LE MATEMATICHE
Vol. LXXIX (2024) – Issue II, pp. 449–467
doi: 10.4418/2024.79.2.11

# A SURVEY ON RECENT RESULTS ON BALANCED GRAPH AND HYPERGRAPH DESIGNS

P. BONACINI - M. GIONFRIDDO - L. MARINO

In this paper we recall some results on balanced type conditions for graph and hypergraph designs. The usual definition of balanced graph design (see [23]) is the following: a $G$-design is balanced if the number of blocks containing any vertex is constant. Later, other balanced type conditions have been introduced for graph designs and extended to hypergraph designs. In the latter case the new notion of edge balanced has been introduced, being related to the number of times that a pair is contained in an edge of a block.

## 1. Introduction

Let $K_v$ be the complete graph on $v$ vertices and let $G = (X, \mathcal{E})$ be a graph on $n$ vertices. A *G-design* of index $\lambda$ is a pair $\Sigma = (V, \mathcal{B})$, where $V$ is a set of $v$ elements and $\mathcal{B} = \{G_1, \dots, G_b\}$, such that:

- $G_i \cong G$ for any $i = 1, \dots, b$
- for any $x, y \in V$, $x \neq y$, there exist $G_{i_1},\dots,G_{i_\lambda}$, $i_1, \dots, i_\lambda \in \{1, \dots, b\}$ pairwise different, such that $\{x, y\} \in E(G_{i_j})$ for $j = 1, \dots, \lambda$.

Received on September 13, 2024

*AMS 2010 Subject Classification:* 05B05, 05C51
*Keywords:* $G$-design, balanced, locally balanced
The authors are both supported by the University of Catania, "PIACERI 2020/22, Linea di intervento 2". Both authors are members of GNSAGA of INDAM (Italy).

The elements of $\mathcal{B}$ are called *blocks* and we say that $\Sigma$ is a $G$-decomposition of $\lambda K_v$. If otherwise stated, we will suppose that $\lambda = 1$.

Given a $G$-design $\Sigma = (V, \mathcal{B})$, the *degree* of a vertex $x \in V$ is the number of blocks of $\mathcal{B}$ containing $x$. $\Sigma$ is called *balanced* if there exists $d \in \mathbb{N}$ such that $d(x) = d$ for all $x \in V$.

Let $G = (X, \mathcal{E})$ be a graph. An *automorphism class* of $G$ is a subset $A \subseteq X$ such that for every $x, y \in A$ there exists an automorphism $\varphi$ of $G$ such that $\varphi(x) = y$. We will denote by $A_1, \ldots, A_s$ the automorphism classes of $G$.

Given a $G$-design $\Sigma = (V, \mathcal{B})$ and an automorphism class $A_i$ of $G$, we denote by $d_i(x)$ the *degree* of a vertex $x \in V$ in $A_i$, which is the number of blocks of $\Sigma$ containing $x$ as an element of $A_i$.

In [2] the notion of strongly balanced design has been introduced. A $G$-design $\Sigma = (V, \mathcal{B})$ is *strongly balanced* if, for every $i = 1, ..., s$, all the vertices of $X$ are contained in the blocks of $\Sigma$ the same number of times as element of $A_i$, i.e. if, for every $i = 1, ..., s$, there exists a constant $d_i \in \mathbb{N}$ such that $d_i(x) = d_i$ for every $x \in V$. A $G$-design $\Sigma$ is called *simply balanced* if it is balanced but not strongly balanced.

In Section 2 we recall some general results on balanced and strongly balanced graph designs. In Section 3 we consider some general cases such as those of cycles, paths, stars and complete bipartite graphs, in which well known results have been proved. In Section 4 we examine the case of graphs with few vertices. Note that in [14], in which the author consider graphs with 5 vertices and in which the concept of orbit-balanced design is equivalent to the one of strongly balanced design, the concept of degree balanced $G$-design has been introduced. This means that the number of times that a vertex appears in a block as an element of degree $d$ is constant for any degree $d$. In Section 5 we recall the concepts of *locally balanced* and *strictly balanced G-design* introduced in [10].

The remaining sections are devoted to the generalization of the previous definitions to the hypergraph design case. It is relevant to note that for a hypergraph design it is possible to give the definition of edge balanced hypergraph design, in which the focus is on the number of times that a pair of vertices is contained in a 3-edge of a block. Note also that edge balanced hypergraph designs have been used in [11] in order to obtain a specific hypergraph design which is colorable (according to Voloshin colorings) with a number of colors that don't determine an interval of integers.

