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ASYMPTOTIC ANALYSIS OF A QUASI-STATIC CONTACT PROBLEM FOR THERMOVISCOELASTIC MATERIALS WITH TRESCA FRICTION AND SOURCE TERM

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This paper deals with the asymptotic behavior of a quasi-static contact problem for thermoviscoelastic materials in a three dimensional thin domain Ω^{ε} with nonlinear friction of Tresca type and nonlinear source term. We will establish a variational formulation for the problem and we give the Theorem of the existence of the weak solution. We then study the asymptotic behavior when one dimension of the domain tends to zero. In which case, the uniqueness result of the displacement and the temperature for the limit problem is also proved.

1. Introduction

Frictional contact represents a phenomenon that arises naturally in many applications. One of the most important examples is sliding systems, such as brakes, clutches, and seals. They are also found in industrial processes and everyday life, such as train wheels on rails, shoes on floors, and tectonic plates, etc. Contact and friction processes are invariably accompanied by heat generation which may be considerable. Thermal effects in contact processes affect the composition and stiffness of the contacting surfaces, and cause thermal stresses in the

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contacting bodies.

Frictional contacts of thermoviscoelastic bodies are complicated, nonlinear and time dependent problems. The introduction of friction with its irreversible character makes the problem more difficult. Predicting the behavior of thermoviscoelastic contacting bodies in such situations is of considerable applied importance in the field of mechanical industry and even the field of biology. In the mathematical literature, contact problems involving elastic or viscoelastic bodies with thermal effects have been widely investigated, yielding various significant results. The majority of these results concern the solvability of contact problems, the uniqueness of solutions, and continuous dependence on data, as well as theoretical results in numerical analysis, simulation, and other computational issues. Let us mention some of these results. Navarro in [21] studied the existence, uniqueness and asymptotic stability of solutions to the evolution equations for a body composed of an inhomogeneous anisotropic linear thermoviscoelastic material. In [1], the authors investigate the asymptotic behavior of solutions to the initial boundary value problem for one-dimensional thermoelastic mixtures, where the exponential stability of the corresponding semigroup and the lack of exponential stability of the corresponding semigroup are established. The numerical approximation of a quasi-static Signorini contact problem for a thin homogeneous viscoelastic rod that undergoes longitudinal deformations due to changes in its temperature has been studied in [11]. Cao-Rial et al. in [8] studied the mathematical analysis and the existence and uniqueness to solution of a dynamic contact problem in thermoelasticity, when the contact is governed by a normal damped response function and the Duhamel-Neumann relation. In paper [5], Bartosz et al. they established the existence of a weak solution to the problem of the evolution of the thermomechanical and electric state of a thermoviscoelastic thermistor that is in frictional contact with a reactive foundation by using time delays, prior estimations and convergence method. Copetti studied in the work [10] the existence, uniqueness and numerical approximation of the solution of a hyperbolic-parabolic problem that models the longitudinal deformations of a thermoviscoelastic rod supported unilaterally by an elastic obstacle. The existence of the weak solution of a model for the dynamic frictional contact between a thermoviscoelastic object and an obstacle was established in ref [20]. In the paper [9], the authors considered a general model for the quasi-static process of thermoviscoelastic contact between a deformable body and a rigid obstacle. They assume that the material behaves according to the Kelvin-Voigt constitutive law with added thermal and diffusion effects. The existence of solutions is proven as the limit of solutions to a penalized problem. They also gave some computational results where the influence of diffusion and viscosity are illustrated in contact.

On the other hand, asymptotic methods are widely applied to derive and justify two-dimensional models for fluid mechanics problems and three-dimensional solids in thin domains, such as beams, sheets, shells and lubrication problems, which in turn have many industrial applications, see ref [22]. In the following, we will cite only some works that use the asymptotic expansion method to derive many models because the literature is very extensive on this topic. In the work [13], the authors are interested in studying a mathematical model describing the quasistatic frictional contact between a piezoelectric body and a deformable foundation with the normal compliance condition and Tresca's friction law in a thin domain, where the limit problem is obtained when one of the dimensions of the domain tends to zero. The asymptotic analysis of a dynamical problem of isothermal elasticity with non linear dissipative term and non linear friction of Tresca type was studied in [14]. In [4], Bayada and Lhalouani studied the asymptotic behavior of a three-body structure consisting of a thin elastic layer placed between a rigid body and an elastic body. The contact between the two elastic bodies is assumed to be one-sided with Coulomb friction and the Lamé coefficients of the thin layer depend on its thickness ε. They demonstrated that when the joint thickness tends towards zero the initial structure can be replaced by a two-body structure with new conditions at the interface. They also provided some numerical simulations to shed light on the theoretically obtained results. Rodríguez-Arós and Viaño in ([23], [24]) justify two models for the bending stretch of a viscoelastic bar using the asymptotic expansion method. The authors in [12], studied the asymptotic analysis of the solutions of the linear viscoelastic problem with nonlinear dissipative and source terms in a three-dimensional thin domain. Recently Dilmi et al. [16] used the asymptotic behavior method to justify the two-dimensional limit problem of an elasticity system model with nonlinear source and dissipative terms in a three-dimensional thin domain. Some research papers have been written dealing with both the asymptotic analysis of an incompressible fluid in a three-dimensional thin domain, when one dimension of the fluid domain tends to zero can be found in ([3], [15]).

The goal of this paper is to study the asymptotic behavior of a model for the quasi-static frictional contact between a thermoviscoelastic body and an obstacle in a three dimensional thin domain Ω^{ε} with non linear friction of Tresca type and non linear source term $|u^{\varepsilon}|^{p-2}u^{\varepsilon}$. We use asymptotic analysis to justify the convergence to a 2D limit problem when the thickness tends to zero.

This article is organized as follows. In Section 2, we introduce the physical setting and the classical formulation of the problem we recall the weak formulation of our problem considered, after which we give the Theorem of the existence of the solution to this problem under suitable hypotheses on the data using the

Faedo-Galerkin strategy. In Section 3, we first prove some estimates on displacement and temperature, independent of the small parameter ε, using several inequalities. Then, the passage to the limit on ε, permits us to the obtention of a two dimensional limit problem and all the properties of our original problem.

2. The contact problem and weak formulation

Let us consider a thermoviscoelastic body occupying a bounded domain Ω^{ε} of the space $\mathbb{R}^3$. The boundary of the domain Γ^{ε} is assumed to be composed of three portions : Λ the bottom of the domain, Γ_1^{ε} the upper surface, and Γ_L^{ε} the lateral surface. We suppose that the Dirichlet boundary conditions are satisfied on $\Gamma_1^{\varepsilon} \cup \Gamma_L^{\varepsilon}$, for the displacement and the temperature. On the bottom surface, the normal velocity is null. On the potential contact surface Λ the body may come in frictional contact with a reactive thermally active foundation, so we assume that the friction is sufficiently large and the tangential velocity is unknown and satisfies the Tresca boundary condition.

2.1. Mathematical model

First we define the physical domain Ω^{ε}. Let Λ be a fixed bounded domain of $\mathbb{R}^3$ of equation $x_3 = 0$. We suppose that Λ has a Lipschitz continuous boundary and is the bottom of the domain. The upper surface Γ_1^{ε} is defined by $x_3 = \varepsilon h(x')$. $\Gamma_L^{\varepsilon} = \{(x', x_3) \in \Omega^{\varepsilon}, x' \in \partial\Lambda, 0 < x_3 < \varepsilon h(x')\}$ the lateral surface . We introduce a small parameter ε, that will tend to zero, and a function $h(\cdot)$ on the closure of Λ such that

$$0 < \underline{\mathrm{h}} \leq h(x') \leq \bar{h}, \forall x' \in \Lambda.$$

The domain Ω^{ε} is defined by

$$\Omega^{\varepsilon} = \{(x', x_3) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \Lambda, 0 < x_3 < \varepsilon h(x_1, x_2)\},$$

where $\Gamma^{\varepsilon} = \Gamma_1^{\varepsilon} \cup \Gamma_L^{\varepsilon} \cup \Lambda$ its boundary. Let $T > 0$, we denote by $u^{\varepsilon} : \Omega^{\varepsilon} \times [0, T] \longmapsto \mathbb{R}^3$ the displacement vectors and θ^{ε}: $\Omega^{\varepsilon} \times [0, T] \longmapsto \mathbb{R}$ the temperature field. Also we denote by $\mathbb{D}(\cdot)$ the displacement-strain relation

$$\mathbb{D}_{ij}(u^{\varepsilon}) = \frac{1}{2}\left(\partial_{x_j} u_i^{\varepsilon} + \partial_{x_i} u_j^{\varepsilon}\right), \ 1 \leq i, j \leq 3.$$

The stress-strain-temperature relation is given by (see [2], [7])

$$\sigma_{ij}^{\varepsilon}(u^{\varepsilon}, \theta^{\varepsilon}) = 2(\mu_0 + \mu(\theta^{\varepsilon}))\mathbb{D}_{ij}(u^{\varepsilon}) + 2\nu(\theta^{\varepsilon})\mathbb{D}_{ij}(\partial_t u^{\varepsilon}) - \eta_0 \theta^{\varepsilon} \delta_{ij},$$

where $\mu_0 + \mu(\cdot)$ is the Lamé coefficient, $\nu(\cdot)$ is the viscosity coefficient, η_0 the thermal expansion coefficient and δ_{ij} is the Krönecker symbol. To simplify we will put $\mu_0 \equiv 1$.

