcal{F} := \alpha_* L_0$. Pick a very ample line bundle A on Y. Then $\mathcal{F} \otimes A^{\otimes m}$ is generated by global sections for m sufficiently large, i.e. there is a surjection

$$\mathcal{O}_Y^{\oplus N} \to \mathcal{F} \otimes A^{\otimes m} \to 0.$$

By twisting with A, we get

$$A^{\oplus N} \to \mathcal{F} \otimes A^{\otimes(m+1)} \to 0.$$

Thus, we find the following fiber-wise embeddings:

$$X \hookrightarrow \mathbb{P}_Y(\mathcal{F}) \cong \mathbb{P}_Y(\mathcal{F} \otimes A^{\otimes(m+1)}) \hookrightarrow \mathbb{P}_Y(A^{\otimes N}).$$

Then, define L as the restriction to X, via the above embeddings, of the tautologic line bundle of $\mathbb{P}_Y(A^{\otimes N})$. $\square$

(1.2) Conic fibrations over curves. Let X be a conic fibration over a smooth curve B. Then any singular fiber F_0 splits in a couple of distinct lines with self intersection -1. In particular, we can express X as a geometrically ruled surface X_0, of base B, blown-up at points belonging to different fibers.

The contrary is also true, i.e. any blow-up of a geometrically ruled surface at points belonging to different fibers is a conic fibration. Indeed, if F is a fiber, the line bundle

$$L := -K_X + mF$$

is ample for sufficiently large m and induces on all fibers their embeddings as plane conics. Thus, Theorem A may be equivalently reformulated for irrational ruled surfaces whose reducible fibers consist exactly of two components.

(1.3) Quadric fibrations of dimension $\geq$ 3 over curves. Let X be a quadric fibration of dimension ≥ 3 over a smooth curve B. It is a well known fact that all its singular fibers are reduced and irreducible (e.g. see [8], pag. 461).

Proposition (1.3.1). *Let X be as above. Then $\rho(X) = 2$ or 3, the case $\rho = 3$ occurring if and only if X is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over B with trivial monodromy. In particular, if $\rho = 3$, then X admits two distinct $\mathbb{P}^1$-bundle fibrations over geometrically ruled surfaces S_1 and S_2. Both the surfaces S_i are ruled over B.*

Proof. The morphism $\pi : X \to B$ is a Fano-Mori contraction. It is enough to control the dimension of $N_1(X/B)$, since, by Kawamata's Contraction Theorem,

$$\rho(X) = \rho(B) + \rho(X/B).$$

If $\dim X \geq 4$, then $\rho(F) = 1$ for every fiber of π, hence $\rho(X/B) = 1$.

So, consider the case $\dim X = 3$. Let $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth fiber, and $C_1, C_2 \subset F$ two lines belonging, respectively, to the two rulings of F. If there exists a singular fiber F_0, we can move the C_i flatly through smooth fibers making them degenerate into F_0. Numerical equivalence is maintained and $\rho(F_0) = 1$, thus we conclude that $\rho(X/B) = 1$. Analogously, if X is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over B with non-trivial monodromy, then, by a turn along a suitable path $\gamma \subset B$ with base-point p, we return on $F = \alpha^{-1}(p)$ with an automorphism inverting the two rulings. Hence $C_1 \equiv C_2$, implying $\rho(X/B) = 1$.

Conversely, assume X to be a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over B with trivial monodromy. Let L be a very ample line bundle inducing the quadric fibration structure on X, and take two general divisor D_1 and D_2 in the linear system $|L|$. Then, for general fiber F, the intersection $D_1 \cap D_2 \cap F$ consists of two points q_1 and q_2 that don't belong to a same line of F. Starting from one of these fibers, select a ruling and pick up the two lines l_1 and l_2, belonging to such ruling and passing, respectively, through q_1 and q_2. Define in this way

$$Y := \bigcup_{\substack{l_i \subset F \\ F \text{ general}}} (l_1 \cup l_2).$$

It is well-defined due to the triviality of the monodromy, and its Zariski closure $\overline{Y}$ in X is a divisor inducing $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}}(2{,}0)$ on each fiber. Therefore, $\overline{Y} \cdot C_1 = 0$ and $\overline{Y} \cdot C_2 = 2$, implying $\rho(X/B) = 2$. In particular, the numerical classes of C_1 and C_2 generate two negative extremal rays of the cone of curves of X. □

2. Preliminary results.

The basic idea for proving the theorems is inspired to the proof of [11, Theorem A]. Here is a sketch. Step by step, we report the results we will apply.

