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WEAKLY ARF PROPERTY FOR AMALGAMATION OF SEMIGROUPS AND RINGS

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We provide a characterization of the weakly Arf property for the amalgamation of numerical semigroups and for amalgamation algebras.

1. Introduction

The notion of Arf ring was formally introduced by J. Lipman in [10] generalizing a class of rings defined by C. Arf in order to study curve singularities (see [1]). The definition of Arf ring requires, in particular, that the ring is Noetherian semi-local and that every localization with respect to a maximal ideal is one-dimensional and Cohen-Macaulay; this concept has been used to classify the multiplicity trees of curve singularities, that, in turn, determine the equisingularity classes with respect to formal equivalence (see e.g. [2]). In the above mentioned context, one possible definition of an Arf ring A is the following:

Definition 1. Let A be a Noetherian semi-local ring such that $A_{\mathfrak{m}}$ is a one dimensional Cohen-Macaulay local ring for every $\mathfrak{m}$ maximal ideal of A. Then A is called an *Arf ring* if the following conditions hold:

1. Every integrally closed ideal I in A that contains a non-zerodivisor has a principal reduction, i.e., $I^{n+1} = aI^n$ for some $n \geq 0$ and $a \in I$;

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2. if $x,y,z \in A$ are such that x is a non-zerodivisor on A and $y/x, z/x \in Q(A)$ are integral over A, then $yz/x \in A$, where $Q(A)$ denotes the total ring of fractions of A.

Since Condition 1 becomes superfluous when A is also local with infinite residue field (see [9, Proposition 8.3.7]), it was natural for the authors of [4] to explore deeply Condition 2. So they propose the following more general definition, without any assumption on A:

Definition 2. Let A be a commutative and unitary ring, A is said to be *weakly Arf* if A satisfies the Condition 2 of the definition above.

In the same paper the authors develop a general theory of weakly Arf rings, studying many interesting properties of the rings in this class. In particular, they study when some ring constructions as the idealization, the amalgamated duplication and some particular fiber products produce a weakly Arf ring (cf. [4, Theorem 5.2, Theorem 6.2, Proposition 6.11]). A related problem was considered also in [3] where the author studies when a quadratic quotient of the Rees algebra of an algebroid branch, with respect to an ideal I, produces an Arf singularity.

Since idealization and amalgamated duplication are both related to a recent construction introduced in [6], called amalgamation, considering that the amalgamation is a particular kind fiber product, it is natural to investigate when this construction produces a weakly Arf ring.

More precisely, let $f : R \to S$ be a ring homomorphism and let J be an ideal of S; we will call the *amalgamation of R with S along J with respect to f* the ring

$$R \bowtie^f J = \{(r, f(r) + j) \mid r \in R \text{ and } j \in J\}$$

As said above, this ring can be constructed as a fiber product; indeed, if we consider the canonical projection $\pi\colon S \to S/J$ and $\check{f} = \pi \circ f$ we get the following commutative diagram

$$\begin{array}{ccc} R \bowtie^f I & \xrightarrow{\rho_1} & R \\ {\scriptstyle \rho_2}\downarrow & & \downarrow{\scriptstyle \check{f}} \\ S & \xrightarrow[\pi]{} & S/J \end{array}$$

where ρ_1 and ρ_2 are the canonical projections from $R \bowtie^f J$ to R and S, respectively. It is easy to check that $R \bowtie^f J = \check{f} \times_{S/J} \pi$. Therefore in this paper we will study under which hypotheses the amalgamation is a weakly Arf ring.

A first step in this direction was motivated by the theory of numerical and good semigroups, where it is possible to naturally define the notion of Arf semigroups, a notion that reproduces additively the definition of weakly Arf rings.

In [5], the author defines the amalgamation of two numerical semigroups, a construction that produces a good subsemigroup of $\mathbb{N}^2$; he shows that the amalgamation of the value semigroups of two algebroid branches produces the value semigroup of the amalgamation of the branches, that, in this case is an algebroid curve with two branches.

Hence we start our investigation studying when the amalgamation of two numerical semigroups gives an Arf semigroup and this is the content of Section 2. The main result of this section is Theorem 1, that completely characterize when the amalgamation is an Arf semigroup.

Theorem 1. 1 Let S and T be two numerical semigroups, let $g\colon S \to T$ a semigroups homomorphism and let E an ideal semigroup of T. Then $S \bowtie^g E$ is an Arf semigroup if and only if S and $g(S) \cup E$ are Arf semigroups and E is integrally closed in $g(S) \cup E$.

In Section 3 we investigated the amalgamation algebra $R \bowtie^f J$, where J is an ideal of S and $f\colon R \to S$ is a ring homorphism. We obtained a result similar to the case of semigroups, under the following hypotheses:

- R and $f(R)+J$ are domains.
- J and $f^{-1}(J)$ are non-zero ideals of S and R, respectively.
- $\overline{f(R)+J}$ is a DVR.

Under these hypotheses we can prove the following result, that is the main theorem of the paper.

Theorem 2. 3 $R \bowtie^f J$ is a weakly Arf ring if and only if R and $f(R)+J$ are weakly Arf rings and J is an integrally closed ideal in $f(R)+J$.