Assuming small deformations, the system of the equation of motion and the energy balance, respectively, are:

$$\left\{ \begin{array}{l} -\mathrm{div}\sigma^{\varepsilon}\left(u^{\varepsilon},\theta^{\varepsilon}\right)+\alpha^{\varepsilon}\left|u^{\varepsilon}\right|^{p-2}u^{\varepsilon}=f^{\varepsilon}, \\ \partial_t\theta^{\varepsilon}-\mathrm{div}\left(\beta^{\varepsilon}\left(x\right)\nabla\theta^{\varepsilon}\right)+\eta_1\mathrm{div}\left(\partial_t u^{\varepsilon}\right)=q^{\varepsilon}, \end{array} \right. \quad \text{in } \Omega^{\varepsilon}\times]0,T[, \tag{1}$$

where $f^{\varepsilon}=(f_i^{\varepsilon})_{1\le i\le 3}$ stands for the exterior volume force, q^{ε} is a given volume heat source, $\beta^{\varepsilon}\left(\cdot\right)$ represents the thermal conductivity fonction and $p>1$, α^{ε}, $\eta_1>0$ are constants.

We turn to describe the boundary conditions, so for every element u^{ε} we denote by u_n^{ε} and u_{τ}^{ε} the normal and the tangential components of u^{ε} on the boundary Λ given by

$$u_n^{\varepsilon}=u^{\varepsilon}.n, \quad u_{\tau}^{\varepsilon}=u^{\varepsilon}-u^{\varepsilon}.n.$$

Also, for a regular function σ^{ε}, we define its normal and tangential components by

$$\sigma_n^{\varepsilon}=\left(\sigma_n^{\varepsilon}\right).n, \;\; \sigma_{\tau}^{\varepsilon}=\sigma^{\varepsilon}.n-\left(\sigma_n^{\varepsilon}\right).n,$$

where $n=(n_1,n_2,n_3)$ be the unit outward normal vector to Γ^{ε}.

• We assume that the displacement is known on $\Gamma_1^{\varepsilon}\times]0,T[$ and on $\Gamma_L^{\varepsilon}\times]0,T[$

$$\begin{array}{rcl} u^{\varepsilon} & = & 0 \text{ on } \Gamma_1^{\varepsilon}\times]0,T[, \\ u^{\varepsilon} & = & 0 \text{ on } \Gamma_L^{\varepsilon}\times]0,T[. \end{array} \tag{2}$$

• On $\Lambda\times]0,T[$ the normal velocity satisfies the following condition

$$\partial_t u^{\varepsilon}.n=0 \text{ on } \Lambda\times]0,T[. \tag{3}$$

• The tangential velocity on $\Lambda\times]0,T[$ is unknown and satisfies the Tresca friction law with a friction coefficient k^{ε}

$$\left\{ \begin{array}{l} \left|\sigma_{\tau}^{\varepsilon}\right|<k^{\varepsilon}\left(x'\right)\Rightarrow\left(\partial_t u^{\varepsilon}\right)_{\tau}=0, \\ \left|\sigma_{\tau}^{\varepsilon}\right|=k^{\varepsilon}\left(x'\right)\Rightarrow\exists\rho>0, \text{ such that } \left(\partial_t u^{\varepsilon}\right)_{\tau}=-\rho\sigma_{\tau}^{\varepsilon}. \end{array} \right. \quad \text{on } \Lambda\times]0,T[, \tag{4}$$

Here $|\cdot|$ denotes the $\mathbb{R}^2$ Euclidean norm.

• For the temperature, we suppose that

$$\theta^{\varepsilon}=0 \text{ on } \Gamma_1^{\varepsilon}\times]0,T[\cup\Gamma_L^{\varepsilon}\times]0,T[, \tag{5}$$

and

$$\beta^{\varepsilon}\left(x\right)\nabla\theta^{\varepsilon}.n=-\gamma^{\varepsilon}\theta^{\varepsilon} \text{ on } \Lambda\times]0,T[, \tag{6}$$

where $\gamma^{\varepsilon}>0$. It is important to note that the frictional heat generation term is not included in the thermal boundary condition (6). This omission is justified

under the assumption that the heat produced by friction is negligible in comparison to the volumetric heat source q^ε and the bulk thermo-mechanical coupling effects in the system.
The problem is to find $(u^\varepsilon,\theta^\varepsilon)$ satisfying the problem $(1)-(6)$ with the following initial conditions

$$u^\varepsilon(x,0)=u_0(x),\quad \theta^\varepsilon(x,0)=\theta_0(x), \forall x\in\Omega^\varepsilon. \tag{7}$$

2.2. Weak formulation

Now, to derive the variational formulation of the problem, let us first, we introduce the functional spaces

$$\mathcal{K}^\varepsilon=\left\{v\in H^1(\Omega^\varepsilon)^3 : v=0 \text{ on } \Gamma_1^\varepsilon\cup\Gamma_L^\varepsilon,\, v.n=0 \text{ on } \Lambda\right\},$$

$$\mathcal{H}^\varepsilon_{\Gamma_1^\varepsilon\cup\Gamma_L^\varepsilon}=\left\{\psi\in H^1(\Omega^\varepsilon) : \psi=0 \text{ on } \Gamma_1^\varepsilon\cup\Gamma_L^\varepsilon\right\},$$

which are the Hilbert spaces of admissible displacements and temperatures, respectively.
Let $(u^\varepsilon,\theta^\varepsilon)$ be the solution of $(1)-(7)$, multiplying the first equation of (1) by $(\varphi-\partial_t u^\varepsilon)$ and the second equation of (1) by ψ, then integrate on Ω^ε. Using Green's formula it is easy to obtain the following contrast problem.

Problem. Find a pair $t\mapsto(u^\varepsilon,\theta^\varepsilon)$ of $[0,T]\longrightarrow\mathcal{K}^\varepsilon\times\mathcal{H}^\varepsilon_{\Gamma_1^\varepsilon\cup\Gamma_L^\varepsilon}$ verifying

$$\begin{aligned}&\mathcal{A}_{\mu(\cdot)+1}\left(\theta^\varepsilon;u^\varepsilon,\varphi-\partial_t u^\varepsilon\right)+\mathcal{A}_{\nu(\cdot)}\left(\theta^\varepsilon;\partial_t u^\varepsilon,\varphi-\partial_t u^\varepsilon\right)\\&-\left(\eta_0\theta^\varepsilon,\operatorname{div}\left(\varphi-\partial_t u^\varepsilon\right)\right)+\left(\alpha^\varepsilon\left|u^\varepsilon\right|^{p-2}u^\varepsilon,\varphi-\partial_t u^\varepsilon\right)+j^\varepsilon(\varphi)-j^\varepsilon\left(\partial_t u^\varepsilon\right)\\&\geq\left(f^\varepsilon,\varphi-\partial_t u^\varepsilon\right),\quad\forall\varphi\in\mathcal{K}^\varepsilon;\end{aligned}\tag{8}$$

$$\begin{aligned}&\left(\partial_t\theta^\varepsilon,\psi\right)+\mathcal{B}\left(\theta^\varepsilon,\psi\right)+\left(\eta_1\operatorname{div}\left(\partial_t u^\varepsilon\right),\psi\right)+\left(\gamma^\varepsilon\theta^\varepsilon,\psi\right)_\Lambda\\&=\left(q^\varepsilon,\psi\right),\forall\psi\in\mathcal{H}^\varepsilon_{\Gamma_1^\varepsilon\cup\Gamma_L^\varepsilon},\end{aligned}\tag{9}$$

$$u^\varepsilon(x,0)=u_0(x),\quad\theta^\varepsilon(x,0)=\theta_0(x).\tag{10}$$

With

$$\mathcal{A}_{\kappa(\cdot)}(w;u,v)=2\int_{\Omega^\varepsilon}\kappa(w)\mathbb{D}(u):\mathbb{D}(v)\,dx;\ \mathbb{D}(u):\mathbb{D}(v)=\sum_{i,j=1}^{3}\mathbb{D}_{ij}(u):\mathbb{D}_{ij}(v),$$

$$\mathcal{B}(\theta,\psi)=\int_{\Omega^\varepsilon}\beta^\varepsilon(x)\nabla\theta.\nabla\psi dx;\quad\left(\gamma^\varepsilon\theta^\varepsilon,\psi\right)_\Lambda=\int_\Lambda\gamma^\varepsilon\theta\psi dx',$$

and

$$j^\varepsilon(v)=\int_\Lambda k^\varepsilon\left(x'\right)\sqrt{(v_1)^2+(v_2)^2}dx';\ (f,v)=\sum_{i=1}^{3}\int_{\Omega^\varepsilon}f_iv_idx.$$

