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A TWO-FUNCTION EXTENSION OF A MINIMAX THEOREM

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In this note we extend a topological minimax theorem due to Ricceri ([2]) to the case of two functions.

1. Introduction and statement of the main result

Let X, Y be two non-empty sets and let φ be a real-valued function on $X \times Y$. Set

$$\varphi_* = \sup_{y \in Y} \inf_{x \in X} \varphi(x, y)$$

and

$$\varphi^* = \inf_{x \in X} \sup_{y \in Y} \varphi(x, y)$$

It is clear that

$$\varphi_* \leq \varphi^*.$$

This is called the trivial minimax inequality. The opposite inequality

$$\varphi^* \leq \varphi_*$$

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is called non-trivial minimax inequality and of course it is equivalent to the minimax equality

$$\varphi_* = \varphi^* \tag{1}$$

Starting from the pioneristic work of von Neumann ([8]), many results ensuring (1) were established. For an introductory bibliography see, for example, the classical survey of Simons ([5]) .

Now, let $f,g : X \times Y \to \mathbb{R}$, with $f(x,y) \leq g(x,y)$ for every $x \in X, y \in Y$. We call non-trivial minimax inequality involving f,g the following

$$f^* \leq g_* \tag{2}$$

So, if $f = g$, (2) is equivalent to (1). For a given minimax theorem for one function, it is an usual fact to see whether it is possible to find a two-function version of it. The most natural way to get this is to split the hypotheses on φ to f and g. For example, the two-function version of the most classical Fan-Sion's theorem ([7]) (Theorem 1.1 below) has been obtained by Simons ([6], Th. 1.4) (Theorem 1.2 below).

Theorem 1.1. *Let X be a nonempty compact convex subset of a topological vector space, Y a nonempty convex subset of a topological vector space, and let* $\Psi : X \times Y \to \mathbb{R}$ *be quasi-convex and lower semicontinuous in X, and quasi-concave and upper semicontinuous in Y. Then, (1) holds.*

Theorem 1.2. *Let X be a nonempty compact convex subset of a topological vector space, Y a nonempty convex subset of a topological vector space, let* $f : X \times Y \to \mathbb{R}$ *be quasi-concave in Y and lower semicontinuous in X, and let* $g : X \times Y \to \mathbb{R}$ *be upper semicontinuous in Y and quasi-convex in X, with* $f \leq g$ *on* $X \times Y$ *. Then, (2) holds.*

In [2], Ricceri proved the following result:

Theorem 1.3. *Let X be a topological space,* $I \subseteq \mathbb{R}$ *an open interval and* $\Psi : X \times I \to \mathbb{R}$ *a function satisfying the following conditions:*

a) for each $x \in X$*, the function* $\Psi(x,\cdot)$ *is quasi-concave and continuous*

b) for each $\lambda \in I$*, the function* $\Psi(\cdot,\lambda)$ *is lower semicontinuous and inf-compact*

c) for every $\lambda^* \in I$*, the function* $\Psi(\cdot,\lambda^*)$ *has only one global minimum point*

Under such hypotheses, (1) holds.

The aim of the present paper is to establish the following extension of Theorem 1.3 to two functions.

Theorem 1.4. *Let X be a topological space, $I \subseteq \mathbb{R}$ an interval and $f, g : X \times I \to \mathbb{R}$ two functions satisfying the following conditions:*

H1) for every $(x,\lambda) \in X \times I$ one has $f(x,\lambda) \leq g(x,\lambda)$

H2) the function g is lower semicontinuous in $X \times I$

H3) for every $x \in X$, the function $g(x,\cdot)$ is continuous

H4) for every $\lambda \in I$, the function $f(\cdot,\lambda)$ is lower semicontinuous and inf-compact

H5) for every $x \in X$, the function $f(x,\cdot)$ is quasi-concave

H6) for every $\lambda \in I$, the function $g(\cdot,\lambda)$ has only one global minimum point

Under such hypotheses, (2) holds.

To realize that when $f = g = \Psi$ Theorem 1.4 gives Theorem 1.3, one has to observe that conditions a), b) of Theorem 1.3, by Lemma 4 of [4], imply that the function Ψ is lower semicontinuous in $X \times I$.

Finally, for the reader's convenience, we recall the following result ([1], Th. 2.3) that will be the main tool used to prove Theorem 1.4.

For a generic set $S \subseteq X \times I$, for each $(x,\lambda) \in X \times I$, we set

$$S_x = \{\mu \in I : (x,\mu) \in S\}$$

$$S^\lambda = \{u \in X : (u,\lambda) \in S\}$$

Theorem 1.5. *Let X be a topological space, $I \subseteq \mathbb{R}$ a compact interval and $S, T \subseteq X \times I$. Assume that S is connected and $S^\lambda \neq \emptyset$ for all $\lambda \in I$, while T_x is non-empty and connected for all $x \in X$, and T^λ is open for all $\lambda \in I$.*

Then, one has $S \cap T \neq \emptyset$.

Remark 1.6. In [3], Theorem 1.3 has been extended to the case where I is an arbitrary convex set in a topological vector space. It is an open challenging problen to know whether the same holds for Theorem 1.4.