2. Arf amalgamation of numerical semigroups

A numerical semigroup is a submonoid S of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. Let E be a subset of a numerical semigroup S; we say that E is a semigroup ideal (or simply an ideal) of S if $E + S \subseteq E$. If E is a semigroup ideal of a numerical semigroup S, let $\tilde{e} = \min E$. We call integral closure of E in S the semigroup ideal $\overline{E} = \{s \in S\colon s \geq \tilde{e}\}$ of S and we will say that E is integrally closed if $\overline{E} = E$.

For an element $a \in \mathbb{N}^2$ we denote by a_1 and a_2 its first and second coordinate, respectively. Given $a, b \in \mathbb{N}^2$, we say that $a \leq b$ if $a_1 \leq b_1$ and $a_2 \leq b_2$ with respect to the usual ordering of $\mathbb{N}$, and we denote the infimum of the set $\{a,b\}$ by $a\wedge b=(\min(a_1,b_1),\min(a_2,b_2))$.

Let S a submonoid of $\mathbb{N}^2$; we say that S is a *good semigroup* if (see [2]):

1. for all $a,b\in S$, then $a\wedge b\in S$;
2. if $a\neq b\in S$ and $a_i=b_i$ for some $i\in\{1,2\}$, then there exists $c\in S$ such that $c_i>a_i=b_i$ and $c_j=\min\{a_j,b_j\}$, with $j\in\{1,2\}\setminus\{i\}$;
3. there exists $c\in S$ such that $c+\mathbb{N}^2\subseteq S$.

Let S a numerical semigroup, we say that S is an *Arf semigroup* if for all $a,b,c\in S$ with $a\leq b$ and $a\leq c$ we have $b+c-a\in S$. Similary; let S a good semigroup, we say that is an *Arf semigroup* if for all $a,b,c\in S$ with $a\leq b$ and $a\leq c$ we have $b+c-a\in S$.

Let now S and T be two numerical semigroups and let $g\colon S\to T$ a semigroup homomorphism, i.e. the multiplication by a positive integer s such that $sa\in T$ for all $a\in S$. Fix an ideal E of T; as it was done in [5], we define the *amalgamation of S with T along E with respect to g* as the following subset of $\mathbb{N}^2$:

$$S\bowtie^g E=D\cup(g^{-1}(E)\times E)\cup\{a\wedge b\colon a\in D,\ b\in g^{-1}(E)\times E\}$$

where $D=\{(a,sa)\colon a\in S\}$. It is not difficult to check that $S\bowtie^g E$ is a good semigroup.

In what follows we want to give a characterization of when $S\bowtie^g E$ is an Arf semigroup.

Remark 2.1. We observe that $g(S)\cup E$ is closed with respect to the sum and, since $\mathbb{N}\setminus E$ is finite, we have that $\mathbb{N}\setminus(g(S)\cup E)$ is finite, that is $g(S)\cup E$ is a numerical semigroup. Now, we call $\pi_1: S\bowtie^g E\to S$ and $\pi_2\colon S\bowtie^g E\to T$ the canonical projections on the first and on the second component, respectively. We get $\pi_1(S\bowtie^g E)=S$ and $\pi_2(S\bowtie^g E)=g(S)\cup E$. Indeed, for the first equality it's enough to observe that $\pi_1(D)=S$; meanwhile for the second equality, if we take an element of the form $a\wedge b$, with $a\in D$ and $b\in g^{-1}(E)\times E$, then $(a\wedge b)_2=a_2\in g(S)$ or $(a\wedge b)_2=b_2\in E$. It follows immediately that $\pi_2(a\wedge b)=(a\wedge b)_2\in g(S)\cup E$.

The next lemma should be well known, but we did not find a direct reference for it, so we include its proof for the sake of completeness.

Lemma 1. Let $S\subseteq\mathbb{N}^2$ be a good semigroup and let $\rho_1: S\to\mathbb{N}$ and $\rho_2\colon S\to\mathbb{N}$ be the canonical projections on the first and on the second component, respectively.

Set $S_1 = \rho_1(S)$ and $S_2 = \rho_2(S)$; if S is an Arf semigroup, then S_1 and S_2 are Arf semigroups.

In particular, if $S \bowtie^g E$ is an Arf semigroup, then both S and $g(S) \cup E$ are Arf semigroups.

Proof. It is enough to prove the lemma for one projection, since it is independent on the choice of the component. Let $a_1, b_1, c_1 \in S_1$ be such that $a_1 \leq b_1 \leq c_1$. By definition, there are $a_2, b_2, c_2 \in S_2$ such that $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in S$. Moreover, we may assume $(a_1, a_2) \leq (b_1, b_2)$ and $(a_1, a_2) \leq (c_1, c_2)$. Indeed, if $(a_1, a_2) \not\leq (b_1, b_2)$ since $a_1 \leq b_1$ it follows that $b_2 < a_2$; therefore $(a_1, a_2) \wedge (b_1, b_2) = (a_1, b_2) \in \rho^{-1}(a_1)$ and $(a_1, b_2) \leq (b_1, b_2)$. Similarly, we have $(a_1, c_2) \leq (c_1, c_2)$ with $(a_1, c_2) \in \rho^{-1}(a_1)$. Hence, it is enough to replace (a_1, a_2) with

$$\big[(a_1, a_2) \wedge (b_1, b_2)\big] \wedge \big[(a_1, a_2) \wedge (c_1, c_2)\big]$$

to get $(a_1, a_2) \leq (b_1, b_2)$ and $(a_1, a_2) \leq (c_1, c_2)$.