Theorem 2.1. *Under the following assumptions*

$$f^{\varepsilon} \in L^2\left(0,T;L^2(\Omega^{\varepsilon})^3\right);\; q^{\varepsilon}, \partial_t q^{\varepsilon} \in L^2\left(0,T;L^2(\Omega^{\varepsilon})\right),$$

$$u_0 \in H^1(\Omega^{\varepsilon})^3 \cap L^p(\Omega^{\varepsilon})^3;\quad \theta_0 \in H^1(\Omega^{\varepsilon}),$$

$$\begin{gathered}\beta^{\varepsilon}(\cdot) \in C^1(\Omega^{\varepsilon}),\, 0<\beta^* \le \beta^{\varepsilon}(\cdot) \le \beta^{**},\\ k^{\varepsilon}(\cdot) \in C^1(\Lambda),\, 0<k_*^{\varepsilon} \le k^{\varepsilon}(\cdot) \le k_{**}^{\varepsilon},\\ \mu(\cdot), \nu(\cdot) \in C^1(\mathbb{R}),\; 0<\mu^* \le \mu(\cdot) \le \mu^{**},\\ 0<\nu^* \le \nu(\cdot) \le \nu^{**},\ \text{such that } \nu^* > (12\eta_1)^2.\end{gathered}$$

Then, there exists a pair $(u^{\varepsilon}, \theta^{\varepsilon})$ *solution to Problem* $(8)-(10)$, *such that*

$$\begin{aligned} u^{\varepsilon} &\in L^{\infty}\left(0,T;H^1(\Omega^{\varepsilon})^3\right) \cap L^2\left(0,T;L^p(\Omega^{\varepsilon})^3\right),\\ \partial_t u^{\varepsilon} &\in L^2\left(0,T;H^1(\Omega^{\varepsilon})^3\right),\end{aligned}$$

$$\theta^{\varepsilon} \in L^{\infty}\left(0,T;H^1(\Omega^{\varepsilon})\right),\ \partial_t\theta^{\varepsilon} \in L^2\left(0,T;L^2(\Omega^{\varepsilon})\right).$$

Proof. The proof is based on the regularization method, which is based on an approximation of non-differentaible term $j^{\varepsilon}(\cdot)$ by a family of differentaible once $j_{\zeta}^{\varepsilon}(\cdot)$, where

$$j_{\zeta}^{\varepsilon}(v) = \int_{\Lambda} k^{\varepsilon}(x')\,\Phi_{\zeta}\left(|v_{\tau}|^2\right)dx' \text{ with } \Phi_{\zeta}(\lambda) = \frac{1}{1+\zeta}|\lambda|^{1+\zeta},\ \zeta>0,$$

and we build a problem approximate

$$\begin{aligned}&\mathcal{A}_{\mu(\cdot)+1}\left(\theta_{\zeta}^{\varepsilon}; u_{\zeta}^{\varepsilon}, \varphi\right) + \mathcal{A}_{\nu(\cdot)}\left(\theta_{\zeta}^{\varepsilon}; \partial_t u_{\zeta}^{\varepsilon}, \varphi\right) - \left(\eta_0\theta_{\zeta}^{\varepsilon}, \operatorname{div}(\varphi)\right)\\ &+ \left(\alpha^{\varepsilon}\left|u_{\zeta}^{\varepsilon}\right|^{p-2} u_{\zeta}^{\varepsilon}, \varphi\right) + \left(j_{\zeta}^{\varepsilon}\right)'\left(\partial_t u_{\zeta}^{\varepsilon}\right)\\ &= (f^{\varepsilon}, \varphi),\ \ \forall\varphi \in \mathcal{K}^{\varepsilon};\end{aligned}$$

$$\begin{aligned}&\left(\partial_t\theta_{\zeta}^{\varepsilon}, \psi\right) + \mathcal{B}\left(\theta_{\zeta}^{\varepsilon}, \psi\right) + \left(\eta_1\operatorname{div}\left(\partial_t u_{\zeta}^{\varepsilon}\right), \psi\right) + \left(\gamma^{\varepsilon}\theta_{\zeta}^{\varepsilon}, \psi\right)_{\Lambda}\\ &= (q^{\varepsilon}, \psi), \forall\psi \in \mathcal{H}_{\Gamma_1^{\varepsilon}\cup\Gamma_L^{\varepsilon}}^{\varepsilon},\end{aligned}$$

$$u_{\zeta}^{\varepsilon}(x,0) = u_0(x),\quad \theta_{\zeta}^{\varepsilon}(x,0) = \theta_0(x).$$

Using Galerkin's method, we show that there exists a solution $\left(u_{\zeta}^{\varepsilon}, \theta_{\zeta}^{\varepsilon}\right)$ of this last approximate problem. Then, the limit of $\left(u_{\zeta}^{\varepsilon}, \theta_{\zeta}^{\varepsilon}\right)$ when ζ tends to zero is a solution of $(8)-(10)$. See the works of Duvaut and Lions [17], Lions [19].

□

3. Asymptotic analysis

For the asymptotic analysis of problem (8) – (10), we use the approach which consist in transposing the problem initially posed in the domain Ω^{ε} which depend on a small parameter ε in an equivalent problem posed in the fixed domain Ω which is independent of ε. For that, we introduce a change of the variable $z = x_3/\varepsilon$, so for (x, x_3) in Ω^{ε} we have (x, z) in

$$\Omega = \{(x', z) \in \mathbb{R}^3 : \left(x', 0\right) \in \Lambda,\ 0 < z < h(x')\},$$

and we denote by $\Gamma = \Gamma_1 \cup \Gamma_L \cup \Lambda$ its boundary, then we define the following functions in Ω

$$\begin{aligned} &\hat{u}_i^{\varepsilon}\left(x', z, t\right) = u_i^{\varepsilon}\left(x', x_3, t\right), i = 1, 2, \\ &\hat{u}_3^{\varepsilon}\left(x', z, t\right) = \varepsilon^{-1} u_3^{\varepsilon}\left(x', x_3, t\right), \\ &\hat{\theta}^{\varepsilon}\left(x', z, t\right) = \theta^{\varepsilon}\left(x', x_3, t\right). \end{aligned} \tag{11}$$

For the data of problem (8) – (10), we suppose that they depend of ε in the following manner

$$\begin{aligned} \hat{f}\left(x', z, t\right) &= \varepsilon^2 f^{\varepsilon}\left(x', x_3, t\right), \\ \hat{q}\left(x', z, t\right) &= \varepsilon^2 q^{\varepsilon}\left(x', x_3, t\right), \\ \hat{k}\left(x'\right) &= \varepsilon k^{\varepsilon}\left(x'\right), \\ \hat{\beta}\left(x', z\right) &= \beta^{\varepsilon}\left(x', x_3\right), \\ \hat{\alpha} &= \varepsilon^2 \alpha^{\varepsilon}, \\ \hat{\gamma} &= \varepsilon \gamma^{\varepsilon}, \end{aligned} \tag{12}$$

with $\hat{f}$, $\hat{q}$, $\hat{k}$, $\hat{\beta}$, $\hat{\alpha}$ and $\hat{\gamma}$ independent of ε.
Now we introduce the functional framework on Ω. For this, we note

$$\mathcal{K} = \left\{\varphi \in H^1(\Omega)^3 : \varphi = 0 \text{ on } \Gamma_L \cup \Gamma_1 \text{ and } \varphi.n = 0 \text{ on } \Lambda\right\},$$

$$\mathcal{H}_{\Gamma_1 \cup \Gamma_L} = \left\{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_1 \cup \Gamma_L\right\},$$

$$\Pi = \left\{\varphi \in H^1(\Omega)^2 : \varphi = (\varphi_1, \varphi_2), \varphi_i = 0 \text{ on } \Gamma_L \cup \Gamma_1, i = 1, 2\right\},$$

$$\mathcal{V}_z = \left\{v : v \in L^2(\Omega);\ \partial_z v \in L^2(\Omega) \text{ and } v = 0 \text{ on } \Gamma_1\right\},$$

$\mathcal{V}_z$ is a Banach space for the norm

$$\|v\|_{\mathcal{V}_z} = \left(\|v\|^2_{L^2(\Omega)} + \|\partial_z v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Assuming $(11)-(12)$, then problem $(8)-(10)$ leads to the following form