Firstly, we want to extend the fibration structure of Z over B to the whole X. It is here that we use the irrationality of B. As remarked in (0.5), if we assume the statement of the following proposition, Theorems A and B hold also if B is rational.

Proposition (2.1). *Let X, $\mathcal{E}$ and Z be as in the assumptions of Theorems A or B. Then the morphism $\pi : Z \to B$ extends to a morphism $\alpha : X \to B$.*

Proof. By (0.3.1)

$$\begin{cases} h^0(\Omega^1_X) = h^0(\Omega^1_Z) = q(Z) > 0 \\ h^0(\Omega^2_X) \leq h^0(\Omega^2_Z) = p_g(Z) = 0. \end{cases}$$

Therefore, the Albanese map $\alpha : X \to \mathrm{Alb(X)}$ is a morphism with connected fibers over a curve (of genus $q = q(Z)$). Since π coincides with the Albanese map of Z modulo isomorphism of the base curve B, by functoriality and isomorphism (0.3.3) we conclude that α is extension of π to X. $\square$

Secondly, we focus on a general fiber of the morphism α, leading the problem to the situation already considered in [9]. Here are the theorems we will apply:

Theorem (2.2). ([9], Theorem A, case $n - r = 1$). *Let X, $\mathcal{E}$ and Z as in $(*)$ and assume that $Z \cong \mathbb{P}^1$. Then the pair $(X, \mathcal{E})$ is one of the following:*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)})$,
(2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$,
(3) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)})$,
(4) $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{F})$ *for some vector bundle $\mathcal{F}$ of rank n on $\mathbb{P}^1$ and* $\mathcal{E} \cong \bigoplus_{j=1}^{n-1}(H \otimes \beta^*\mathcal{O}_{\mathbb{P}^1}(b_j))$, *where H stands for the tautologic bundle of $\mathcal{F}$ and $\beta : X \to \mathbb{P}^1$ is the bundle projection.*

Theorem (2.3). ([9], Theorem B). *Let X, $\mathcal{E}$ and Z as in $(*)$ and assume that $Z \cong \mathbb{Q}^k$ with $k \geq 2$. Then the pair $(X, \mathcal{E})$ is one of the following:*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-k-1)})$,
(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-k)})$,
(3) $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{F})$ *for some vector bundle $\mathcal{F}$ of rank n on $\mathbb{P}^1$ and* $\mathcal{E} \cong \bigoplus_{j=1}^{n-2}(H \otimes \beta^*\mathcal{O}_{\mathbb{P}^1}(b_j))$, *where H stands for the tautologic bundle of $\mathcal{F}$ and $\beta : X \to \mathbb{P}^1$ is the bundle projection.*

Subsequently, when we need to extend our analysis from general fiber to all the fibers of α, we will use the following semi-continuity property of the Δ-genus. Let (X, H) be a polarized manifold of dimension n. Its Δ-genus is defined by

$$\Delta(X, H) := n + H^n - h^0(X, \mathcal{O}_X(H)).$$

Proposition (2.4). ([4], Corollary 1.10). $\Delta(X, H) \geq 0$.

Proposition (2.5). ([4], Propositions 2.1 and 2.2).

(1) $(X, H) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1))$ *if and only if* $H^n = 1$ *and* $\Delta(X, H) = 0$.

(2) X *is isomorphic to a (non-necessarily smooth) hyperquadric* Q *of* $\mathbb{P}^{n+1}$ *and* $H \cong \mathcal{O}_Q(1)$ *if and only if* $H^n = 2$ *and* $\Delta(X, H) = 0$.