Now, if S is an Arf semigroup we have $(b_1 + c_1 - a_1, b_2 + c_2 - a_2) \in S$, therefore $b_1 + c_1 - a_1 \in S_1$, thus S_1 is an Arf semigroup. □

Remark 2.2. Let E be an integrally closed ideal of $g(S) \cup E$ and let $a = b \wedge c$ with $b \in D$ and $c \in g^{-1}(E) \times E$. If we assume $a \neq b$ and $a \neq c$, then $a = (b_1, c_2)$ or $a = (c_1, b_2)$.

In the first case we have $c_2 \leq b_2 = sb_1$ so $b_2 \in E$, therefore $b_1 \in g^{-1}(E)$. In the second case we have $c_1 \leq b_1$, so $sc_1 \leq sb_1 = b_2$, with $sc_1 \in E$; therefore $b_2 \in E$ and $b_1 \in g^{-1}(E)$.

In conclusion, we have $a = b \in D$ or $a \in g^{-1}(E) \times E$ and hence

$$S \bowtie^g E = D \cup (g^{-1}(E) \times E).$$

Remark 2.3. Let E be an integrally closed ideal in $g(S) \cup E$ and let $a, b \in S \bowtie^g E$ such that $a \leq b$. If $a \in g^{-1}(E) \times E$, then $b \in g^{-1}(E) \times E$.

Indeed, if $b \in D$ and $a \in g^{-1}(E) \times E$ we have $a_2 \leq b_2 = sb_1$ so that $b_2 \in E$. Therefore $b_1 \in g^{-1}(E)$, hence $b \in g^{-1}(E) \times E$.

Proposition 2.4. *Assume that S and $g(S) \cup E$ are both Arf semigroups and let E be an integrally closed ideal of $g(S) \cup E$. Then $S \bowtie^g E$ is an Arf semigroup.*

Proof. Let $a, b, c \in S \bowtie^g E$ be such that $a \leq b$ and $a \leq c$; since S and $g(S) \cup E$ are Arf semigroups we have $b_1 + c_1 - a_1 \in S$ and $b_2 + c_2 - a_2 \in g(S) \cup E$. By the above remarks $S \bowtie^g E = D \cup (g^{-1}(E) \times E)$ and, if $a \in g^{-1}(E) \times E$ then $b, c \in g^{-1}(E) \times E$ too. So we need to consider the following three cases:

- if $a, b, c \in D$, then $b_2 + c_2 - a_2 = s(b_1 + c_1 - a_1) \in g(S)$, hence $b + c - a \in D \subseteq S \bowtie^g E$;

- if $a \in D$ and at least one of b and c is in $g^{-1}(E) \times E$, for instance $c \in g^{-1}(E) \times E$; we have $s(b_1 + c_1 - a_1) \in g(S) \cup E$ and $s(b_1 + c_1 - a_1) \geq sc_1$, with $sc_1 \in E$. Since E is integrally closed in $g(S) \cup E$, it follows that $s(b_1 + c_1 - a_1) \in E$, so $b_1 + c_1 - a_1 \in g^{-1}(E)$. Hence $b + c - a \in g^{-1}(E) \times E \subseteq S \bowtie^g E$;

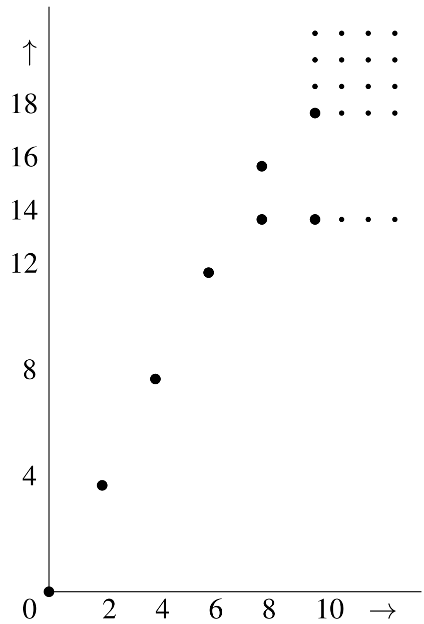
- if $a, b, c \in g^{-1}(E) \times E$ we have $s(b_1 + c_1 - a_1) \geq sc_1$ and $b_2 + c_2 - a_2 \geq c_2$ with $sc_1, c_2 \in E$, therefore $b_1 + c_1 - a_1 \in g^{-1}(E)$ and $b_2 + c_2 - a_2 \in E$. Hence $b + c - a \in g^{-1}(E) \times E \subseteq S \bowtie^g E$ (notice that, trivially, if E is integrally closed, also $g^{-1}(E)$ is integrally closed). □

The condition E integrally closed in $g(S) \cup E$ cannot be removed in order to obtain that $S \bowtie^g E$ is an Arf semigroup, as the following example shows.

Example 1. Let $S = \{0, 2, 4, 6, 8, 10, \rightarrow\}$ and $T = \{0, 4, \rightarrow\}$; both S and T are Arf semigroups. Let $E = \{14, 18, \rightarrow\} \subset T$ and let $g \colon S \rightarrow T$ the multiplication by 2, that is $g(a) = 2a$ for all $a \in S$. We have $g(S) = \{0, 4, 8, 12, 16, 20, 22, \dots\}$ and therefore

$$g(S) \cup E = \{0, 4, 8, 12, 14, 16, 18, \rightarrow\}.$$

Notice that $16 \notin E$, so E is not integrally closed in $g(S) \cup E$. Computing $g^{-1}(E)$ we get $\{10, \rightarrow\}$; thus $S \bowtie^g E$ is the good semigroup depicted in the following picture.