Problem. Find the pair $t\longmapsto\left(\hat{u}^{\varepsilon},\hat{\theta}^{\varepsilon}\right)$ of $[0,T]\longrightarrow\mathcal{K}\times\mathcal{H}_{\Gamma_1\cup\Gamma_L}$ verifying

$$\begin{aligned}
&\mathcal{A}_{\hat{\mu}(\cdot)+1}\left(\hat{\theta}^{\varepsilon};\hat{u}^{\varepsilon},\hat{\varphi}-\partial_t\hat{u}^{\varepsilon}\right)+\mathcal{A}_{\hat{\nu}(\cdot)}\left(\hat{\theta}^{\varepsilon};\hat{u}^{\varepsilon},\hat{\varphi}-\partial_t\hat{u}^{\varepsilon}\right)\\
&-\eta_0\sum_{i=1}^{2}\left(\varepsilon^2\hat{\theta}^{\varepsilon},\partial_{x_i}\left(\hat{\varphi}_i-\partial_t\hat{u}_i^{\varepsilon}\right)\right)-\eta_0\left(\varepsilon^2\hat{\theta}^{\varepsilon},\partial_z\left(\hat{\varphi}_3-\partial_t\hat{u}_3^{\varepsilon}\right)\right)\\
&+\hat{\alpha}\sum_{i=1}^{2}\left(|\hat{u}_i^{\varepsilon}|^{p-2}\,\hat{u}_i^{\varepsilon},\hat{\varphi}_i-\partial_t\hat{u}_i^{\varepsilon}\right)+\hat{\alpha}\varepsilon^{p}\left(|\hat{u}_3^{\varepsilon}|^{p-2}\,\hat{u}_3^{\varepsilon},\hat{\varphi}_3-\partial_t\hat{u}_3^{\varepsilon}\right)\\
&+\hat{J}(\hat{\varphi})-\hat{J}(\partial_t\hat{u}^{\varepsilon})\\
&\geq\sum_{i=1}^{2}\left(\hat{f}_i,\hat{\varphi}_i-\partial_t\hat{u}_i^{\varepsilon}\right)+\varepsilon\left(\hat{f}_3,\hat{\varphi}_3-\partial_t\hat{u}_3^{\varepsilon}\right),\ \forall\hat{\varphi}\in\mathcal{K};
\end{aligned}\tag{13}$$

$$\begin{aligned}
&\left(\varepsilon^2\partial_t\hat{\theta}^{\varepsilon},\hat{\psi}\right)+\hat{\mathcal{B}}\left(\hat{\theta}^{\varepsilon},\hat{\psi}\right)+\eta_1\sum_{i=1}^{2}\left(\varepsilon^2\partial_{x_i}\left(\partial_t\hat{u}_i^{\varepsilon}\right),\hat{\psi}\right)\\
&+\eta_1\left(\varepsilon^2\partial_z\left(\partial_t\hat{u}_3^{\varepsilon}\right),\hat{\psi}\right)+\left(\hat{\gamma}\hat{\theta}^{\varepsilon},\hat{\psi}\right)_{\Lambda}\\
&=\left(\hat{q},\hat{\psi}\right),\ \forall\hat{\psi}\in\mathcal{H}_{\Gamma_1\cup\Gamma_L},
\end{aligned}\tag{14}$$

$$\hat{u}^{\varepsilon}(0)=\hat{u}_0,\quad\hat{\theta}^{\varepsilon}(0)=\hat{\theta}_0,\tag{15}$$

where

$$\hat{J}(\hat{\varphi})=\int_{\Lambda}\hat{k}\,|\hat{\varphi}_{\tau}|\,dx',$$

$$\begin{aligned}
\mathcal{A}_{\kappa(\cdot)}\left(\hat{w};\hat{u},\hat{v}\right)=\ &\varepsilon^2\sum_{i,j=1}^{2}\int_{\Omega}\kappa\left(\hat{\theta}^{\varepsilon}\right)\left(\partial_{x_j}\hat{u}_i+\partial_{x_i}\hat{u}_j\right)\partial_{x_j}\hat{v}_i\,dx'dz\\
&+\sum_{i=1}^{2}\int_{\Omega}\kappa\left(\hat{\theta}^{\varepsilon}\right)\left(\partial_z\hat{u}_i^{\varepsilon}+\varepsilon^2\partial_{x_i}\hat{u}_3^{\varepsilon}\right)\left(\partial_z\hat{v}_i+\varepsilon^2\partial_{x_i}\hat{v}_3\right)dx'dz\\
&+2\varepsilon^2\int_{\Omega}\kappa\left(\hat{\theta}^{\varepsilon}\right)\partial_z\hat{u}_3^{\varepsilon}\partial_z\hat{v}_3\,dx'dz,
\end{aligned}$$

and

$$\hat{\mathcal{B}}\left(\hat{\theta}^{\varepsilon},\hat{\psi}\right)=\sum_{i=1}^{2}\int_{\Omega}\varepsilon^2\hat{\beta}\,\partial_{x_i}\hat{\theta}^{\varepsilon}\partial_{x_i}\hat{\psi}\,dx'dz+\int_{\Omega}\hat{\beta}\,\partial_z\hat{\theta}^{\varepsilon}\partial_z\hat{\psi}\,dx'dz.$$

3.1. Some estimates

In the next, we will obtain estimates on $\left(\hat{u}^{\varepsilon},\hat{\theta}^{\varepsilon}\right)$. These estimates will be useful in order to prove the convergence of $\left(\hat{u}^{\varepsilon},\hat{\theta}^{\varepsilon}\right)$ toward the expected functions.

Theorem 3.1. *Under the assumptions of Theorem 2.1, there exists a positive constant c independent of ε such that*

$$
\begin{aligned}
&\sum_{i=1}^{2}\left(\|\partial_z\hat{u}_i^\varepsilon\|^2_{L^2(\Omega)}+\left\|\varepsilon^2\partial_{x_i}\hat{u}_3^\varepsilon\right\|^2_{L^2(\Omega)}+\|\hat{u}_i^\varepsilon\|^p_{L^p(\Omega)}\right) \\
&+\sum_{i,j=1}^{2}\left\|\varepsilon\partial_{x_j}\hat{u}_i^\varepsilon\right\|^2_{L^2(\Omega)}+\|\varepsilon\partial_z\hat{u}_3^\varepsilon\|^2_{L^2(\Omega)}+\|\varepsilon\hat{u}_3^\varepsilon\|^p_{L^p(\Omega)}\leq c,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&\sum_{i=1}^{2}\left(\|\partial_z(\partial_t\hat{u}_i^\varepsilon)\|^2_{L^2(0,T;L^2(\Omega))}+\left\|\varepsilon^2\partial_{x_i}(\partial_t\hat{u}_3^\varepsilon)\right\|^2_{L^2(0,T;L^2(\Omega))}\right) \\
&+\sum_{i,j=1}^{2}\left\|\varepsilon\partial_{x_j}(\partial_t\hat{u}_i^\varepsilon)\right\|^2_{L^2(0,T;L^2(\Omega))}+\|\varepsilon\partial_z(\partial_t\hat{u}_3^\varepsilon)\|^2_{L^2(0,T;L^2(\Omega))}\leq c,
\end{aligned}
\tag{17}
$$

$$
\left\|\varepsilon\partial_t\hat{\theta}^\varepsilon\right\|^2_{L^2(0,T;L^2(\Omega))}+\left\|\partial_z\hat{\theta}^\varepsilon\right\|^2_{L^2(\Omega)}+\sum_{i=1}^{2}\left\|\varepsilon\partial_{x_i}\hat{\theta}^\varepsilon\right\|^2_{L^2(\Omega)}\leq c. \tag{18}
$$

Proof. Let $(u^\varepsilon,\theta^\varepsilon)$ be a solution to problem $(8)-(10)$, we choose $\varphi=0$ and $\psi=\partial_t\theta^\varepsilon$ as test functions, we get

$$
\begin{aligned}
&\mathcal{A}_{\mu(\cdot)+1}(\theta^\varepsilon;u^\varepsilon,\partial_tu^\varepsilon)+\mathcal{A}_{\nu(\cdot)}(\theta^\varepsilon;\partial_tu^\varepsilon,\partial_tu^\varepsilon)-(\eta_0\theta^\varepsilon,\operatorname{div}(\partial_tu^\varepsilon)) \\
&+\alpha^\varepsilon\left(|u^\varepsilon|^{p-2}u^\varepsilon,\partial_tu^\varepsilon\right)+j^\varepsilon(\partial_tu^\varepsilon) \\
&\leq(f^\varepsilon,\partial_tu^\varepsilon);
\end{aligned}
$$

$$
(\partial_t\theta^\varepsilon,\partial_t\theta^\varepsilon)+\mathcal{B}(\theta^\varepsilon,\partial_t\theta^\varepsilon)+(\eta_1\operatorname{div}(\partial_tu^\varepsilon),\partial_t\theta^\varepsilon)+\gamma^\varepsilon(\theta^\varepsilon,\partial_t\theta^\varepsilon)_\Lambda=(q^\varepsilon,\partial_t\theta^\varepsilon).
$$

As $j^\varepsilon(\partial_tu^\varepsilon)$ is positive, then

$$
\begin{aligned}
&\frac{d}{dt}\|\mathbb{D}(u^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}+2\nu_*\|\mathbb{D}(\partial_tu^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}-\eta_0(\theta^\varepsilon,\operatorname{div}(\partial_tu^\varepsilon)) \\
&+\frac{\alpha^\varepsilon}{p}\frac{d}{dt}\|u^\varepsilon\|^p_{L^p(\Omega^\varepsilon)^3} \\
&\leq(f^\varepsilon,\partial_tu^\varepsilon)-\int_{\Omega^\varepsilon}\mu(\theta^\varepsilon)\mathbb{D}(u^\varepsilon):\mathbb{D}(\partial_tu^\varepsilon)\,dx;
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\|\partial_t\theta^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}+\frac{1}{2}\frac{d}{dt}\left\|\sqrt{\beta^\varepsilon}\nabla\theta^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^3}+\eta_1(\operatorname{div}(\partial_tu^\varepsilon),\partial_t\theta^\varepsilon) \\
&+\frac{1}{2}\gamma^\varepsilon\frac{d}{dt}\|\theta^\varepsilon\|^2_{L^2(\Lambda)} \\
&=(q^\varepsilon,\partial_t\theta^\varepsilon).
\end{aligned}
\tag{20}
$$