## 2. Necessary conditions for graph designs

In this section we will show some conditions that hold for balanced and strongly balanced graph designs. The first immediate condition is given by:

**Theorem 2.1.** *Let $G$ be a graph and let $n = |V(G)|$ and $m = |E(G)|$. Let $\Sigma = (V, \mathcal{B})$ be a balanced $G$-design of order $v$. Then $v(v-1) \equiv 0 \mod 2m$ and $n(v-1) \equiv 0 \mod 2m$.*

*Proof.* The first condition is immediate, since we know that $|\mathcal{B}| = \dfrac{v(v-1)}{2m}$. If $d = d(x)$ for any $x \in V$, then:

$$d \cdot v = n \cdot |\mathcal{B}|$$

and this leads us to the conclusion. □

It is immediate to see that a strongly balanced design is also balanced and the converse, in general, is not true. For example:

**Example 2.2.** The graph called *kite* is the graph $C_3 + P_2$, i.e. it is the graph with vertices $\{a,b,c,d\}$ and edges $\{a,b\}$, $\{b,c\}$, $\{c,a\}$ and $\{c,d\}$. We denote it with $(a,b,c) - d$. Note that this graph has 3 automorphisms classes, $A_1 = \{a,b\}$, $A_2 = \{c\}$ and $A_3 = \{d\}$.

Let us consider the design of order 9 on the vertex set $\{0,1,\dots,7\} \cup \{\infty\}$, with $\infty \notin \{0,1,\dots,7\}$, having set of blocks:

$$(\infty,0,4)-1,\ (\infty,5,1)-2,\ (\infty,6,2)-3,\ (\infty,7,3)-4,$$
$$(5,4,2)-0,\ (6,5,3)-1,\ (6,4,7)-2,\ (7,5,0)-3,\ (6,0,1)-7.$$

This design is balanced, because any vertex has degree 4 in the design. However, it is not strongly balanced, since the vertex $\infty$ appears only as element of $A_1$.

In general, the following proposition is trivial:

**Proposition 2.3.** *Let $G$ be a graph, let $A_1,\dots,A_s$ be its automorphism classes and let $d_{A_i}$ be the degree of the vertices of $A_i$ in $G$ for $i = 1,\dots,s$. If $\Sigma = (V,\mathcal{B})$ is a $G$-design of order $v$ and for any $x \in V$ and any $i = 1,\dots,s$ $d_i(x)$ is the degree of $x$ in $A_i$, then the following equalities hold for any $x \in V$:*

$$\begin{cases} \sum_{i=1}^{s} d_i(x) = d(x) \\ \sum_{i=1}^{s} d_{A_i} d_i(x) = v-1. \end{cases}$$

Note that if $s = 1$ obviously the concepts of balanced and strongly balanced designs coincide. Moreover, the following result is an immediate consequence of Proposition 2.3 (see also [12]):

**Corollary 2.4.** *Let $G$ be a graph, let $A_1$ and $A_2$ its automorphism classes and let $d_{A_1}$ and $d_{A_2}$ the degrees of the vertices of $A_1$ and $A_2$, respectively, with $d_{A_1} \neq d_{A_2}$. Then any balanced $G$-design is strongly balanced.*

Since it is well known that all regular graphs with at most 6 vertices are vertex transitive, i.e. they have only one automorphism class, we get the following:

**Corollary 2.5.** *Let $G$ a regular graph with at most 6 vertices. Then any balanced $G$-design is strongly balanced.*

It is also immediate to note that:

**Theorem 2.6.** *Let $G$ be a graph and let $m = |E(G)|$. Let $A_1,\dots,A_s$ be its automorphism classes. If $\Sigma = (V,\mathcal{B})$ is a strongly balanced $G$-design of order $v$, then $v(v-1) \equiv 0 \mod 2m$ and $|A_i|(v-1) \equiv 0 \mod 2m$ for any $i = 1,\dots,s$.*

*Proof.* This follows by the obvious fact that $v \cdot d_i(x) = |A_i| \cdot \frac{v(v-1)}{2m}$ for $i = 1,\dots,s$. □

So we see that:

**Corollary 2.7.** *Let $G$ be a graph, with $|E(G)| = m$, and let $A_1$ and $A_2$ be its automorphism classes. Suppose that the vertices of $A_1$ and the vertices of $A_2$ have different degrees. Then any balanced $G$-design of order $v$ is strongly balanced and $gcd(|A_1|,|A_2) \cdot (v-1) \equiv 0 \mod 2m$.*

The following example shows that there exist regular graphs $G$ with two automorphism classes and $G$-designs that are balanced but not strongly balanced.

**Example 2.8.** Let us consider the following 3-regular graph $G$ on the 8 vertices $\{a,b,c,d,e,f,g,h\}$ having edges $\{a,b\}$, $\{b,h\}$, $\{h,c\}$, $\{c,d\}$, $\{d,e\}$, $\{e,a\}$, $\{a,f\}$, $\{f,d\}$, $\{b,g\}$, $\{g,c\}$, $\{e,f\}$ and $\{g,h\}$.

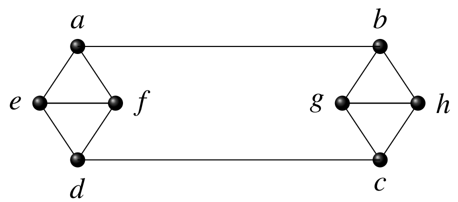



Figure 1: The graph $G$

Note that we have two automorphism classes in $G$, $A_1 = \{a,b,c,d\}$ and $A_2 = \{e,f,g,h\}$.