Let $\alpha : X \to Y$ be surjective proper flat morphism between reduced and irreducible varieties of dimension, respectively, n and m. Note that α is equidimensional. Assume that every fiber of α is reduced and irreducible. Let L be a line bundle on X which is ample relative to α, briefly said α-ample, i.e. such that its restriction to every fiber is an ample line bundle. Let $F = \alpha^{-1}(p)$ stand for the fiber over $p \in Y$. Then

Theorem (2.6). ([4], Theorem 5.2). L_F^{n-m} *is a continuous function of* $p \in Y$, *and* $\Delta(F, L_F)$ *is a lower-semi-continuous function of* $p \in Y$.

Remark (2.6.1). Let Y be a smooth curve and X a reduced and irreducible variety. Then any surjective morphism $\alpha : X \to Y$ is a flat morphism (e.g. see [6], Chapter III, Proposition 9.7).

In the last part of the proof of Theorem B, we will analyze the cone of curves of X in comparison with the one of Z. The idea is to prove that such cones have a common negative extremal ray, and to use this property in order to extend some fibration structure of Z to X. But, to start with, we need to show that $\overline{NE}(X)$ and $\overline{NE}(Z)$ are actually subcones of the same real vector space. This is guaranteed by the following

Lemma (2.7). *Let* X *and* Z *be as in* $(*)$, *and assume* $\dim Z \geq 3$. *Then* $N_1(X) \cong N_1(Z)$.

Proof. By (0.2), $H^2(X, \mathbb{Z}) \cong H^2(Z, \mathbb{Z})$. Recall that, on the Picard groups, numerical and homological equivalence coincide (e.g. see [5], Proposition 3.1). Then, taking quotients with respect to numerical equivalence, isomorphism $\operatorname{Pic}(X) \cong \operatorname{Pic}(Z)$ (0.3.2) still induces an isomorphism between the numerical equivalence class groups. So, by tensoring with $\mathbb{R}$ and using duality, we get $N_1(X) \cong N_1(Z)$. $\square$

We will identify $N_1(X)$ and $N_1(Z)$ via the isomorphism above. Through such identification we can look at $\overline{NE}(Z)$ as a subcone of $\overline{NE}(X)$. The will use following two properties.

Theorem (2.8). ([17], Theorem 1.1). *Let* R *be a negative extremal ray on a manifold* X, $\gamma : X \to W$ *its contraction,* $E := \operatorname{Exc}(f)$ *the locus of the points of the curves belonging to* R, E_w *a general fiber of* $\gamma|_E$. *Then*

$$\dim E + \dim E_w \geq X - 1 + l(R).$$

Lemma (2.9). ([3], Lemma 1.4). *Let Z be a smooth irreducible subvariety of a manifold X, and assume that the inclusion induces an isomorphism $N_1(Z) \cong N_1(X)$. Let R be a negative extremal ray in $\overline{NE}(X)$ and let $\gamma^X : X \to Y$ be the corresponding contraction. Then the restriction $\gamma^X|_Z$ of γ^X to Z is a non-finite morphism if and only if R is an extremal ray in $\overline{NE}(Z)$ as well. In this case, if R is also negative in $\overline{NE}(Z)$, denote by γ^Z the relative contraction of Z; then $\gamma^X|_Z$ factors through γ^Z and a finite morphism.*

3. Proof of Theorem A.

By Proposition (2.1), the morphism $\pi : Z \to B$ extends to a morphism $\alpha : X \to B$. Note that α is a flat morphism (2.6.1). Unless otherwise specified, throughout all the section F will denote a general fiber of α. F is a smooth fiber and, if $s_F \in \Gamma(\mathcal{E}_F)$ is the restriction of the section s to F,

$$f := (s_F)_0 = F \cap Z$$

is a general fiber of π. In particular, f is isomorphic to $\mathbb{P}^1$. Note that the pair $(F, \mathcal{E}_F)$ satisfies the hypothesis of Theorem (2.1). We will use this property later. We split our investigation in two parts, depending on whether $\rho(X)$ is 2 or more.

First part:. $\rho(X) = 2$.