The obtained semigroup $S \bowtie^g E$ is not an Arf semigroup: if we consider $a = (8,14), b = (8,16)$ and $c = (10,14) \in S \bowtie^g E$, we have $a \le b$ and $a \le c$, but $b+c-a = (10,16) \notin S \bowtie^g E$.

We conclude the section showing that the sufficient condition of Proposition 2.4 is in fact a characterization.

Theorem 1. Given two numerical semigroups $S,T \subseteq \mathbb{N}$ let $g\colon S \to T$ a semigroup homomorphism and E a semigroup ideal of T.The following conditions are equivalent.

1. $S \bowtie^g E$ is an Arf semigroup.
2. S and $g(S) \cup E$ are Arf semigroups with E integrally closed in $g(S) \cup E$.

Proof. We need to show only the implication $1 \Rightarrow 2$. By Lemma 1 we know that $S \bowtie^g E$ are S and $g(S) \cup E$ are Arf numerical semigroups. So it remains to prove that E in integrally closed in $g(S) \cup E$.

Suppose, by contradiction, that E is not integrally closed in $g(S) \cup E$ and set $\tilde{e} = \min E$; then there is an element $b_2 \in g(S) \cup E$ such that $b_2 > \tilde{e}$ and $b_2 \notin E$; hence $b_2 \in g(S)$, and so $b_2 = g(b_1)$, for some $b_1 \in S$. Consider $b = (b_1,b_2) \in S \bowtie^g E$ and let c_2 be the maximal element in E such that $c_2 < b_2$; let $c = (c_1,c_2) \in g^{-1}(E) \times E$ be such that $c_1 > b_1$ (such an element exists, since $g^{-1}(E)$ is an ideal of S, so it has finite complement in $\mathbb{N}$). Let $a = b \wedge c \in S \bowtie^g E$; by construction we have $a \le b$, $a \le c$ and $a = (b_1,c_2)$.

We claim that $b+c-a = (c_1,b_2) \notin S \bowtie^g E$. Since $c_1 \neq b_1$ and $b_2 = g(b_1) \notin E$, the unique possibility to have $(c_1,b_2) \in S \bowtie^g E$ is that $(c_1,b_2) = d \wedge e$ for some $d \in D$ and $e \in g^{-1}(E) \times E$. Hence $(c_1,b_2) = (d_1,e_2)$ or $(c_1,b_2) = (e_1,d_2)$.

- If $(c_1,b_2) = (d_1,e_2)$, then $b_2 = e_2 \in E$, that is a contradiction.
- If $(c_1,b_2) = (e_1,d_2)$, then $g(b_1) = sb_1 = b_2 = d_2 = sd_1$, and so $b_1 = d_1$; but then $c_1 = e_1 \le d_1 = b_1$. That is a contradiction, because we chose $c_1 > b_1$.

This shows that $b+c-a \notin S \bowtie^g E$ and it is a contradiction against the assumption that it is Arf. □

3. Weakly Arf property for amalgamated algebras

For a commutative ring R let $W(R)$ be the set of non-zerodivisors of R and denote by $\overline{R}$ the integral closure of R in its total ring of fraction $Q(R)$. We recall that: given an ideal I of R, an element $r \in R$ is said to be *integral over I* if there exist an integer n and elements $a_i \in I^i$ for $i = 1, \dots, n$, such that

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0.$$

The set of all elements that are integral over I is called the *integral closure* of I, and is denoted $\bar{I}$ (for more details see [9]).

A commutative ring with unity R is said to be weakly Arf if: for any $x, y, z \in R$ with $x \in W(R)$ and $y/x, z/x \in \overline{R}$, then $yz/x \in R$.

In this section we are interested in determining when an amalgamation algebra is weakly Arf. So let R and S be commutative rings, let J be an ideal of S and $f\colon R \to S$ a ring homomorphism. As we defined in the introduction, the amalgamation of R with S along J with respect of f is the ring

$$R \bowtie^f J = \{(r, f(r) + j) \mid r \in R, j \in J\}.$$

We recall now some basic properties of the amalgamation that we will need in the sequel.

Remark 3.1. By [6, Proposition 5.1] we have:

1. Let $\iota := \iota_{R,f,J}\colon R \to R \bowtie^f J$ be the natural the ring homomorphism defined by $\iota(a) := (a, f(a))$, for all $a \in R$. Then ι is an embedding, making $R \bowtie^f J$ a ring extension of R (with $\iota(R) = \Gamma(f)$ $(:= \{(a, f(a)) \mid a \in R\}$ subring of $R \bowtie^f J)$.