Now, let $V = \{0,\dots,24\}$ and let us consider the $G$-design $\Sigma = (V,\mathcal{B})$, where $\mathcal{B}$ is the set of all the following blocks $B_i$ for $i = 0,1,\dots,24$:

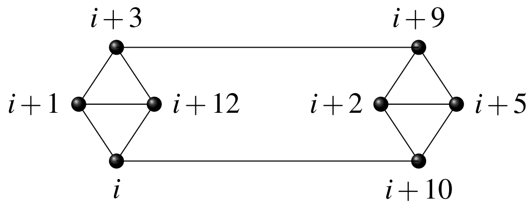



Figure 2: The block $B_i$

Obviously, the system $\Sigma$ is balanced and strongly balanced. Now consider the blocks $B_0$ and $B_2$ and replace the vertex 9 in $B_0$ with 14 and the vertex 14 in $B_2$ with 9. We get a new $G$-design $\Sigma'$ which is still balanced but not strongly balanced.

## 3. First cases

The first graphs $G$ for which the existence of graph designs has been studied are cycles and complete graphs. These graphs, of course, are regular and have both one automorphism class. So, in this cases any $G$-design is, as we said earlier, balanced and strongly balanced.

If $G = K_k$, the complete graph on $k$ vertices, a $K_k$-design of order $v$ and index $\lambda$ is simply a Steiner system $S_\lambda(2,k,v)$, denoted by $S(2,k,v)$ if $\lambda = 1$, also known as a $(v,k,\lambda)$-BIBD. There is an extremely vast literature on BIBDs and it is not in the intention of this survey to explore this subject. So, we suggest the well known [17] as a starting point.

As for cycles, the problem for the existence of $C_k$-designs has been settled by Alspach and Gavlas [1] in the case $k$ odd and by Šajna [33] in the case $k$ even:

**Theorem 3.1.** *Let $k \geq 3$. There exists a $C_k$-design of order $v$ if and only if the $v \geq k$, $v$ is odd and $v(v-1) \equiv 0 \mod 2k$.*

Balanced $G$-designs, which generalize BIBD, have been introduced in [23], in which the authors study the existence of balanced $G$-designs in the case in which $G$ is a path. The case of balanced $P_k$-designs, with index $\lambda \geq 1$, has been completely solved in a series of papers (see [25, 28–31]), where Huang [25] and Hung and Mendelsohn [29] showed the complete solution.

**Theorem 3.2.** *There exists a balanced $P_k$-design of order $v$ and index $\lambda \geq 1$ if and only if the following conditions hold:*

- $\lambda v(v-1) \equiv 0 \mod 2(k-1)$;

- $\lambda k(v-1) \equiv 0 \mod 2(k-1)$;
- $\lambda(v-1) \equiv 0 \mod (k-1)$.

The concept of strongly balanced $G$-designs has been introduced by Huang in [26], but developed starting from path designs by Berardi, Gionfriddo and Rota in [2], in which they introduce the concept of simply balanced graph designs and study the existence of simply balanced $P_5$ and $P_6$-designs.

Note that in a path $P_k$ each automorphism class has two vertices for $k$ even while there esists an automorphism class with just one vertex for $k$ odd. So, since a cyclic design is clearly strongly balanced, we can see that by Lawless([30]) the condition for the existence of strongly balanced path designs with index $\lambda = 1$ has been proved:

**Theorem 3.3.** *There exists a strongly balanced $P_k$-design of order $v > k$ and index $\lambda = 1$ if and only if either $k$ is odd and $v \equiv 1 \mod 2(k-1)$ or $k$ is even and $v \equiv 1 \mod (k-1)$.*

A classical method to construct simply balanced graph designs involves modifying the blocks provided by a cyclic graph design, as done in Example 2.8. In [2] the authors provide iterative constructions for the following result:

**Theorem 3.4.**
1. *A simply balanced $P_5$-design of order $v$ exists if and only if $v \equiv 1 \mod 8$, $v \geq 9$.*
2. *A simply balanced $P_6$-design of order $v$ exists if and only if $v \equiv 1 \mod 5$, $v \geq 6$.*

Other general results have been obtained in the case that the graph $G$ is a star $S_k$, i.e. a connected graph with $k$ vertices and $k-1$ edges all incident in the same vertex. By [32] (see also [24, 26, 27]) we have that:

**Theorem 3.5.** *There exists a balanced $S_k$-design of order $v$ and index $\lambda = 1$ if and only if $v \equiv 1 \mod 2(k-1)$.*

Note that by Corollary 2.4 any balanced $S_k$-design is also strongly balanced and also that the constructions provided above are all cyclic. Indeed, considering in general any complete bipartite graph, in [24, 26] (see also [32]) it was proved that:

**Theorem 3.6.** *There exists a cyclic $K_{p,q}$-design of order $v$ and index $\lambda$ for any $\lambda$ such that either $p \equiv 0 \mod \lambda$ or $q \equiv 0 \mod \lambda$ and $v \equiv 1 \mod (2pq/\lambda)$, with $v \geq pq+1$.*

## 4. Graph designs with few vertices

In this section we consider graphs $G$, connected or not, without isolated points, with at most 6 vertices. First, note that with 3 vertices the only graph to be considered are $P_3$ and $K_3 = C_3$, both covered in the previous section.