Claim (3.1). *Every fiber of α is irreducible and reduced.*

Proof. First of all, note that, if there is a non-reduced component of a fiber of $\alpha : X \to B$, then, by restricting to Z, we would find a non-reduced component of a fiber of $\pi : Z \to B$, but we know that this doesn't happen. Now, assume F to be a reducible fiber of α: write $F = A \cup B$. Since $\mathcal{O}_A(A + B) \cong \mathcal{O}_A(F) \cong \mathcal{O}_A$, A and B are two effective divisors inducing dual (and non-trivial, by connectedness of F) line bundles on A:

$$\mathcal{O}_A(A) \cong \mathcal{O}_A(-B).$$

Therefore A, B and K_X are linear independent divisors, in contradiction with $\rho(X) = 2$. □

We are ready to analyze the possibilities given by Theorem (2.2) for the pair $(F, \mathcal{E}_F)$. We remark that, *a posteriori* of the following arguments, for every smooth fiber F of α the pair $(F, \mathcal{E}_F)$ must come into the same case of Theorem (2.2).

(3.2) Assume $(F, \mathcal{E}_F)$ to be as in the first case on Theorem (2.2). In order to apply the Δ-genus semi-continuity argument to $\alpha : X \to B$, the first step is to find a line bundle on X whose restriction to the smooth fibers of α is isomorphic to $\mathcal{O}_{\mathbb{P}}(1)$. Let $U \subset B$ be the locus of the smooth fibers of α:

$$U = \{p \in B | F_p = \alpha^{-1}(p) \text{ is a smooth fiber}\},$$

and $V = \alpha^{-1}(U) \subset X$. Then $\alpha|_V : V \to U$ is a $\mathbb{P}^{n-1}$-bundle over U. Tsen's Theorem gives the vanishing of the étale cohomology group $H^2(U_{et,}\mathbb{G}_m)$ [13], Chapter III, 2.22(d), thus, at least in the étale topology, V is the projectivization of a vector bundle on U. We apply Hilbert's Theorem ([13], Chapter III, Proposition 4.9) to conclude that produces, in the Zariski topology, a line bundle H_0 on V which induces $\mathcal{O}_{\mathbb{P}}(1)$ on the fibers. Then such line bundle H_0 extends to a line bundle H on X. It is easy to see that H is α-ample using the fact that all the fibers of α are reduced and irreducible. Let F_0 any fiber of α. By Proposition (2.4) and Theorem (2.6), $\Delta(F_0, H_{F_0}) = 0$ and $H_{F_0}^{n-1} = 1$. We conclude, using Proposition (2.5), that $\alpha : X \to B$ is a $\mathbb{P}^{n-1}$-bundle and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for *every* fiber. This gives case (0) of Theorem A. Note, in particular, that all the fibers of π must be smooth. Conversely, if Z has not singular fibers, then $(X, \mathcal{E})$ is as above by [10].

(3.3) Assume now $(F, \mathcal{E}_F)$ to be as in (2.2.2), and define

$$L := -K_X - \det \mathcal{E}. \tag{3.3.1}$$

Then $L_F = -K_F - \det \mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)$. Moreover, L is α-ample. Proceeding as in the second part of (3.2), we conclude that $\alpha : X \to B$ is a $\mathbb{P}^{n-1}$ -bundle and, this time, $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}$ for *every* fiber. This is case (1) of Theorem A. The same kind of computations shows that, if $(F, \mathcal{E}_F)$ is as in (2.2.3) and we define L as in (3.3.1), then $(X, \mathcal{E})$ comes into case (2) of Theorem A. In fact, the line bundle $L + mF$ is ample for m sufficiently large, and it induces on X the quadric fibration structure over B.

(3.4) Now, let $(F, \mathcal{E}_F)$ be as in (2.2.4). We are going to show that this case, that is expected to give the last case of Theorem A, cannot occur in this situation, assuming $\rho(X) = 2$.

Remember that $f = F \cap Z \cong \mathbb{P}^1$ by generality of F. Let $\beta : F \to \mathbb{P}^1$ denote the bundle projection and G be a fiber of β. If

$$s_G \in \Gamma(G, \mathcal{E}_G) \cong \Gamma(\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$$

is the restriction of the section s to G, then

$$f \cap G = (s_G)_0 \neq \phi. \tag{3.4.1}$$

Therefore $f \cap G$ is either a point or all f. Since the event f contained in some G would contradict (3.4.1) if we change G, the intersection $f \cap G$ must be a point. In other words, f is a section of $\beta : F \to \mathbb{P}^1$.