2. Let $p_R\colon R \bowtie^f J \to R$ and $p_S\colon R \bowtie^f J \to S$ be the natural projections of $R \bowtie^f J \subseteq R \times S$ into R and S, respectively. Then p_R is surjective and $\ker(p_R) = \{0\} \times J$. Moreover, $p_S(R \bowtie^f J) = f(R) + J$ and $\ker(p_S) = f^{-1}(J) \times \{0\}$. Hence, the following canonical isomorphisms hold:

$$\frac{R \bowtie^f J}{(\{0\} \times J)} \cong R \text{ and } \frac{R \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(R) + J.$$

3. Let γ: $R \bowtie^f J \to (f(R) + J)/J$ be the natural ring homomorphism, defined by $(a, f(a) + j) \mapsto f(a) + J$. Then γ is surjective and $\ker(\gamma) = f^{-1}(J) \times J$. Thus, there exists a natural isomorphism

$$\frac{R \bowtie^f J}{f^{-1}(J) \times J} \cong \frac{f(R) + J}{J}.$$

Proposition 3.2 ([7, Proposition 3.1])**.** *Assume that J and $f^{-1}(J)$ are regular ideals of S and R, respectively. Then $Q(R \bowtie^f J)$ is canonically isomorphic to $Q(R) \times Q(S)$.*

Proposition 3.3 ([7, Proposition 3.4])**.** *Assume that J and $f^{-1}(J)$ are regular ideals of S and R, respectively. Then $\overline{R \bowtie^f J}$ (i.e., the integral closure of $R \bowtie^f J$ in its total ring of fractions) coincides with $\overline{R} \times \overline{f(R)+J}$. In particular, if f is an integral homomorphism (i.e. $f(R) \subseteq S$ is an integral extension), then $\overline{R \bowtie^f J} = \overline{R} \times \overline{S}$.*

In the light of the last two results, in order to study the weakly Arf property for $R \bowtie^f J$, we will need to always assume that J and $f^{-1}(J)$ are regular ideals of S and R, respectively. Moreover, to control the non-zerodivisors of the amalgamation, we will assume some extra hypothesis.

Proposition 3.4. *Assume J and $f^{-1}(J)$ are regular ideals of S and R, respectively and that $f(a) \in W(S)$ for all $a \in W(R)$. Then, if $R \bowtie^f J$ is a weakly Arf ring, also R is a weakly Arf ring.*

Proof. Let $a,b,c \in R$ be such that $c \in W(R)$ and $a/c, b/c \in \overline{R}$. By hypothesis $f(c) \in W(S)$, then $\alpha = (a/c, f(a)/f(c))$ and $\beta = (b/c, f(b)/f(c))$ are elements of $Q(R) \times Q(S)$. Moreover $(c, f(c)) \in W(R \bowtie^f J)$, therefore

$$\frac{(a,f(a))}{(c,f(c))}, \frac{(b,f(b))}{(c,f(c))} \in Q(R \bowtie^f J)$$

and we can write

$$\alpha = \frac{(a,f(a))}{(c,f(c))} \quad \text{and} \quad \beta = \frac{(b,f(b))}{(c,f(c))}.$$

Since $a/c \in \overline{R}$, there exists a monic polynomial $h(X) \in R[X]$ such that $h(a/c) = 0$, say $h(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$; define $f(h)(X) = X^n + \sum_{i=0}^{n-1} f(a_i) X^i \in S[X]$. Then it is straightforward that $\alpha = (a/c, f(a)/f(c))$ is a root of the polynomial $X^n + \sum_{i=0}^{n-1} (a_i, f(a_i)) X^i$, that is $\alpha \in \overline{R \bowtie^f J}$. Similarly we get that $\beta \in \overline{R \bowtie^f J}$.

By hypothesis $R \bowtie^f J$ is a weakly Arf ring, therefore

$$\frac{(a,f(a))(b,f(b))}{(c,f(c))} = \left(\frac{ab}{c}, \frac{f(ab)}{f(c)} \right) \in R \bowtie^f J$$

and so $ab/c \in R$, that is R is a weakly Arf ring. □

Now, we want to understand under which condition the converse of the previous proposition is true. To start we assume that R and S are integral domains, in this case we have the following.

Proposition 3.5 ([6, Proposition 5.2])**.** *With the notation above, we assume $J \neq \{0\}$. Then, the following conditions are equivalent.*

1. *$R \bowtie^f J$ is an integral domain.*

2. *$f(R)+J$ is an integral domain and $f^{-1}(J)=\{0\}$.*

We note that, if $J=\{0\}$ we have $R \bowtie^f J \cong R$. On the other hand, if $f^{-1}(J)=\{0\}$ we have $R \bowtie^f J \cong f(R)+J$. Hence we will always assume $J \neq \{0\}$ and $f^{-1}(J) \neq \{0\}$. In particular, we have that $R \bowtie^f J$ is not an integral domain. Moreover; since R and S are integral domains, then J and $f^{-1}(J)$ are regular ideals. So that, by Proposition 3.3, we get $\overline{R \bowtie^f J} \cong \overline{R} \times \overline{f(R)+J}$.

The following result will be very useful in the sequel.

Lemma 2. Let I be an ideal of a ring R, let $x,i \in R$ such that $x/i \in \overline{R}$ and $i \in I$. Then $x \in \bar{I}$.

Proof. Let $x,i \in R$ such that $x/i \in \overline{R}$ and $i \in I$. Then $x \in i\overline{R} \subseteq I\overline{R}$; since $\bar{I}=\overline{I\overline{R}} \cap R$ we get $x \in \bar{I}$. □

If we further assume that $\overline{f(R)+J}$ is a valuation domain, then the following holds are true.

Lemma 3. Assume that J is integrally closed in $f(R)+J$, and let $x,y \in f(R)+J$ be such that $x \notin J$ and $y=f(a)+j$, with $a \in R$ and $j \in J$; then $y/x \in \overline{f(R)+J}$ if and only if $f(a)/x \in \overline{f(R)+J}$.