If $G$ has 4 vertices, then $G$ can be one of the following graphs: $2P_2$, i.e. a not connected graph with 2 edges, $C_4$, $P_4$, $S_4$, the so called kite, which is a 3-cycle with a pendant edge, $K_4 - e$ and $K_4$. The cases of cycles, paths stars and complete graphs have been discussed in the previous section. In the other cases, we have:

- if $G = 2P_2$ and a balanced $G$-design of order $v$ exists, then $v \equiv 0,1 \mod 4$ and $v \geq 4$; in the case that $v \equiv 1 \mod 4$, it is immediate to see that for any of these values of $v$ there exists a cyclic $G$-design of order $v$; in the case that $v \equiv 0 \mod 4$, any factorization of $K_v$ induces a $2P_2$-decomposition of $K_v$; moreover, since $G$ is regular, any $G$-design is also strongly balanced;
- if $G$ is a kite and a balanced $G$ design of order $v$ exists, then $v \equiv 1 \mod 8$, $v \geq 9$; in this case, in [4] it is proved the existence of a cyclic $G$-design of order $v$, which so is also strongly balanced;
- if $G = K_4 - e$ and a balanced $G$ design of order $v$ exists, then $v \equiv 1 \mod 5$, $v \geq 6$; in this case, in [4] it is proved the existence of a cyclic $G$-design of order $v$, which so is also strongly balanced.

So, if $G$ has at most 4 vertices, the necessary conditions for the existence of a balanced $G$-design are also sufficient and for any admissible value $v$ there exists a strongly balanced $G$-design that is cyclic.

In [14] Bonvicini consider graphs $G$, connected or not and without isolated points, having 5 vertices. For such graphs the author completes the study of the spectrum of balanced and strongly balanced $G$-designs (called also *orbit balanced*), considering also the condition for the existence of *degree balanced* graph designs, introduced in [12]. A $G$-design $\Sigma = (V, \mathcal{B})$ is called degree balanced if for any $x \in V$ and any $d \in \mathbb{N}$ the number $r_d(x)$ of blocks containing $x$ as a vertex of degree $d$ is independent from $x$. Clearly, any degree balanced graph design is balanced and, if $G$ is a regular graph, a balanced $G$-design is degree balanced.

There are 23 graphs with 5 vertices and without isolated points. Keeping the notation used in [3] and [14], they are denoted by $G_1, \dots, G_{23}$, where $G_3 = P_5$, $G_5 = S_5$, $G_{10} = C_5$, $G_{14} = K_{2,3}$ and $G_{23} = K_5$. In [14] (see also all the references therein) it is proved that for all the graphs $G_i$, for $i = 1, \dots, 23$ the necessary conditions for the existence of a balanced $G_i$-design of order $v$ are also sufficient, with $v > 5$ for $i \neq 6, 10, 23$.

In [14] it is also proved (again, see all the references therein) that the necessary conditions for the existence of strongly balanced $G_i$-designs, for $i = 1,\dots,23$, are all sufficient. Moreover, we can note that there exist cyclic strongly balanced $G_i$-designs of order $v$ in the following cases:

- for $i \in \{1,\dots,13,14,15\} \cup \{17\}$ and for any admissible value $v$;
- for $i = 16$ and for any admissible value $v$, with $15 \le v \le 71$;
- for $i = 18$ and $v = 15,29$;
- for $i = 19$ and $v = 15,29,71$;
- for $i = 20$ and $v = 17,33,49,97,113,177$;
- for $i = 21$ and $v = 17,33,65$;
- for $i = 22$ and $v = 19,37,55,109$.

In the case of $G$-designs, with $G$ graph with 6 vertices we have a few results. In [10] the authors consider the graph $C_4 + P_3$. It has vertex set $V = \{x,y,z,t,w_1,w_2\}$ and edges $\{x,y\}$, $\{y,z\}$, $\{z,t\}$, $\{t,x\}$, $\{x,w_1\}$, $\{w_1,w_2\}$. In [10] the existence of strongly balanced $(C_4+P_3)$-designs is studied and it is proved that:

**Theorem 4.1.** *There exists a strongly balanced $(C_4+P_3)$-design if and only if $v \equiv 1 \mod 12$, $v \ge 13$.*

Note that the designs provided in the above result are all cyclic.