Let $C \subset Z$ be a section of $\pi : Z \to B$. For general fiber f, consider the intersection point $p = f \cap C$ and the fiber G_p of β containing p. Letting f vary among general fibers of π, define

$$Y := \bigcup_{\substack{p = f \cap C \\ f \text{ general}}} G_p.$$

Its Zariski closure $\overline{Y}$ in X is a divisor of X. Then $\overline{Y}$ is linearly independent from K_X and F, as we can check by intersecting with a line l contained in a fiber G of β and with a smooth fiber f of π.

Second part:. $\rho(X) \geq 3$.

Using the adjunction formula (0.1), we see that $K_X + \det \mathcal{E}$ is not numerically effective. Therefore the pair $(X, \mathcal{E})$ satisfies the hypothesis of a result of Maeda [12] that gives a list of admissible cases for $(X, \mathcal{E})$. If we impose the condition $\rho(X) \geq 3$, only two of these cases survive, and we have:

Theorem (3.5). [12, Theorem, case $\rho(X) \geq 3$]. *Let $\mathcal{E}$ be an ample vector bundle of rank $n-2$ on a manifold X of dimension n such that $K_X + \det \mathcal{E}$ is not nef. Assume moreover $\rho(X) \geq 3$. Then $(X, \mathcal{E})$ is one of the following:*

(1) *X is a $\mathbb{P}^{n-2}$-bundle over a smooth surface S and $\mathcal{E}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ for every fiber G of $\psi : X \to S$,*

(2) *there is an effective divisor E on X such that $(E, \mathcal{E}_E) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ and $\mathcal{O}_X(E)|_E \cong \mathcal{O}_{\mathbb{P}}(-1)$.*

(3.6) Assume $(X, \mathcal{E})$ to be as in (3.5.1). For every fiber G of ψ, let

$$s_G \in \Gamma(G, \mathcal{E}_G) \cong \Gamma(\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$$

be the restriction of s to G. Then

$$g := (s_G)_0 = Z \cap G$$

is a non-empty linear subspace of G. Actually, g can be just a point or line of G. Therefore $\psi|_Z : Z \to S$ is a generically one to one surjection whose positive dimensional fibers are (-1)-curve of Z. *A priori* $\psi|_Z$ could be an isomorphism, but

$$\rho(S) = \rho(X) - 1 \leq \rho(Z) - 1$$

implies that at least one (-1)-curves of Z is contracted by ψ. This is case (3) of Theorem A.

(3.7) Assume now $(X, \mathcal{E})$ to be as in (3.5.2). This is the last admissible case. Then X is the blow-up at a point q of a manifold X', and E is the exceptional divisor (see [15]). Let $\sigma : X \to X'$ denote the contraction morphism. [10, Lemma 5.1] tells us that, in this situation, there exists an ample vector bundle $\mathcal{E}'$ on X' such that

$$\mathcal{E} \cong \sigma^* \mathcal{E}' \otimes \mathcal{O}_X(-E). \tag{3.7.1}$$

Let $s' \in \Gamma(X', \mathcal{E}')$ be the section corresponding to $s \in \Gamma(X, \mathcal{E})$ via isomorphism (3.7.1), and $Z' := (s')_0$. By construction, $Z' = \sigma(Z)$. If

$$s_E \in \Gamma(E, \mathcal{E}_E) \cong \Gamma(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$$

is the restriction of s to E,

$$e := (s_E)_0 = Z \cap E$$

is a positive-dimensional linear subspace of G, hence $e \cong \mathbb{P}^1$. This means that $\sigma|_Z : Z \to Z'$ is the contraction morphism of e, that is a (-1)-curve of Z. In particular, $q \in Z'$. We say that $(X', \mathcal{E}', Z')$ is a *reduction* of $(X, \mathcal{E}, Z)$.

Now, X', $\mathcal{E}'$ and Z' satisfy the assumptions of the theorem we are proving. If $(X', \mathcal{E}')$ is again as in (3.6.2), we iterate the reduction. We remark that, in this event, the exceptional divisors $\mathcal{E}'$ of X' cannot contain the point $q = \sigma(E)$. This means that the exceptional divisor of X are disjoint. Eventually, we find a reduction, say $(X', \mathcal{E}', Z')$, satisfying the hypothesis of Theorem A and not containing any exceptional divisor. Thus, $(X', \mathcal{E}')$ should come into one of the previous cases stated in Theorem A. The following claim says that this can not happen, concluding the proof of Theorem A.