By Korn's inequality, there exists a constant $C_K > 0$ independent of ε, such that

$$\|\mathbb{D}(u^\varepsilon)\|^2_{L^2(\Omega)^{3\times 3}} \geq C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega)^{3\times 3}}.$$

Integrating the two formulas (19) and (20) from 0 to t and using Korn's inequality, it follows that

$$\begin{aligned}
&C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + 2\nu_* \int_0^t \|\nabla \partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds + \frac{\alpha^\varepsilon}{p} \|u^\varepsilon\|^p_{L^p(\Omega^\varepsilon)^3} \\
&\leq 2\int_0^t |(f^\varepsilon, \partial_t u^\varepsilon)|\, ds + 2\eta_0 \int_0^t |(\theta^\varepsilon(s), \operatorname{div}(\partial_t u^\varepsilon(s)))|\, ds + \|\nabla u_0\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \\
&+ \frac{\alpha^\varepsilon}{p} \|u_0\|^p_{L^p(\Omega^\varepsilon)^3} + 4\mu^* \int_0^t \int_{\Omega^\varepsilon} |\nabla u^\varepsilon(s)| . |\nabla \partial_t u^\varepsilon(s)|\, dx ds;
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\int_0^t \|\partial_t \theta^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \beta^* \|\nabla \theta^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} \\
&\leq 2\int_0^t (q^\varepsilon, \partial_t \theta^\varepsilon)\, ds + 2\eta_1 \int_0^t |(\operatorname{div}\partial_t u^\varepsilon, \partial_t \theta^\varepsilon)|\, ds \\
&+ \beta^{**} \|\theta_0\|^2_{H^1(\Omega^\varepsilon)} + \gamma^\varepsilon \|\theta_0\|^2_{L^2(\Lambda)}.
\end{aligned} \tag{22}$$

Now adding the two inequalities (21) and (22), we find

$$\begin{aligned}
&C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + 2\nu_* \int_0^t \|\nabla \partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds \\
&+ \int_0^t \|\partial_t \theta^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \beta^* \|\nabla \theta^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{\alpha^\varepsilon}{p} \|u^\varepsilon\|^p_{L^p(\Omega^\varepsilon)^3} \\
&\leq 2\int_0^t |(f^\varepsilon, \partial_t u^\varepsilon)|\, ds + 2\eta_0 \int_0^t |(\nabla \theta^\varepsilon, \partial_t u^\varepsilon)|\, ds + \|\nabla \hat{u}_0\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \\
&+ 4\mu^* \int_0^t \int_{\Omega^\varepsilon} |\nabla u^\varepsilon(s)|\, |\nabla \partial_t u^\varepsilon(s)|\, dx ds + \frac{\alpha^\varepsilon}{p} \|u_0\|^p_{L^p(\Omega^\varepsilon)^3} \\
&+ 2\int_0^t (q^\varepsilon, \partial_t \theta^\varepsilon)\, ds + 2\eta_1 \int_0^t |(\operatorname{div}\partial_t u^\varepsilon, \partial_t \theta^\varepsilon)|\, ds \\
&+ \beta^{**} \|\hat{\theta}_0\|^2_{H^1(\Omega^\varepsilon)} + \gamma^\varepsilon \|\theta_0\|^2_{L^2(\Lambda)}.
\end{aligned} \tag{23}$$

Using the Poincaré inequality

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq \varepsilon \bar{h} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}},$$

and the Cauchy-Schwarz and Young inequalities we get the following estimates

$$\begin{aligned}
2\int_0^t |(f^\varepsilon(s), \partial_t u^\varepsilon(s))|\, ds &\leq \frac{8\varepsilon^2 \bar{h}^2}{\nu_*} \int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3} ds \\
&+ \frac{\nu_*}{2} \int_0^t \|\nabla \partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds;
\end{aligned} \tag{24}$$

$$2\eta_0\int_0^t\left(\theta^\varepsilon(s),\operatorname{div}\left(\partial_t u^\varepsilon(s)\right)\right)ds \le \frac{72\left(\eta_0\bar{h}\right)^2}{\nu_*}\int_0^t \|\nabla\theta^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}ds + \frac{\nu_*}{2}\int_0^t \|\nabla\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}ds, \tag{25}$$

and

$$2\eta_1\int_0^t\left(\operatorname{div}\partial_t u^\varepsilon(s),\partial_t\theta^\varepsilon(s)\right)ds \le \frac{1}{2}\int_0^t \|\partial_t\theta^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds + 72\left(\eta_1\right)^2\int_0^t \|\nabla\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}ds. \tag{26}$$

On the other hand using the integral by parts, we find

$$\begin{aligned} 2\int_0^t\left(q^\varepsilon(s),\partial_t\theta^\varepsilon(s)\right)ds &= 2\left(q^\varepsilon(t),\theta^\varepsilon(t)\right)-2\left(q^\varepsilon(0),\theta^\varepsilon(0)\right)\\ &\quad -2\int_0^t\left(\partial_t q^\varepsilon(s),\theta^\varepsilon(s)\right)ds, \end{aligned}$$

according to the Cauchy-Schwarz, Poincaré and Young inequalities, we get

$$\begin{aligned} 2\int_0^t\left(q^\varepsilon(s),\partial_t\theta^\varepsilon(s)\right)ds &\le \frac{8\varepsilon^2\bar{h}^2}{\beta_*}\|q^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}+\frac{\beta_*}{2}\|\nabla\theta^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)^3}\\ &\quad +\frac{8\varepsilon^2\bar{h}^2}{\beta_*}\|q^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)}+\frac{\beta_*}{2}\|\nabla\theta_0\|^2_{L^2(\Omega^\varepsilon)^3}\\ &\quad +4\varepsilon^2\bar{h}^2\int_0^t\|\partial_t q^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds+\int_0^t\|\nabla\theta^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}ds. \end{aligned} \tag{27}$$

Now, insert the formulas (24)-(27) into (23), we have

$$\begin{aligned} & C_K\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}+\left(\frac{\nu_*}{2}-72\left(\eta_1\right)^2\right)\int_0^t\|\nabla\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}ds\\ & +\frac{1}{2}\int_0^t\|\partial_t\theta^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds+\frac{\beta^*}{2}\|\nabla\theta^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3}+\frac{\alpha^\varepsilon}{p}\|u^\varepsilon\|^p_{L^p(\Omega^\varepsilon)^3}\\ \le\ & \frac{72\mu^*}{\nu_*}\int_0^t\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}ds+\left(1+8\left(\eta_0\right)^2\right)\int_0^t\|\nabla\theta^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}ds\\ & +\frac{8\varepsilon^2\bar{h}^2}{\nu_*}\int_0^t\|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}ds+\frac{8\varepsilon^2\bar{h}^2}{\beta_*}\|q^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}\\ & +4\varepsilon^2\bar{h}^2\int_0^t\|\partial_t q^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds+\frac{8\varepsilon^2\bar{h}^2}{\beta_*}\|q^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)}+\frac{\beta_*}{2}\|\theta_0\|^2_{H^1(\Omega^\varepsilon)}\\ & +\|\nabla u_0\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}+\frac{\alpha^\varepsilon}{p}\|u_0\|^p_{L^p(\Omega^\varepsilon)^3}+\beta^{**}\|\theta_0\|^2_{H^1(\Omega^\varepsilon)}+\gamma^\varepsilon\|\theta_0\|^2_{L^2(\Lambda)}, \end{aligned} \tag{28}$$

as

$$\varepsilon^2 \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} = \varepsilon^{-1}\left\|\hat{f}\right\|^2_{L^2(\Omega)^3}, \varepsilon^2 \|q^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1}\|\hat{q}\|^2_{L^2(\Omega)}.$$