In [13] the authors consider the 6 trees with 6 vertices. The 4 trees that are not the star or the path are the following:

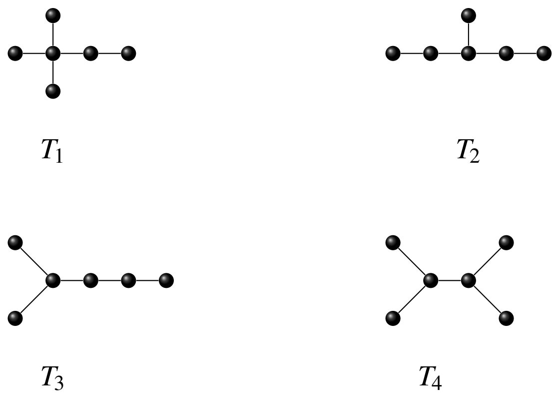



Figure 3

In [13] the following results are proved:

**Theorem 4.2.** *Let $i \in \{1,2,3,4\}$ and let $v \in \mathbb{N}$, $v > 1$. Then:*

- *for $i = 1,2,3$ there exists a simply balanced $T_i$-design of order $v$ if and only if $v \equiv 1 \mod 10$, with $v \geq 31$ for $i = 1$;*
- *for $i = 1,2,3$ there exists a strongly balanced $T_i$-design if and only if $v \equiv 1 \mod 10$;*
- *there exists a strongly balanced $T_4$-design of order $v$ if and only if $v \equiv 1 \mod 5$.*

Note that by Corollary 2.4 any balanced $T_4$-design is also strongly balanced.

## 5. Locally balanced designs

In [10] the notion of locally balanced $G$-designs has been introduced:

**Definition 5.1.** Let $G$ be a graph and let $A_1,\dots,A_s$ its automorphism classes. Let $\Sigma = (V,\mathcal{B})$ be a $G$-design and let $T \subseteq \{A_1,\dots,A_s\}$, $T \neq \emptyset$. The system $\Sigma$ is called:

- *locally $A_i$-balanced* if there exists a constant $c_i \in \mathbb{N}$ such that $d_i(x) = c_i$, for every $x \in V$;
- *locally $T$-balanced* if $\Sigma$ is locally $A_i$-balanced for any $A_i \in T$;
- *strictly $T$-balanced* if $\Sigma$ is locally $A_i$-balanced for any $A_i \in T$ and if $\Sigma$ is not locally $A_j$-balanced for any $A_j \notin T$.

The obvious necessary condition for a $G$-design in order to be $T$-balanced is given by the following:

**Theorem 5.2.** *Let $G$ be a graph with automorphism classes $A_1,\dots,A_s$ and let $m = |E(G)|$. Let $T = \{A_{i_1},\dots,A_{i_h}\}$ for some $i_1,\dots,i_h \in \{1,\dots,s\}$. Let $\Sigma = (V,\mathcal{B})$ be a $T$-balanced $G$-design and let $g = \mathrm{GCD}(|A_{i_1}|,\dots,|A_{i_h}|)$. Then $g(v-1) \equiv 0 \mod 2m$.*

It is also worthy to remark that:

**Theorem 5.3.** *Let $G$ be a graph and let $A_1,\dots,A_s$ be its automorphism classes. Let $\Sigma = (V,\mathcal{B})$ be a strictly $T$-balanced $G$-design. Then either $\Sigma$ is strongly balanced or $s \geq 3$ and $|T| \leq s-2$.*

*Proof.* Let $v = |V|$, $x \in V$, $d_i(x)$ the degree of $x$ in $A_i$ for any $i = 1, \ldots, s$ and $d_{A_i}$ the degree of the vertices of $A_i$ in $G$. The statement follows by the fact that:

$$\begin{cases} \sum_{i=1}^{s} d_{A_i} d_i(x) = v - 1 \\ \sum_{i=1}^{s} d_i(x) = d(x) \end{cases}$$

for any $x \in V$. This implies that either $T = \{A_1, \ldots, A_s\}$, i.e. $\Sigma$ is strongly balanced, or $|T| \leq s - 2$. □

In [10] the authors start the study of locally balanced graphs designs by considering the graphs $C_4 + e$ and $C_4 + P_3$.

The graph $C_4 + e$ has vertex set $V = \{x, y, z, t, w\}$ and edges $\{x, y\}$, $\{y, z\}$, $\{z, t\}$, $\{t, x\}$ and $\{x, w\}$. This graph has 4 automorphism classes: $A_1 = \{w\}$, $A_2 = \{x\}$, $A_3 = \{y, t\}$ and $A_4 = \{z\}$. The study of locally balanced $(C_4 + e)$-designs focuses on the so-called *central class* $A_2$, determining the spectrum of strictly $T$-balanced $(C_4 + e)$-designs in the case that $|T| \leq 2$ and $A_2 \in T$. So in [10] it is proved the following:

**Theorem 5.4.** *Let $T \subseteq \{A_1, A_2, A_3, A_4\}$, with $|T| \leq 2$ and $A_2 \in T$. Then there exists a strictly $T$-balanced $(C_4 + e)$-design of order $v > 1$ if and only if $v \equiv 1$* mod 10*, with $v \geq 21$ in the case that $A_1 \notin T$.*