Claim (3.7.3). *Assume $(X', \mathcal{E}')$ to be as in one case stated in Theorem A. Let $\sigma : X \to X'$ the blow-up of X' at any point q, and $\mathcal{E}$ the vector bundle on X defined by* (3.7.1). *Then $\mathcal{E}$ is not ample.*

Proof. Let F' be the fiber in X' containing q and $F = \sigma^{-1}(F')$. Note that $\mathcal{E}'_{F'}$, hence $\mathcal{E}_F$, is decomposable, and $\mathcal{E}_F$ has at least one summand isomorphic to

$$\text{summand of } \mathcal{E}_F \cong \begin{cases} \sigma^*\mathcal{O}_{\mathbb{P}^{n-1}}(1) \otimes \mathcal{O}_F(-E) & \text{in cases (0) or (1)} \\ \sigma^*\mathcal{O}_{\mathbb{Q}^{n-1}}(1) \otimes \mathcal{O}_F(-E) & \text{in case (2)} \\ \sigma^*(H(\mathcal{F}) \otimes \beta^*\mathcal{O}_{\mathbb{P}^1}(b_j)) \otimes \mathcal{O}_F(-E) & \text{in case (3).} \end{cases}$$

Take then $C' \subset F'$ to be a curve passing through p: precisely, choose C' to be, respectively, a line of $F' \cong \mathbb{P}^{n-1}$ in cases (0) and (1), a line of $F' \cong \mathbb{Q}^{n-1}$ in case (2), and a line of the fiber $G \cong \mathbb{P}^{n-1}$ of β containing p in the last case. In any event, the summand of $\mathcal{E}_F$ previously selected has intersection zero with the strict transform of the curve C'. $\square$

4. Proof of Theorem B.

By Proposition (2.1), the morphism $\pi : Z \to B$ extends to a morphism $\alpha : Z \to B$, which is flat, as we remarked in (2.6.1). Let F be a general fiber of α: then the pair $(F, \mathcal{E}_F)$ satisfies the hypothesis of Theorem (2.3). In the following we analyze the possibilities listed in (2.3).

(4.1) Assume $(F, \mathcal{E})$ to be as in one of the first two cases of Theorem (2.3). Due to the isomorphism between the Picard groups (0.3.2), every fiber of α is (reduced and) irreducible, $\rho(Z) = 2$, and we can choose a line bundle L on X whose restriction L_Z to Z induces the quadric fibration structure. Then $L_F \cong \mathcal{O}_{\mathbb{P}}(1)$ or $\mathcal{O}_{\mathbb{Q}}(1)$ respectively, and, using the Δ-genus semi-continuity argument in analogous way as we did in the previous section, precisely in (3.3), we obtain cases (1) and (2) of Theorem B.

(4.2) So, let $(F, \mathcal{E})$ to be as in the last case of Theorem (2.3): F admits a $\mathbb{P}^{n-2}$-bundle fibration $\beta : F \to \mathbb{P}^1$, and $\mathcal{E}_G \cong \mathcal{O}_{\mathbb{P}}(1)^{\otimes(n-3)}$ for every fiber G of β.

Claim (4.2.1). $\rho(X) = 3$

Proof. Let F be a general fiber of α and $f = F \cap Z$. Then, for every fiber G of $\beta : F \to \mathbb{P}^1$,

$$f \cap G = (s_G)_0$$

is a positive-dimensional linear subspace of G, being

$$s_G \in \Gamma(G, \mathcal{E}_G) \cong \Gamma(\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)})$$

the restriction of s to G. Therefore, the restriction $\beta|_f : f \to \mathbb{P}^1$ must be the projection of $f \cong \mathbb{P}^1 \times \mathbb{P}^1$ onto one of the two factors. Here we repeat an argument similar to the one given in the proof of (1.3.1). Let L be a line bundle on X whose restriction L_Z to Z induces the quadric fibration structure: we may assume L_Z to be very ample by (1.1.1). If D and D' are two general divisor in the linear system $|L_Z|$, then the intersection $D \cap D' \cap f$ consists of two points q_1 and q_2, not belonging to a same line of f. Let G_1 and G_2 denote the two fibers of β containing q_1 and q_2 respectively, and define:

$$Y := \bigcup_{\substack{G_i \subset F \\ F \text{ general}}} (G_1 \cup G_2).$$

Let $\overline{Y}$ be its Zariski closure in X. By intersecting with two curves belonging to the two different rulings of a fiber $f \cong \mathbb{P}^1 \times \mathbb{P}^1$, we see that $\overline{Y}$, F and K_X are linear independent line bundles. $\square$