Proof. Since $j \in J$ and $x \in f(R)+J$, we have $x/j \in \overline{f(R)+J}$ if and only if $x \in j(\overline{f(R)+J}) \cap (f(R)+J) \subseteq \overline{J}=J$; thus $x/j \notin \overline{f(R)+J}$, since $x \notin J$; therefore $j/x \in \overline{f(R)+J}$. Now, by

$$\frac{y}{x}=\frac{f(a)}{x}+\frac{j}{x}$$

the thesis follows immediately. □

Now, we are ready to show the following result.

Proposition 3.6. *Let R and S be two integral domains, let $(0) \neq J \subset R$ be an ideal of S such that $f^{-1}(J) \neq (0)$. We also assume that $\overline{f(R)+J}$ is a valuation domain. Then, if R and $f(R)+J$ are weakly Arf and J is integrally closed in $f(R)+J$, $R \bowtie^f J$ is weakly Arf.*

Proof. Let $\alpha,\beta,\gamma \in R \bowtie^f J$ such that $\gamma \in W(R \bowtie^f J)$ and $\alpha/\gamma, \beta/\gamma \in \overline{R \bowtie^f J}$, we write $\alpha=(a_1,a_2)$, $\beta=(b_1,b_2)$ and $\gamma=(c_1,c_2)$. So $a_1/c_1, b_1/c_1 \in \overline{R}$ and $a_2/c_2, b_2/c_2 \in \overline{f(R)+J}$. Since R and $f(R)+J$ are weakly Arf, we have $\frac{a_1b_1}{c_1} \in R$ and $\frac{a_2b_2}{c_2} \in f(R)+J$, hence $\alpha\beta/\gamma \in R \times (f(R)+J)$. Since J is integrally closed in $f(R)+J$, then $f^{-1}(J)$ is also integrally closed; thus, if $c_2 \in J$ (that implies also $c_1 \in f^{-1}(J)$), by Lemma 3.6 we get $\alpha, \beta, \gamma \in f^{-1}(J) \times J$. Furthermore, we note that

$$\frac{a_1 b_1}{c_1} = c_1 \frac{a_1}{c_1} \frac{b_1}{c_1} \in c_1 \overline{R} \cap R \subseteq \overline{(f^{-1}(J))\overline{R}} \cap R = \overline{f^{-1}(J)}$$

$$\frac{a_2 b_2}{c_2} = c_2 \frac{a_2}{c_2} \frac{b_2}{c_2} \in c_2(\overline{f(R)+J}) \cap (f(R)+J) \subseteq \overline{J(\overline{f(R)+J})} \cap (f(R)+J) = \overline{J}$$

hence

$$\frac{\alpha\beta}{\gamma} \in f^{-1}(J) \times J \subseteq R \bowtie^f J.$$

On the other hand, if $c_2 \notin J$, by Lemma 3.7, we can assume that $a_2, b_2, c_2 \in f(R)$. By definition of $R \bowtie^f J$, it follows that a pair $(x,y) \in R \bowtie^f J$ if and only if $f(x) - y \in J$, therefore we want to show that $f(a_1 b_1 / c_1) - f(a_1) f(b_1) / f(c_1) \in J$. Indeed

$$f\left(\frac{a_1 b_1}{c_1}\right) - \frac{f(a_1)f(b_1)}{f(c_1)} = \frac{f(c_1) f(\frac{a_1 b_1}{c_1}) - f(a_1) f(b_1)}{f(c_1)} =$$

$$= \frac{f(c_1 \frac{a_1 b_1}{c_1}) - f(a_1 b_1)}{f(c_1)} = 0 \in J$$

Hence, also in this case we have $\alpha\beta/\gamma \in R \bowtie^f J$. □

Now we would like to understand when the converse of the above proposition holds. Before doing that, we need to show some more preliminary results. First of all we recall that, if R is a local, noetherian, one-dimensional domain, whose integral closure $\overline{R}$ is a DVR, then, every ideal I has a principal reduction (x), with x an element of minimal value in I; moreover $\overline{I} = x\overline{R} \cap R$ (see [9, Proposition 1.6.1]), and thus, for any $i \in I$, $i/x \in \overline{R}$.

Lemma 4. Let R be a local noetherian domain, such that $\overline{R}$ is a DVR and $(R\colon \overline{R}) \neq (0)$; denote the associated valuation by v. Let I be an ideal of R and let $e_0 \in I$ be a minimal reduction of I. Let $c \in (R\colon \overline{R}) \setminus \{0\}$.

Then, for any $a \in \overline{R}$ such that $v(a) > v(ce_0)$, we have $a \in I$.

Proof. By $c \in (R\colon \overline{R})$ it follows that $c\overline{R} \subseteq R$; therefore, if $v(a) > v(ce_0)$, we have $a \in e_0(c\overline{R}) \subseteq I$. Thus $a \in I$. □

Lemma 5. Let R be a local noetherian domain such that $\overline{R}$ is a DVR and $(R\colon \overline{R}) \neq (0)$; denote the associated valuation by v. Denote by $\mathfrak{n}$ the maximal ideal of $\overline{R}$ and by $\mathfrak{m} = \mathfrak{n} \cap R$ the maximal ideal of R. Let I be an ideal of R not integrally closed and let $e_0 \in I$ be a minimal reduction of I.

If $\overline{R}/\mathfrak{n} \cong R/\mathfrak{m}$, then there exists $a \in R \setminus I$ such that $a/e_0 \in \overline{R}$ and $v(a) \notin v(I) = \{v(i) | i \in I \setminus \{0\}\}$.