Multiplying (28) by ε $(0<\varepsilon<1)$, we have

$$\begin{aligned}\varepsilon\Bigg[&C_K\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + \frac{\beta^*}{2}\|\nabla\theta^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{\alpha^\varepsilon}{p}\|u^\varepsilon\|^p_{L^p(\Omega^\varepsilon)^3} \\ &+\frac{1}{2}\int_0^t \|\partial_t\theta^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds + \left(\frac{\nu_*}{2} - 72(\eta_1)^2\right)\int_0^t \|\nabla\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}ds\Bigg] \\ &\le A + \int_0^t \varepsilon\left[\frac{32\mu^*}{\nu_*}\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + \left(1+8(\eta_0)^2\right)\|\nabla\theta^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}\right]ds,\end{aligned} \tag{29}$$

where A is a constant does not depend on ε such that

$$\begin{aligned}A = {}& \frac{8\bar{h}^2}{\nu_*}\int_0^t \left\|\hat{f}(s)\right\|^2_{L^2(\Omega)^3}ds + \frac{8\bar{h}^2}{\beta_*}\|\hat{q}(t)\|^2_{L^2(\Omega)} \\ &+4\bar{h}^2\int_0^t \|\partial_t\hat{q}(s)\|^2_{L^2(\Omega)}ds + \frac{8\bar{h}^2}{\beta_*}\|\hat{q}(0)\|^2_{L^2(\Omega)} + \frac{\beta_*}{2}\left\|\hat{\theta}_0\right\|^2_{H^1(\Omega)} \\ &+\|\nabla\hat{u}_0\|^2_{L^2(\Omega)^{3\times 3}} + \frac{\hat{\alpha}}{p}\|\hat{u}_0\|^p_{L^p(\Omega)^3} + \beta^{**}\left\|\hat{\theta}_0\right\|^2_{H^1(\Omega)} + \hat{\gamma}\left\|\hat{\theta}_0\right\|^2_{L^2(\Lambda)}.\end{aligned}$$

Applying Gronwall's inequality to the inequality (29), we obtain there exists a constant c does not depend on ε such that

$$\varepsilon\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + \varepsilon\|\nabla\theta^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} \le c. \tag{30}$$

From (30) and the inequality (29) can be deduced

$$\varepsilon\|\partial_t\theta^\varepsilon\|^2_{L^2(0,T;L^2(\Omega^\varepsilon))} + \varepsilon\|\nabla\partial_t u^\varepsilon\|^2_{L^2\left(0,T;L^2(\Omega^\varepsilon)^{3\times 3}\right)} + \frac{1}{\varepsilon}\|u^\varepsilon\|^p_{L^p(\Omega^\varepsilon)^3} \le c.$$

Which completes the proof. □

3.2. The limit problem and the main results

Theorem 3.2. *Under the same assumptions of Theorem 3.1, then there exists* $\theta^* \in L^2(0,T;\mathcal{V}_z)$ *and* $u_i^* \in L^2(0,T;\mathcal{V}_z\cap L^p(\Omega))$, $i=1,2$, *such that*

$$\left.\begin{array}{l}\hat{u}_i^\varepsilon \rightharpoonup u_i^* \text{ weakly in } L^2(0,T;\mathcal{V}_z)\cap L^2(0,T;L^p(\Omega)),\ i=1,2, \\ \hat{\theta}^\varepsilon \rightharpoonup \theta^* \text{ weakly in } L^2(0,T;\mathcal{V}_z),\end{array}\right\} \tag{31}$$

$$\partial_t\hat{u}_i^\varepsilon \rightharpoonup \partial_t u_i^* \text{ weakly in } L^2(0,T;\mathcal{V}_z),\ i=1,2, \tag{32}$$

$$|\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon \rightharpoonup |u_i^*|^{p-2}u_i^*\ i=1,2 \text{ weakly in } L^2\left(0,T;L^{p'}(\Omega)\right),\ p'=\frac{p}{p-1}, \tag{33}$$

$$\varepsilon^p|\hat{u}_3^\varepsilon|^{p-2}\hat{u}_3^\varepsilon \rightharpoonup 0 \textit{ weakly in } L^2\left(0,T;L^{p'}(\Omega)\right), \tag{34}$$

$$\left.\begin{array}{l}\varepsilon\partial_{x_j}\hat{u}_i^\varepsilon \rightharpoonup 0\\ \varepsilon\partial_{x_j}\left(\partial_t\hat{u}_i^\varepsilon\right) \rightharpoonup 0\end{array} \; i,j=1,2\right\} \textit{ weakly in } L^2\left(0,T;L^2(\Omega)\right), \tag{35}$$

$$\left.\begin{array}{l}\varepsilon^2\partial_{x_i}\hat{u}_3^\varepsilon \rightharpoonup 0\\ \varepsilon^2\partial x_i\left(\partial_t\hat{u}_3^\varepsilon\right) \rightharpoonup 0\end{array} \; i=1,2\right\} \textit{ weakly in } L^2\left(0,T;L^2(\Omega)\right), \tag{36}$$

$$\left.\begin{array}{l}\varepsilon^2\partial_z\hat{u}_3^\varepsilon \rightharpoonup 0\\ \varepsilon^2\partial_z\left(\partial_t\hat{u}_3^\varepsilon\right) \rightharpoonup 0\end{array}\right\} \textit{ weakly in } L^2\left(0,T;L^2(\Omega)\right), \tag{37}$$

$$\left.\begin{array}{l}\varepsilon\partial_t\hat{\theta}^\varepsilon \rightharpoonup 0\\ \varepsilon\partial_{x_i}\hat{\theta}^\varepsilon \rightharpoonup 0\end{array}\right\} \textit{ weakly in } L^2\left(0,T;L^2(\Omega)\right). \tag{38}$$

Proof. From the Theorem 3.1, there exists a constant c independent of ε such that

$$\|\partial_z\hat{u}_i^\varepsilon\|_{L^2(\Omega)}^2 \leq C; \quad \|\hat{u}_i^\varepsilon\|_{L^p(\Omega)}^p \leq C, \;\; i=1,2; \quad \left\|\partial_z\hat{\theta}^\varepsilon\right\|_{L^2(\Omega)}^2 \leq C.$$

Using this estimates and the Poincaré inequality, we have

$$\|\hat{u}_i^\varepsilon\|_{L^2(\Omega)}^2 \leq \bar{h}\,\|\partial_z\hat{u}_i^\varepsilon\|_{L^2(\Omega)}^2, i=1,2; \quad \left\|\hat{\theta}^\varepsilon\right\|_{L^2(\Omega)}^2 \leq \bar{h}\left\|\partial_z\hat{\theta}^\varepsilon\right\|_{L^2(\Omega)}^2,$$

so, we deduce that the sequence $(\hat{u}_i^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;\mathcal{V}_z\cap L^p(\Omega))$ and the sequence $\left(\hat{\theta}^\varepsilon\right)_\varepsilon$ is bounded in $L^2(0,T;\mathcal{V}_z)$, so the weak convergence result (31). Similarly, according to (17) and the Poincaré inequality for $\partial_t\hat{u}_i^\varepsilon$, we deduce that $\partial_t\hat{u}_i^\varepsilon$ is bounded in $L^2(0,T;\mathcal{V}_z)$ and hence converges to a limit v, and as $\hat{u}_i^\varepsilon \rightharpoonup u_i^*$, so $v=\partial_t u_i^* \in L^2(0,T;\mathcal{V}_z)$.

We have

$$\int_\Omega \left||\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon\right|^{p'} dx'dz = \int_{\Omega^\varepsilon} |\hat{u}_i^\varepsilon|^p dx'dz \leq C, i=1,2 \text{ where } \frac{1}{p}+\frac{1}{p'}=1.$$

So $|\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon$ is bounded in $L^2(0,T;L^{p'}(\Omega))$, then we get (33). The convergences of $(34)-(38)$ are a direct result of inequalities $(16)-(17)$ and $(31)-(32)$. □

Proposition 3.3. *With the same assumptions of Theorem 3.2, the pair* (u^*,θ^*) *satisfy*

$$\begin{aligned}&\sum_{i=1}^{2}\int_\Omega\left[\left(1+\mu\left(\theta^*\right)\right)\partial_z u_i^* + \nu\left(\theta^*\right)\partial_z\left(\partial_t u_i^*\right)\right]\frac{\partial}{\partial z}\left(\hat{\varphi}_i-\partial_t u_i^*\right)dx'dz \\ &+\hat{\alpha}\sum_{i=1}^{2}\int_\Omega |u_i^*|^{p-2}u_i^*\left(\hat{\varphi}_i-\partial_t u_i^*\right)dx'dz+\hat{J}(\hat{\varphi})-\hat{J}\left(\partial_t u^*\right) \\ &\geq \sum_{i=1}^{2}\left(\hat{f}_i,\hat{\varphi}_i-\partial_t u_i^*\right), \;\; \forall\hat{\varphi}\in\Pi;\end{aligned} \tag{39}$$

$$\int_{\Omega}\hat{\beta}\partial_z\hat{\theta}^{\varepsilon}\partial_z\hat{\psi}dx'dz+\hat{\gamma}\int_{\Lambda}\hat{\theta}^{\varepsilon}\hat{\psi}dx'=\int_{\Omega}\hat{q}\hat{\psi}dx'dz,\ \forall\hat{\psi}\in\mathcal{H}_{\Gamma_1\cup\Gamma_L},\tag{40}$$

and the limit problem

$$\left.\begin{array}{l}-\partial_z\left[\left(1+\mu\left(\theta^*\right)\right)\partial_z u_i^*\left(t\right)+\nu\left(\theta^*\right)\partial_z\left(\partial_t u_i^*\right)\left(t\right)\right]+\hat{\alpha}\left|u_i^*\right|^{p-2}u_i^*\left(t\right)\\ =\hat{f}_i\left(t\right),\ in\ L^2\left(\Omega\right),\ i=1,2;\\ \\ -\partial_z\left(\hat{\beta}\left(x',z\right)\partial_z\theta^*\right)\left(t\right)=\hat{q}\left(t\right),\quad in\ L^2\left(\Omega\right);\\ \\ u^*\left(x',z,0\right)=u_0\left(x',z\right).\end{array}\right\}\tag{41}$$