Therefore $\rho(Z) = 3$, hence Z is a $\mathbb{P}^1 \times \mathbb{P}^1$- bundle over B with trivial monodromy by Proposition (1.3.1), and two lines C_1 and C_2 belonging to the two different rulings of a fiber $f \cong \mathbb{P}^1 \times \mathbb{P}^1$ generate two (negative) extremal rays of $\overline{NE}(Z)$. Let $R_i := \mathbb{R}_{\geq 0} \cdot [C_i]$ denote such rays and $\phi_i : Z \to S_i$ be the corresponding contraction morphisms.

Claim (4.2.2). *One of the two morphisms ϕ_i, say $\phi_1 : Z \to S_1$, extends to a morphism $\gamma : X \to S_1$.*

Proof. In order to prove the claim, we will show that $\overline{NE}(X/B)$ and $\overline{NE}(Z/B)$, hence $\overline{NE}(X)$ and $\overline{NE}(Z)$, have a common extremal ray R.

First of all, $(K_X + \det \mathcal{E})_Z = K_Z$ by adjunction formula (0.1). This implies that $K_X + \det \mathcal{E}$ is negative, via intersection, on $\overline{NE}(Z/B)$. By the inclusion of the cones, we see that $K_X + \det \mathcal{E}$ has negative intersection with some element of $\overline{NE}(X/B)$. Using convexity of the cone $\overline{NE}(X/B)$, we deduce that at least one of its two extremal rays has negative intersection with $K_X + \det \mathcal{E}$. *A fortiori* R has negative intersection with K_X, being $\mathcal{E}$ ample. Let C be a rational curve on X generating R and such that $l(R) = -K_X \cdot C$. Then

$$\Lambda(X, \mathcal{E}, R) := -(K_X + \det \mathcal{E}) \cdot C \geq 1.$$

This inequality, together with the property that, for any curve $C \subset X$, $\det \mathcal{E} \cdot C \geq \mathrm{rk}\mathcal{E}$ because of the ampleness of $\mathcal{E}$, give the following lower-bound to the length of R:

$$l(R) \geq \Lambda(X, \mathcal{E}, R) + \mathrm{rk}\mathcal{E} \geq n - 2.$$

Thus, due to Theorem (2.8), we find a range for the dimension of the exceptional locus of the contraction $\gamma : X \to W$ of the ray R. If $E := \mathrm{Exc}(\gamma)$ is the locus of the points of the curves belonging to R, and E_w the general fiber of $\gamma|_E$, we have in fact

$$2n \geq \dim E + \dim E_w \geq \dim X - 1 + l(R) \geq 2n - 3.$$

If we assume γ to be birational, then E is a divisor and $\dim(\gamma(E)) \leq 1$. There are two possibilities: either $E \supset Z$, or $E \supset Z$ is a effective divisor of Z, due to (0.3.2). In any case $\gamma|_Z$ is not finite, hence, by Lemma (2.9), it factors through a contraction of an extremal ray of $\overline{NE}(Z)$ and a finite morphism. but we know that the only contractions of extremal rays of Z are the two surjective morphisms $\phi_i : Z \to S_i$, and these don't agree with the cases above.