Proof. Since I is not integrally closed, there exists $a \in \overline{I} \setminus I \subseteq R$; the choice of e_0 implies that $v(a) - v(e_0) > 0$, and so $a/e_0 \in \overline{R}$.

If $v(a) \in v(I)$, that is $v(a) = v(i_1)$ for some $i_1 \in I$, $a/i_1 \in \overline{R} \setminus \mathfrak{n}$; by hypothesis there exists $b_1 \in R \setminus \mathfrak{m}$ such that $b_1 - a/i_1 \in \mathfrak{n}$, that implies $v(b_1 i_1 - a) = v(i_1(b_1 - a/i_1)) > v(i_1) = v(a)$; furthermore $a - b_1 i_1 \in R \setminus I$ and $(a - b_1 i_1)/e_0 \in \overline{R}$.

If $v(a - b_1 i_1) \notin v(I)$ we get the thesis replacing a with $a - b_1 i_1$. Otherwise, as above, we can find $i_2 \in I$ and $b_2 \in R \setminus \mathfrak{m}$ such that $v(a - b_1 i_1 - b_2 i_2) > v(a - b_1 i_1)$; furthermore $a - b_1 i_1 - b_2 i_2 \in R \setminus I$ and $(a - b_1 i_1 - b_2 i_2)/e_0 \in \overline{R}$.

Inductively, for any $r \in \mathbb{N}_+$, if $v(a_{r-1}) \in v(I)$, we can find $i_r \in I$ and $b_r \in R \setminus I$ such that the element $a_r = a - b_1 i_1 - \dots - b_r i_r \in R \setminus I$, $a_r/e_0 \in \overline{R}$ and $v(a_r) > v(a_{r-1})$.

We claim that, after a finite number of steps we get $v(a_r) \notin v(I)$. Indeed, if $c \in (R \colon \overline{R}) \setminus \{0\}$, since the sequence of the $v(a_i)$ is strictly increasing, there exists $r \in \mathbb{N}_+$ such that $v(a_r) > v(ce_0)$ and, by Lemma 4, we get $a_r \in I$ that is a contradiction. □

Now, we assume that both R and S are integral domains and $\overline{f(R)+J}$ is a DVR with valuation v. We assume that J is a finitely generated ideal of S and that $f^{-1}(J) \neq \{0\}$. Then, by [7, Proposition 3.1], we have $\overline{R \bowtie^f J} = \overline{R} \times \overline{f(R)+J}$ Assume also that $(f(R)+J \colon \overline{f(R)+J}) \neq (0)$ and that the residue fields of $\overline{f(R)+J}$ and $f(R)+J$ are isomorphic (so we can apply the previous lemma).

With this setting, we have the following.

Theorem 2. If J is not integrally closed in $f(R)+J$, then $R \bowtie^f J$ is not weakly Arf.

Proof. Let $e_0 \in J$ be such that $J(\overline{f(R)+J}) = e_0(\overline{f(R)+J})$; since J is not integrally closed in $f(R)+J$ we can take $y \in f(R)+J$ such that $y/e_0 \in \overline{f(R)+J}$ and, by Lemma 5, we can assume that $v(y) \notin v(J)$; if we write $y = f(x)+j$ with $x \in R$ and $j \in J$, we set $a = (x,y) \in R \bowtie^f J$. Let $i \in J$ be such that $v(y) > v(i)$ (e.g. take $i = e_0$) and set $b = (0,i) \in R \bowtie^f J$. Let $c = a+b \in R \bowtie^f J$ and we observe that $a/c, b/c \in \overline{R \bowtie^f J}$; indeed we have

$$\frac{a}{c} = \frac{(x,y)}{(x,y+i)} = \left(1, \frac{y}{y+i}\right) \qquad \text{and} \qquad \frac{b}{c} = \frac{(0,i)}{(x,y+i)} = \left(0, \frac{i}{y+i}\right)$$

clearly $0, 1 \in \overline{R}$; moreover $v(y) - v(y+i) = v(y) - v(i) > 0$ and $v(i) - v(y+i) = 0$, thus $\frac{y}{y+i}, \frac{i}{y+i} \in \overline{f(R)+J}$.

Now, we want to show that

$$\frac{ab}{c} = \left(0, \frac{yi}{y+i}\right) \notin R \bowtie^f J$$

that is $\frac{yi}{y+i} \notin J$. If $\frac{yi}{y+i} \in J$ we would have $yi \in (y+i)J$, so $yi = (y+i)h$ for some $h \in J$; then $v(y)+v(i) = v(y+i)+v(h)$; since $v(y+i) = v(i)$ we get $v(y) = v(h) \in v(J)$, that is a contradiction. Thus $ab/c \notin R \bowtie^f J$, that is $R \bowtie^f J$ is not weakly Arf. □

Theorem 3. Under the standing assumptions, the following are equivalent:

(i) $R \bowtie^f J$ is weakly Arf.

(ii) R and $f(R)+J$ are weakly Arf and J is integrally closed in $f(R)+J$.

Proof. By Proposition 3.6 we need only prove that *(i)* implies *(ii)*. Furthermore, if *(i)* holds, then J is integrally closed in $f(R)+J$ by Theorem 2. So it remains to prove only that R and $f(R)+J$ are weakly Arf.