Proof. By the convergence results of Theorem 3.2, the continuity of $\mu(\cdot)$ and $\nu(\cdot)$, the Egorov Theorem as in [18] and the fact that $\hat{J}(\cdot)$ is convex and lower semi-continuous the variational problem $(13)-(14)$ became

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega}\left[\partial_z\left(1+\mu\left(\theta^*\right)\right)\partial_z u_i^*+\nu\left(\theta^*\right)\partial_z\left(\partial_t u_i^*\right)\right]\partial_z\left(\hat{\varphi}_i-\partial_t u_i^*\right)dx'dz\\ &+\hat{\alpha}\sum_{i=1}^{2}\int_{\Omega}\left|u_i^*\right|^{p-2}u_i^*\left(\hat{\varphi}_i-\partial_t u_i^*\right)dx'dz+\hat{J}\left(\hat{\varphi}\right)-\hat{J}\left(\partial_t u^*\right)\\ &\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\left(\hat{\varphi}_i-\partial_t u_i^*\right)dx'dz;\end{aligned}\tag{42}$$

$$\int_{\Omega}\hat{\beta}\partial_z\theta^*\partial_z\hat{\psi}dx'dz+\hat{\gamma}\int_{\Lambda}\theta^*\hat{\psi}dx'=\int_{\Omega}\hat{q}\hat{\psi}dx'dz.\tag{43}$$

Now, we choose in the variational inequality (42)

$$\hat{\varphi}_i=\partial_t u_i^*\pm w_i,\text{ such that }w_i\in H_0^1\left(\Omega\right)\ \ i=1,2,$$

we find

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega}\left[\left(1+\mu\left(\theta^*\right)\right)\partial_z u_i^*+\nu\left(\theta^*\right)\partial_z\left(\partial_t u_i^*\right)\right]\partial_z w_i dx'dz\\ &+\hat{\alpha}\sum_{i=1}^{2}\int_{\Omega}\left|u_i^*\right|^{p-2}u_i^*.w_i dx'dz\\ &=\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i w_i dx'dz,\end{aligned}$$

by the Green formula, we deduce

$$\begin{aligned}&-\int_{\Omega}\left[\partial_z\left(\left(1+\mu\left(\theta^*\right)\right)\partial_z u_i^*+\nu\left(\theta^*\right)\partial_z\left(\partial_t u_i^*\right)\right)+\hat{\alpha}\left|u_i^*\right|^{p-2}u_i^*\right]w_i dx'dz\\ &=\int_{\Omega}\hat{f}_i w_i dx'dz,\ i=1,2.\end{aligned}$$

Similarly for the formula (43), we choose $\hat{\psi} \in H_0^1(\Omega)$ and use Green's formula to get

$$-\int_\Omega \partial_z\left(\hat{\beta}\partial_z\theta^*\right).\hat{\psi}dx'dz = \int_\Omega \hat{q}\hat{\psi}dx'dz.$$

Then

$$\left.\begin{array}{l} -\partial_z\left[(1+\mu(\theta^*))\partial_z u_i^* + \nu(\theta^*)\partial_z\partial_t u_i^*\right] + \hat{\alpha}\left|u_i^*\right|^{p-2}u_i^* \\ = \hat{f}_i,\ i=1,2, \\ \\ -\partial_z\left[\hat{\beta}(x',z)\partial_z\theta^*\right] = \hat{q}, \end{array}\right\} \text{ in } H^{-1}(\Omega). \quad (44)$$

We know that $\hat{f}_i$, $\hat{q} \in L^2(\Omega)$, $i=1,2$, then (44) is true in $L^2(\Omega)$. □

Proposition 3.4. *Let*

$$\tau^*(x',t) = \partial_z u^*(x',0,t) \text{ and } s^*(x',t) = u^*(x',0,t)$$

the traces of the displacement u^* *on* Λ*. The traces* τ^* *and* s^* *satisfy the following inequality*

$$\int_\Lambda \hat{k}\left(|\phi+\partial_t s^*| - |\partial_t s^*|\right)dx' - \int_\Lambda \left((1+\mu(\zeta^*))\tau^* + \nu(\zeta^*)\partial_t\tau^*\right)\phi dx' \quad (45)$$
$$\geq 0,\ \forall\phi \in L^2(\Lambda)^2,$$

where $\zeta^* = \theta^*(x',0,t)$*, and the following limit form of the Tresca boundary conditions*

$$\left\{\begin{array}{l} |(1+\mu(\zeta^*))\tau^* + \nu(\zeta^*)\partial_t\tau^*| \\ < \hat{k} \Rightarrow \partial_t s^* = 0; \\ \\ |(1+\mu(\zeta^*))\tau^* + \nu(\zeta^*)\partial_t\tau^*| = \hat{k} \\ \Rightarrow \exists\rho > 0 \text{ such that} \\ \partial_t s^* = \rho((1+\mu(\zeta^*))\tau^* + \nu(\zeta^*)\partial_t\tau^*) \end{array}\right. \quad a.e \text{ on } \Lambda\times]0,T[. \quad (46)$$

Proof. Choosing in (39) $\hat{\varphi} = (\partial_t u_1^* + \phi_1, \partial_t u_2^* + \phi_2)$, where $\phi \in \Pi$, we obtain

$$\int_\Lambda \hat{k}\left(|\phi+\partial_t s^*| - |\partial_t s^*|\right)dx'$$
$$\geq -\sum_{i=1}^2 \int_\Omega \left[(1+\mu(\theta^*))\partial_z u_i^*(t) + \nu(\theta^*)\partial_z(\partial_t u_i^*(t))\right]\partial_z\phi_i dx'dz$$
$$+\sum_{i=1}^2 \int_\Omega \left(\hat{\alpha}\left|u_i^*\right|^{p-2}u_i^*(t) - \hat{f}_i\right)\phi_i dx'dz,$$

using now the Green formula, the equality (41) and the fact that $\phi_i = 0$ on $\Gamma_1 \cup \Gamma_L$, we get

$$\begin{aligned}&\int_\Lambda \hat{k}\left(|\phi+\partial_t s^*| - |\partial_t s^*|\right)dx' - \int_\Lambda \left((1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)\right).\phi\, dx' \\ &\geq 0, \forall \phi \in \Pi.\end{aligned}$$

This inequality remain valid for any $\phi \in \mathcal{D}(\Lambda)^2$, and by density of $\mathcal{D}(\Lambda)^2$ in $L^2(\Lambda)^2$, it also remain valid for any $\phi \in L^2(\Lambda)^2$.

For (46), we choose $\phi = \pm\partial_t s^*$ in (45), we obtain

$$\int_\Lambda \left[\hat{k}\,|\partial_t s^*| - \left((1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)\right)\partial_t s^*\right]dx' = 0, \tag{47}$$

taking $\phi = \mathcal{Y} - \partial_t s^*$ with $\mathcal{Y} \in L^2(\Lambda)^2$, in (45), we obtain

$$\begin{aligned}&\int_\Lambda \left[\hat{k}\,|\mathcal{Y}| - \left((1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)\right).\mathcal{Y}\right]dx' \\ &\geq \int_\Lambda \left[\hat{k}\,|\partial_t s^*| - \left((1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)\right)\partial_t s^*\right]dx', \forall \mathcal{Y} \in L^2(\Lambda)^2,\end{aligned}$$

from (47), we deduce

$$\int_\Lambda \left[\hat{k}\,|\mathcal{Y}| - \left((1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)\right)\mathcal{Y}\right]dx' \geq 0. \tag{48}$$

In (48), we take $\mathcal{Y}_i \geq 0$, $(i = 1,2)$, we obtain

$$\int_\Lambda \left[\hat{k} - \left\{\begin{array}{l} \left(|(1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)|\right) \\ \times\cos\left(\left((1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)\right), \mathcal{Y}\right)\end{array}\right\}\right]|\mathcal{Y}|\,dx' \geq 0,$$

then

$$\begin{aligned}&\left\{\begin{array}{l} \left(|(1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)|\right) \\ \times\cos\left(\left((1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)\right), \mathcal{Y}\right)\end{array}\right\} \\ &\leq \hat{k}, \text{ a.e on } \Lambda.\end{aligned} \tag{49}$$