Therefore γ is a fiber-type morphism. Moreover, $\dim W \leq 3$. Let F be a general fiber of α, and compare the restriction $\gamma|_F$ of γ to F with $\beta : F \to \mathbb{P}^1$. Since both are contractions of extremal rays of $\overline{NE}(F)$ and the sum of the dimensions of respective general fibers exceeds the dimension of F, by the Kawamata Contraction Theorem we deduce that they are the same morphism:

$$\gamma|_F = \beta : F \to \mathbb{P}^1.$$

This implies that R is an extremal ray of $\overline{NE}(Z)$, in particular of $\overline{NE}(Z/B)$, say R_1. By applying the second part of (2.9), we see that restriction $\gamma|_Z$ of γ to Z factors through $\phi_1 : Z \to S_1$ and, possibly, a finite morphism $\nu : S_1 \to S'$. But R being an extremal ray of $\overline{NE}(X/B)$ means that γ is a factor of $\alpha : X \to B$, hence that it is a fiber-wise morphism over B. On the other hand, $\gamma|_F = \beta$ implies firstly that $\dim W = 2$, and secondly that ν should be $1:1$ relatively to the fibers of S_1 over B. We conclude in this way that $\nu : S_1 \to W$ is a finite birational morphism. But W is normal, as stated in Kawamata's Contraction Theorem, hence we have that, actually, $\gamma : X \to S_1$ by Zariski's Main Theorem. $\square$

If now G denotes a general fiber of γ, $G \cong \mathbb{P}^{n-2}$ and $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}$. We have to extend this property from the general G to *every* fiber of γ in order to conclude the proof of Theorem B. The following claim permits us to apply the Δ-genus semi-continuity argument.

Claim (4.2.3). *$\gamma : X \to S_1$ is an equidimensional morphism with reduced and irreducible fibers.*

Proof. The general fiber has dimension $n-2$. A bigger dimensional fiber would be either a divisor of X, that it is impossible being γ a surjection onto a surface, or a component of X, in contrast with the non-singularity of X. This proves the first part of the claim.

Let G be any fiber of γ. Then $g = Z \cap G \cong \mathbb{P}^1$ is a fiber of $\gamma|_Z = \phi_1 : Z \to S_1$. Let A be an irreducible component of G. Then $g \cap A = Z \cap A = (s_A)_0$ is positive dimensional, because $s_A \in \Gamma(A, \mathcal{E}_A)$ and $\mathrm{rk}\mathcal{E}_A < \dim A$. In fact $g \subset A$, being g an irreducible curve. We deduce that any component A is reduced, so being g. To show that G is irreducible, note that $\gamma|_Z$ is a smooth morphism onto S_1, hence, for any $p \in g$,

$$(\gamma|_Z)_{*,p} = \gamma_{*,p}|_{T_pZ} : T_pZ \to T_{\gamma(p)}S_1$$

is surjective. On the other hand, if G is reducible, then p belongs to the intersection of the irreducible components of G. Thus G is singular in p, and this implies that the map

$$\gamma_{*,p} : T_pX \to T_{\gamma(p)}S_1$$

is not surjective. Then contradiction follows by the inclusion $T_{Z,p} \subset T_{X,p}$. $\square$

To conclude the proof, let L be the line bundle on X defined, via isomorphism (0.3.2), as extension of a tautological line bundle of $\phi_1 : Z \to S_1$. Then $L_G \cong \mathcal{O}_{\mathbb{P}}(1)$ for a general fiber G of γ. Since $\overline{NE}(X/S_1)$ is generated by the numerical class of a line in G, we check easily that L is γ-ample by using the relative version of Kleiman criterion. To see that γ is a flat morphism, for any point $p \in S_1$ pick a smooth and general curve $C \subset S_1$ passing through p, and define $Y := \gamma^{-1}(p)$. Y is necessarily (reduced and) irreducible: indeed, $\gamma|_Y$ being equidimensional implies that there are not irreducible components over a point of C, and the existence of smooth fibers guarantees that there are no more than one irreducible component dominating C. Therefore $\gamma|_Y$ is flat by (2.6.1), hence γ does, since we can take p in S_1 arbitrarily. Thus, we can apply the Δ-genus semi-continuity (2.6) and Proposition (2.5) to conclude that every fiber G of γ is isomorphic to $\mathbb{P}^{n-2}$. Furthermore, $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}$, as follows by Theorem (2.2) applied to $(G, \mathcal{E}_G)$, and the first part of Theorem (2.5) applied to $(G, \det \mathcal{E}_G)$.

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Dipartimento di Matematica,
Università degli Studi di Genova
Via Dodecaneso 35,
16146 Genova (ITALY)
e-mail: defernex@dima.unige.it
defernex@math.uic.edu