The graph $C_4 + P_3$ has vertex set $V = \{x, y, z, t, w_1, w_2\}$ and edges $\{x, y\}$, $\{y, z\}$, $\{z, t\}$, $\{t, x\}$, $\{x, w_1\}$, $\{w_1, w_2\}$. In this case, we have 5 automorphism classes: $A_1 = \{w_2\}$, $A_2 = \{w_1\}$, $A_3 = \{x\}$, $A_4 = \{y, t\}$ and $A_5 = \{z\}$. Again, the focus is on the *central class* $A_3$ and the authors in [10] prove that:

**Theorem 5.5.** *Let $T$ be either $\{A_1, A_3, A_4\}$ or $\{A_2, A_3, A_5\}$ or $\{A_1, A_3, A_5\}$. Then there exists a strictly $T$-balanced $(C_4 + P_3)$-design of order $v > 1$ if and only if $v \equiv 1 \mod 12$, with $v \geq 25$ if $T = \{A_2, A_3, A_5\}$.*

**Remark 5.6.** Clearly, the notions of balanced, strongly balanced and locally balanced designs can be easily extended to decompositions of multipartite graphs. This, for example, has been done in [10] and it is remarkable to note that in [10] the existence of strongly balanced decompositions of bipartite graphs has been used in order to get locally balanced designs.

## 6. Hypergraph designs case

The notion of balanced graph design can be easily extended to the hypergraph design case. Let $K_v^{(3)}$ be the complete 3-uniform hypergraph on $v$ vertices and let $\mathcal{H} = (X, \mathcal{E})$ be a 3-uniform hypergraph, with $n = |X|$ and $m = |\mathcal{E}|$. An $\mathcal{H}$-*design* of index $\lambda$ is a pair $\Sigma = (V, \mathcal{B})$, where $V$ is a set of $v$ elements and $\mathcal{B} = \{\mathcal{H}_1, \ldots, \mathcal{H}_b\}$, such that:

- $\mathcal{H}_i \cong \mathcal{H}$ for any $i = 1,\dots,b$
- for any $x,y,z \in V$, pairwise different, there exist $\mathcal{H}_{i_1},\dots,\mathcal{H}_{i_\lambda}$, with $i_1,\dots,i_\lambda \in \{1,\dots,b\}$ pairwise different, such that $\{x,y,z\} \in E(\mathcal{H}_{i_j})$ for $j = 1,\dots,\lambda$.

The elements of $\mathcal{B}$ are called *blocks* and we say that $\Sigma$ is an $\mathcal{H}$-decomposition of $\lambda K_v^{(3)}$. The obvious necessary condition for the existence of an $\mathcal{H}$-design of order $v$ is that $\frac{1}{m}\lambda\binom{v}{3} \in \mathbb{N}$. As done for the graph case, if otherwise stated, we will suppose that $\lambda = 1$.

The definition of balanced (strongly balanced) hypergraph design is analogous to the one given in the case of graph designs. So, we continue to denote by $d(x)$ the *degree* of any vertex $x \in V$, i.e. the number of blocks of $\mathcal{B}$ containing $x$. If there exists $d \in \mathbb{N}$ such that $d(x) = d$ for any $x \in V$, the system $\Sigma$ is called *balanced*.

Clearly, if we consider the automorphism classes $A_1,\dots,A_s$ for $\mathcal{H}$, we can similarly get the definition of *strongly balanced* $\mathcal{H}$-design. In this case, the necessary conditions for the existence of a balanced and strongly balanced hypergraph designs are the following:

**Theorem 6.1.** *Let $\mathcal{H}$ be a 3-uniform hypergraph, let $n = V(\mathcal{H})$ and $m = |\mathcal{E}(\mathcal{H})|$. Let $\Sigma = (V,\mathcal{B})$ be a balanced $\mathcal{H}$-design of order $v$. Then $v(v-1)(v-2) \equiv 0 \mod 6m$ and $n(v-1)(v-2) \equiv 0 \mod 6m$.*

**Theorem 6.2.** *Let $\mathcal{H}$ be a 3-uniform hypergraph, let $m = |\mathcal{E}(\mathcal{H})|$ and let $A_1,\dots,A_s$ be its automorphism classes. If $\Sigma = (V,\mathcal{B})$ is a strongly balanced $\mathcal{H}$-design of order $v$, then $v(v-1)(v-2) \equiv 0 \mod 6m$ and $|A_i|(v-1)(v-2) \equiv 0 \mod 6m$ for $i = 1,\dots,s$.*

We can also note that, as in the case of graphs, we have:

**Proposition 6.3.** *Let $\mathcal{H}$ be a 3-uniform hypergraph and let $A_1,\dots, A_s$ be its automorphism classes and let $d_{A_i}$ be the degree of the vertices of $A_i$ for $i = 1,\dots,s$ in $\mathcal{H}$. If $\Sigma = (V,\mathcal{B})$ is an $\mathcal{H}$-design of order $v$ and for any $x \in V$ $d_i(x)$ is the degree of $x$ in $A_i$, for $i = 1,\dots,s$, then the following equalities hold for any $x \in V$:*

$$\begin{cases} \sum_{i=1}^{s} d_i(x) = d(x) \\ \\ \sum_{i=1}^{s} d_{A_i} d_i(x) = \binom{v-1}{2}. \end{cases}$$