2. Proof of Theorem 1.4

We argue by contradiction. So, assume that

$$g_* < f^* \tag{3}$$

and fix $r \in \mathbb{R}$ satisfying

$$g_* < r < f^* \tag{4}$$

For each $\lambda \in I$, let x_λ be the only global minimum point of $g(\cdot,\lambda)$. Let us show that the function $\lambda \to x_\lambda$ is continuous. To this end, it is clearly enough to show that if $\{\lambda_n\}$ is a sequence in I converging to $\bar{\lambda} \in I$, then $x_{\bar{\lambda}}$ is a cluster point of x_{λ_n}. Let $[a,b] \subseteq I$ a compact interval containing the sequence $\{\lambda_n\}$. From H5) it follows that

$$\cup_{\lambda\in[a,b]} \{x \in X : f(x,\lambda) \leq r\} \subseteq \{x \in X : f(x,a) \leq r\} \cup \{x \in X : f(x,b) \leq r\}$$

and so, from H1)

$$\cup_{\lambda\in[a,b]} \{x \in X : g(x,\lambda) \leq r\} \subseteq \{x \in X : f(x,a) \leq r\} \cup \{x \in X : f(x,b) \leq r\} \tag{5}$$

Since, for every $n \in \mathbb{N}$, x_{λ_n} belongs to the left-hand side of (5), from H4) and (5) it follows that the sequence $\{x_{\lambda_n}\}$ is contained in a compact set and so it has a cluster point $\bar{x}$. Then, $(\bar{x},\bar{\lambda})$ is a cluster point of $\{(x_{\lambda_n},\lambda_n)\}$ in $X \times [a,b]$. Let us show that

$$g(\bar{x},\bar{\lambda}) \leq \limsup_n g(x_{\lambda_n},\lambda_n) \tag{6}$$

Assume the contrary. Choose η such that

$$\limsup_n g(x_{\lambda_n},\lambda_n) < \eta < g(\bar{x},\bar{\lambda})$$

This implies that there exist $\alpha \in \mathbb{N}$ and, by H2), a neighbourhood U of $(\bar{x},\bar{\lambda})$ such that, for every $(x,\lambda) \in U$ and every $n > \alpha$, one has

$$g(x_{\lambda_n},\lambda_n) < \eta < g(x,\lambda) \tag{7}$$

Since $(\bar{x},\bar{\lambda})$ is a cluster point of $\{(x_{\lambda_n},\lambda_n)\}$, there exists $\bar{n} > \alpha$ such that $(x_{\lambda_{\bar{n}}},\lambda_{\bar{n}}) \in U$ and so, by (7)

$$g(x_{\lambda_{\bar{n}}},\lambda_{\bar{n}}) < \eta < g(x_{\lambda_{\bar{n}}},\lambda_{\bar{n}})$$

that is absurde.

Now, let us fix $x \in X$. Taking (6) and H3) into account, we have

$$g(\bar{x}, \bar{\lambda}) \leq \limsup_{n} g(x_{\lambda_n}, \lambda_n) \leq \lim_{n} g(x, \lambda_n) = g(x, \bar{\lambda})$$

Thus, $\bar{x}$ is a global minimum point for $g(x, \bar{\lambda})$, and so $\bar{x} = x_{\bar{\lambda}}$. This prove the claim.

Now, let $\{I_n\}$ be an increasing sequence of compact intervals whose union is I. We claim that there exists $n \in \mathbb{N}$ such that

$$\sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \tag{8}$$

Arguing by contradiction, suppose that, for every $n \in \mathbb{N}$, one has

$$\inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \leq \sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda)$$

For every $n \in \mathbb{N}$, let us put

$$C_n = \left\{ x \in X : \sup_{\lambda \in I_n} f(x, \lambda) \leq r \right\}$$

Each set C_n is non-empty: otherwise, by (4), one would have

$$r \leq \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \leq \sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda) \leq g_* < r$$

Since the sequence $\{I_n\}$ is increasing, the sequence $\{C_n\}$ is decreasing. Summarizing, $\{C_n\}$ is a decreasing sequence of non-empty closed and compact sets. So, there exists $x^* \in \bigcap_{n \in \mathbb{N}} C_n$.

From the fact that for every $n \in \mathbb{N}$ and for every $\lambda \in I_n$ one has $f(x^*, \lambda) \leq r$, it follows that $f(x^*, \lambda) \leq r$ for every $\lambda \in I$ and so one can conclude that

$$\inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda) \leq r$$

against (4). Now, fix $n \in \mathbb{N}$ for which (8) is satisfied and choose s such that

$$\sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda) < s < \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \tag{9}$$

Let us put

$$S = \{(x_\lambda, \lambda) : \lambda \in I_n\}$$

$$T = \{(x,\lambda) \in X \times I_n : f(x,\lambda) > s\}$$

Thanks to the continuity of the function $\lambda \to x_\lambda$ we can say that the set S is connected in $X \times I_n$. Observe now that, for every $x \in X$, the set T_x is non-empty from (9) and connected from H5) and that, for every $\lambda \in I_n$, the set T^λ is open from H4). Then, thanks to Theorem 1.5, one has $S \cap T \neq \emptyset$. But, for every $\lambda \in I_n$, one has, from (9) and H1)

$$f(x_\lambda,\lambda) \leq g(x_\lambda,\lambda) = \inf_{x\in X} g(x,\lambda) \leq \sup_{\lambda\in I_n} \inf_{x\in X} g(x,\lambda) < s$$

and so, $S \cap T = \emptyset$. This contradiction completes the proof.

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