Let $a,b,c \in R$ such that $c \neq 0$ and $a/c, b/c \in \overline{R}$; we set $\alpha = (a, f(a))$ and $\beta = (b, f(b)) \in R \bowtie^f J$. Two cases can occur:

1. $f(c) \neq 0$; in this case $\gamma = (c, f(c)) \in W(R \bowtie^f J)$ and $\alpha/\gamma, \beta/\gamma \in \overline{R \bowtie^f J}$. Since $R \bowtie^f J$ is weakly Arf, we get $\alpha\beta/\gamma \in R \bowtie^f J$. Thus $ab/c \in R$;

2. $f(c) = 0$; we have $c \in f^{-1}(J)$, therefore $a \in c\overline{R} \subseteq \overline{f^{-1}(J)} = f^{-1}(J)$. Since J is a finitely generated ideal, there exists $e \in J$ such that $v(e) = \min v(J)$; note that $v(f(a)) \geq v(e)$ and $v(f(b)) \geq v(e)$, hence $f(a)/e, f(b)/e \in \overline{f(R)+J}$. If we set $\gamma = (c,e) \in W(R \bowtie^f J)$, then $\alpha/\gamma, \beta/\gamma \in \overline{R \bowtie^f J}$; by hypothesis we get $\alpha\beta/\gamma \in R \bowtie^f J$, thus $ab/c \in R$.

This shows that R is weakly Arf.

Now, we check that $f(R)+J$ is weakly Arf. Let $x,y,z \in f(R)+J$ be such that $x \neq 0$ and $y/x, z/x \in \overline{f(R)+J}$; set $x = f(c)+h$, $y = f(a)+j$ and $z = f(b)+i$, with $a,b,c \in R$ and $j,i,h \in J$. Again two cases can occur:

1. $x \in J$; by $y/x, z/x \in \overline{f(R)+J}$, we have $y,z \in J$. Pick $d \in J$ such that $d \neq 0$ and set $\alpha = (d,y)$, $\beta = (d,z)$ and $\gamma = (d,x)$; then $\alpha, \beta \in R \bowtie^f J$ and $\gamma \in W(R \bowtie^f J)$. Therefore:

$$\frac{\alpha}{\gamma} = \left(\frac{d}{d}, \frac{y}{x}\right) = \left(1, \frac{y}{x}\right) \in \overline{R \bowtie^f J}$$

$$\frac{\beta}{\gamma} = \left(\frac{d}{d}, \frac{z}{x}\right) = \left(1, \frac{z}{x}\right) \in \overline{R \bowtie^f J};$$

since $R \bowtie^f J$ is weakly Arf, we get $\alpha\beta/\gamma \in R \bowtie^f J$, thus $yz/x \in f(R)+J$;

2. $x \notin J$; by Lemma 3 we have $f(a)/x \in \overline{f(R)+J}$, therefore

$$f(a) \in x(\overline{f(R)+J}) \subseteq f(c)(\overline{f(R)+J}) + J.$$

Then there is $k \in J$ such that $(f(a)+k)/f(c) \in \overline{f(R)+J}$; again by Lemma 3 we get $f(a)/f(c) \in \overline{f(R)+J}$. Hence, we can assume $x, y, z \in f(R)$. Set $\alpha = (a,y), \beta = (b,z), \gamma = (c,x) \in R \bowtie^f J$; since $x \notin J$ we obtain $c \neq 0$ so that $\gamma \in W(R \bowtie^f J)$; by hypothesis, $y/x \in \overline{f(R)+J}$ therefore there is a monic polynomial $P(t) \in (f(R)+J)[t]$ such that $P(\frac{y}{x}) = 0$. We write

$$P(t) = t^n + \sum_{i=0}^{n-1} w_i t^i$$

where $w_i = f(m_i) + k_i$ with $m_i \in R$ and $k_i \in J$; we set

$$Q(t) = t^n + \sum_{i=0}^{n-1} m_i t^i \in R[t]$$

by $f(Q(t)) = P(t) - \sum_{i=0}^{n-1} k_i t^i$ we get $f(Q(\frac{a}{c})) \in J(\overline{f(R)+J}) \subseteq J$, therefore $Q(t) - Q(\frac{a}{c}) \in R[t]$ and $(Q(t) - Q(\frac{a}{c}))(\frac{a}{c}) = 0$; thus $a/c \in \overline{R}$; similarly $b/c \in \overline{R}$. Then $\alpha/\gamma, \beta/\gamma \in \overline{R \bowtie^f J}$, so that $\alpha\beta/\gamma \in R \bowtie^f J$ and thus $yz/x \in f(R)+J$.

This shows that $f(R)+J$ is weakly Arf. □

Example 2. Let $R = k[X_1, \ldots, X_n]$ and $S = k[[Y_1, \ldots, Y_m]]$, let $f\colon R \to S$ the ring homomorphism defined by $f(X_i) = Y_1$ for any $i = 1, \ldots, n$ and $f(1) = 1$. Let J be the ideal of S generated by Y_1. It's easy to check that all hypothesis of Theorem 3 are satisfied observing that $f(R)+J = k[[Y_1]]$ is a DVR. Now; because R and $k[[Y_1]]$ are UFDs, by [9, Proposition 2.1.5] they are integrally closed, and hence they are weakly Arf rings. Moreover (Y_1) is an ideal integrally closed in $k[[Y_1]]$, therefore J is integrally closed in $f(R)+J$. Hence, by Theorem 3 the ring $R \bowtie^f J$ is a weakly Arf ring.

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