This implies that, as for graphs:

**Corollary 6.4.** *Let $\mathcal{H}$ be a 3-uniform hypergraph, let $A_1$ and $A_2$ be its automorphism classes and let $d_{A_1}$ and $d_{A_2}$ be the degrees of the vertices of $A_1$ and $A_2$, respectively, with $d_{A_1} \neq d_{A_2}$. Then any balanced $\mathcal{H}$-design is also strongly balanced.*

In the case of hypergraphs a new balance-type condition has been introduced in [19]. In order to do this, we introduce the degrees of 2-edges. Let $\mathcal{H}$ be a 3-uniform hypergraph and let $\Sigma = (V, \mathcal{B})$ be an $\mathcal{H}$-design. For any $x, y \in V$, $x \neq y$, the degree $d(x,y)$ of the 2-edge $[x,y]$ is the number of blocks $B \in \mathcal{B}$ containing $x$ and $y$ as elements of an edge (3-edge) in $B$.

Let $A'_1, \ldots, A'_q$ the automorphism classes of the 2-edges of $\mathcal{H}$. If $\Sigma = (V, \mathcal{B})$ is an $\mathcal{H}$-design, for any $x, y \in V$, $x \neq y$, and for any $i = 1, \ldots, q$ we can consider the degree $d_i(x,y)$ of $[x,y]$ in $A'_i$, which is the number of blocks of $\mathcal{B}$ containing $x, y$ as elements of $A'_i$.

**Definition 6.5.** Let $\mathcal{H}$ be a 3-uniform hypergraph and let $\Sigma = (V, \mathcal{B})$ be an $\mathcal{H}$-design. $\Sigma$ is called *edge balanced* if there exists $d \in \mathbb{N}$ such that $d(x,y) = r$ for any $x, y \in V$, $x \neq y$.

Clearly, we have also the definition of strongly edge balanced $\mathcal{H}$-design:

**Definition 6.6.** Let $\mathcal{H}$ be a 3-uniform hypergraph and let $\Sigma = (V, \mathcal{B})$ be an $\mathcal{H}$-design. $\Sigma$ is called *strongly edge balanced* if for any $i = 1, \ldots, q$ there exists $d_i \in \mathbb{N}$ such that $d_i(x,y) = d_i$ for any $x, y \in V$, $x \neq y$.

We obviously have the following results:

**Theorem 6.7.** *Let $\mathcal{H}$ be a 3-uniform hypergraph and let $m = |\mathcal{E}(\mathcal{H})|$ and let $p$ be the number of 2-edges in $\mathcal{H}$. If $\Sigma = (V, \mathcal{B})$ is an edge balanced $\mathcal{H}$-design of order $v$, then $v(v-1)(v-2) \equiv 0 \mod 6m$ and $p(v-2) \equiv 0 \mod 3m$.*

**Theorem 6.8.** *Let $\mathcal{H}$ be a 3-uniform hypergraph and let $m = |\mathcal{E}(\mathcal{H})|$. Let $A'_1, \ldots, A'_q$ be its automorphism classes of the 2 edges. If $\Sigma = (V, \mathcal{B})$ is a strongly edge balanced $\mathcal{H}$-design of order $v$, then $v(v-1)(v-2) \equiv 0 \mod 6m$ and $|A'_i|(v-2) \equiv 0 \mod 3m$ for $i = 1, \ldots, q$.*

and similarly:

**Proposition 6.9.** *Let $\mathcal{H}$ be a 3-uniform hypergraph and let $A'_1, \ldots, A'_q$ be its automorphism classes of 2-edges and let $d_{A'_i}$ be the degree of the 2-edges of $A'_i$ for $i = 1, \ldots, q$. If $\Sigma = (V, \mathcal{B})$ is an $\mathcal{H}$-design of order $v$ and for any $x, y \in V$, $x \neq y$, $d_i(x,y)$ is the degree of $[x,y]$ in $A_i$, for $i = 1, \ldots, q$, then the following equalities hold for any $x, y \in V$, $x \neq y$:*

$$\begin{cases} \sum_{i=1}^{q} d_i(x,y) = d(x,y) \\ \sum_{i=1}^{q} d_{A'_i} d_i(x,y) = v-2. \end{cases}$$

This implies that:

**Corollary 6.10.** *Let $\mathcal{H}$ be a 3-uniform hypergraph, let $A'_1$ and $A'_2$ be its 2-edges automorphism classes and let $d_{A'_1}$ and $d_{A'_2}$ be the degrees of the 2-edges of $A_1$ and $A_2$, respectively, with $d_{A'_1} \neq d_{A'_2}$. Then any edge balanced $\mathcal{H}$-design is also strongly edge balanced.*

## 7. Constructions of hypergraph designs

In [18] Gionfriddo introduced a matrix (see also [21, 22]) useful for the construction of cyclic, and so strongly balanced, 3-uniform hypergraph designs of order $v$. We need to distinguish the cases $v \equiv 1,2 \mod 3$ and $v \equiv 0 \mod 3$.