In (48), we take $-\mathcal{Y}$, such that $\mathcal{Y}_i \geq 0$, $(i = 1,2)$, we obtain

$$\int_\Lambda \left[\hat{k} + \left\{\begin{array}{l} \left(|(1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)|\right) \\ \times\cos\left(\left((1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)\right), \mathcal{Y}\right)\end{array}\right\}\right]|\mathcal{Y}|\,dx' \geq 0,$$

then

$$\begin{aligned}&\left\{\begin{array}{l} \left(|(1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)|\right) \\ \times\cos\left(\left((1+\mu(\zeta^*))\,\tau^*(t) + \nu(\zeta^*)\,\partial_t\tau^*(t)\right), \mathcal{Y}\right)\end{array}\right\} \\ &\geq -\hat{k}, \text{ a.e on } \Lambda.\end{aligned} \tag{50}$$

From (49) and (50), we get

$$|(1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t)|\leq\hat{k}.$$

Moreover, we have

$$\begin{aligned}\hat{k}|\partial_t s^*| &\geq (|(1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t)|)|\partial_t s^*|\\ &\geq ((1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t)).\partial_t s^*,\text{ a.e on }\Lambda,\end{aligned}$$

then

$$\hat{k}|\partial_t s^*|-((1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t)).\partial_t s^*\geq 0,\text{ a.e on }\Lambda,$$

and from (47), we deduce that

$$\hat{k}|\partial_t s^*|-((1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t)).\partial_t s^*=0,\text{ a.e on }\Lambda.\tag{51}$$

So, if $\hat{k}=|((1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t))|$, then from (51), we have

$$\begin{aligned}&|(1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t)||\partial_t s^*|\\ &=((1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t)).\partial_t s^*,\text{ a.e on }\Lambda,\end{aligned}$$

then, there exists $\rho\geq 0$ such that

$$\partial_t s^*=\rho((1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t)),\text{ a.e on }\Lambda,$$

and if $\hat{k}>|(\mu(\zeta^*)\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t))|$, then from (51), we have

$$\begin{aligned}0 &= \hat{k}|\partial_t s^*|-((1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t)).\partial_t s^*\\ &\geq \left(\hat{k}-|((1+\mu(\zeta^*))\tau^*(t)+\nu(\zeta^*)\partial_t\tau^*(t))|\right)|\partial_t s^*|,\text{ a.e on }\Lambda,\end{aligned}$$

that implies $\partial_t s^*=0$ a.e on Λ. Then (46) follows. □

Theorem 3.5. *Assuming that* $p\leq 4$. *The solution* (u^*,θ^*) *of our limit problem is unique.*

Proof. Let $\left(u^{*1},\theta^{*1}\right)$, $\left(u^{*2},\theta^{*2}\right)$ be two solutions of our limit (39)-(40). Then θ^{*1} and θ^{*2} solve (40), so $\Theta=\theta^{*1}-\theta^{*2}$ satisfies the problem

$$\beta_*\int_\Omega(\partial_z\Theta)^2dx'dz+\hat{\gamma}\int_\Lambda\Theta^2dx'\leq 0.$$

So $\Theta=0$, thus $\theta^{*1}=\theta^{*2}=\theta^*$.

Taking $\varphi = \partial_t u_i^{*2}$ and $\varphi = \partial_t u_i^{*1}$, respectively as test functions in (39), we get

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}(1+\mu(\theta^*))\left(\partial_z u_i^{*1}-\partial_z u_i^{*2}\right)\partial_z\left(\partial_t u_i^{*1}-\partial_t u_i^{*2}\right)dx'dz\\
&+\sum_{i=1}^{2}\int_{\Omega}\nu(\theta^*)\left|\partial_z\left(\partial_t u_i^{*1}-\partial_t u_i^{*2}\right)\right|^2 dx'dz\\
&+\hat{\alpha}\sum_{i=1}^{2}\int_{\Omega}\left(\left|u_i^{*1}\right|^{p-2}u_i^{*1}-\left|u_i^{*2}\right|^{p-2}u_i^{*2}\right)\left(\partial_t u_i^{*1}-\partial_t u_i^{*2}\right)\\
&\leq 0,
\end{aligned}$$

this leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\partial_z u^{*1}-\partial_z u^{*1}\right\|_{L^2(\Omega)^2}^2+\nu^*\left\|\partial_z\left(\partial_t u^{*1}-\partial_t u^{*1}\right)\right\|_{L^2(\Omega)^2}^2\\
&\leq\frac{4(\mu^*)^2}{\nu^*}\left\|\partial_z u^{*1}-\partial_z u^{*1}\right\|_{L^2(\Omega)^2}^2+\frac{\nu^*}{4}\left\|\partial_z\left(\partial_t u_i^{*1}-\partial_t u_i^{*1}\right)\right\|_{L^2(\Omega)^2}^2\\
&+\hat{\alpha}\sum_{i=1}^{2}\int_{\Omega}\left(\left|u_i^{*1}\right|^{p-2}u_i^{*1}-\left|u_i^{*2}\right|^{p-2}u_i^{*2}\right)\left(\partial_t u_i^{*1}-\partial_t u_i^{*2}\right).
\end{aligned}\tag{52}$$

The last member of (52) is bounded in absolute value by

$$\sum_{i=1}^{2}\int_{\Omega}\left(\left|u_i^{*1}\right|^{p-2}+\left|u_i^{*2}\right|^{p-2}\right)\left|u_i^{*1}-u_i^{*2}\right|\left|\partial_t u_i^{*1}-\partial_t u_i^{*2}\right|,$$

which is bounded according to the Hölder inequality by

$$\begin{aligned}
&\left[\left\|\left|u^{*1}\right|^{p-2}\right\|_{L^2(\Omega)^2}+\left\|\left|u^{*2}\right|^{p-2}\right\|_{L^2(\Omega)^2}\right]\\
&\times\left(\left\|u^{*1}-u^{*2}\right\|_{L^4(\Omega)^2}\left\|\partial_t u^{*1}-\partial_t u^{*2}\right\|_{L^4(\Omega)^2}\right).
\end{aligned}$$

We have $2(p-2)\leq 4$, then

$$\begin{aligned}
\left\|\left|u^{*1}\right|^{p-2}\right\|_{L^2(\Omega)^2}+\left\|\left|u^{*2}\right|^{p-2}\right\|_{L^2(\Omega)^2} &= \left\|u^{*1}\right\|_{L^{2(p-2)}(\Omega)^2}^{(p-2)}+\left\|u^{*2}\right\|_{L^{2(p-2)}(\Omega)^2}^{(p-2)}\\
&\leq c\left(\left\|u^{*1}\right\|_{L^4(\Omega)^2}^{(p-2)}+\left\|u^{*2}\right\|_{L^4(\Omega)^2}^{(p-2)}\right).
\end{aligned}$$

Using the embedding of $\mathcal{V}_z$ in $L^4(\Omega)$ (see [6]), we find

$$\begin{aligned}
&\left[\left\| |u^{*1}|^{p-2} \right\|_{L^2(\Omega)^2} + \left\| |u^{*2}|^{p-2} \right\|_{L^2(\Omega)^2}\right] \\
&\times \left(\|u^{*1} - u^{*2}\|_{L^4(\Omega)^2} \|\partial_t u^{*1} - \partial_t u^{*2}\|_{L^4(\Omega)^2} \right) \\
&\leq \tilde{c}_{(\bar{h})} \left(\|u^{*1}\|_{(\mathcal{V}_z)^2}^{(p-2)} + \|u^{*2}\|_{(\mathcal{V}_z)^2}^{(p-2)} \right) \times \\
&\|\partial_z u^{*1}(s) - \partial_z u^{*1}(s)\|_{L^2(\Omega)^2} \|\partial_z \partial_t u^{*1} - \partial_z \partial_t u^{*2}\|_{L^2(\Omega)^2}.
\end{aligned}$$

Now, integrating the inequality (52) over $(0,t)$ and using the fact that $\|u^{*1}\|_{(\mathcal{V}_z)^2}^{(p-2)} + \|u^{*2}\|_{(\mathcal{V}_z)^2}^{(p-2)} \leq \hat{c}_p$, we get

$$\begin{aligned}
&\|\partial_z u^{*1} - \partial_z u^{*1}\|_{L^2(\Omega)^2}^2 \\
&\leq \left[\frac{(2\mu^{**})^2}{\nu^*} + \left(\frac{2\tilde{c}_{(\bar{h})} \cdot \hat{c}_p}{\nu^*} \right)^2 \right] \int_0^t \|\partial_z u^{*1}(s) - \partial_z u^{*1}(s)\|_{L^2(\Omega)^2}^2 \, ds.
\end{aligned}$$

Applying Gronwall's Lemma, gives

$$\|\partial_z u^{*1} - \partial_z u^{*1}\|_{L^2(\Omega)^2}^2 \leq 0.$$

Then, using the Poincaré inequality, we find

$$\|u^{*1} - u^{*1}\|_{L^2\left(0,T;(\mathcal{V}_z)^2\right)} = 0.$$

□

4. Conclusion

We have found and mathematically justified a two-dimensional limit model for a thermoviscoelastic shell that may come into contact with a deformable foundation. The contact process is quasistatic and modeled using Tresca's friction law. The proofs were based on insights provided by the asymptotic analysis method and justified this approach by obtaining convergence theorems and the limit problem. We have also proven the uniqueness of the two-dimensional problem solution when $p \leq 4$.

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