Suppose that $v \equiv 1,2 \mod 3$, so that either $v = 3h+1$ or $v = 3h+2$ for some $h \in \mathbb{N}$. We can associate to $v$ the following matrix $M(v)$:

$$M(v) = \begin{bmatrix} (1,1) & (1,v-2) & (v-2,1) \\ (1,2) & (2,v-3) & (v-3,1) \\ \vdots & \vdots & \vdots \\ (1,v-3) & (v-3,2) & (2,1) \\ (2,2) & (2,v-4) & (v-4,2) \\ \vdots & \vdots & \vdots \\ (2,v-5) & (v-5,3) & (3,2) \\ (3,3) & (3,v-6) & (v-6,3) \\ \vdots & \vdots & \vdots \\ (3,v-7) & (v-7,4) & (4,3) \\ \vdots & \vdots & \vdots \\ (h,h) & (h,v-2h) & (v-2h,h) \\ (h,v-2h-1) & (v-2h-1,h+1) & (h+1,h) \end{bmatrix}$$

Consider a triple $T = \{x,y,z\} \subseteq \mathbb{Z}_v$, with $x<y<z$. Let $a = y-x$ and $b = z-y$. It is possible to see that there exists a row in $M(v)$ containing the pair $(a,b)$. Moreover, the triple $T$ is obtained by the base triple $\{0,a,a+b\}$ by adding $i = x \in \mathbb{Z}_v$ to each element:

$$x = i,\ y = a+i,\ z = y+b = a+b+i.$$

Conversely, if we take any $\{x,y,z\}$, with $x<y<z$, translate of $\{0,a,a+b\}$, the pair $(y-x, z-y)$ can be any of the 3 pairs in the row of $(a,b)$.

Let $v = 3h$, for some $h \in \mathbb{N}$. In this case the matrix $M(v)$ is following:

$$M(v) = \begin{bmatrix} (1,1) & (1,v-2) & (v-2,1) \\ (1,2) & (2,v-3) & (v-3,1) \\ \vdots & \vdots & \vdots \\ (1,v-3) & (v-3,2) & (2,1) \\ (2,2) & (2,v-4) & (v-4,2) \\ \vdots & \vdots & \vdots \\ (2,v-5) & (v-5,3) & (3,2) \\ (3,3) & (3,v-6) & (v-6,3) \\ \vdots & \vdots & \vdots \\ (3,v-7) & (v-7,4) & (4,3) \\ \vdots & \vdots & \vdots \\ (h-1,h-1) & (h-1,h+2) & (h+2,h-1) \\ (h-1,h) & (h,h+1) & (h+1,h-1) \\ (h-1,h+1) & (h+1,h) & (h,h-1) \\ (h,h) & (h,h) & (h,h) \end{bmatrix}$$

In this case, the use of the matrix is as above, the only difference being that the pair $(h,h)$ is repeated three times in the last row.

Let us first consider the balance condition for the vertices. The first hypergraphs considered are the two connected 3-uniform hypergraphs with two edges. We denote them by $P^{(3)}(2,4)$ and $P^{(3)}(1,5)$. $P^{(3)}(2,4)$ has vertices $\{a,b,c,d\}$ and edges $\{a,b,c\}$ and $\{a,b,d\}$; $P^{(3)}(1,5)$ has vertices $\{a,b,c,d,e\}$ and edges $\{a,b,c\}$ and $\{a,d,e\}$. Note that both of them have two automorphism classes of different degrees for the vertices, so that in both cases a balanced design is also strongly balanced.

In [22] the spectrum of balanced $P^{(3)}(2,4)$-designs is determined:

**Theorem 7.1.** *There exists a balanced $P^{(3)}(2,4)$-design of order $v \geq 4$ if and only if either $v \equiv 1,5 \mod 12$ or $v \equiv 2,4 \mod 6$.*

Note that the case $v \equiv 2,4 \mod 6$ is an immediate consequence of the existence of $SQS(v)$, while the constructions in [22] provided for the cases $v \equiv 1,5 \mod 12$ is a cyclic one, based on the matrix $M(v)$.

In [6] the spectrum of balanced $P^{(3)}(1,5)$-designs is determined. In this case all the constructions are cyclic and based on the matrix $M(v)$:

**Theorem 7.2.** *There exists a balanced $P^{(3)}(1,5)$-design of order $v$ if and only if $v \equiv 1,2,5,10 \mod 12$, $v \geq 5$.*

In [8] some non-cyclic constructions of balanced $P^{(3)}(1,5)$-designs are given and the following results are proved.

**Theorem 7.3.** *If* $\Sigma_1 = (X_1, \mathcal{B}_1)$, $\Sigma_2 = (X_2, \mathcal{B}_2)$ *are two balanced* $P^{(3)}(1,5)$-*designs of order* $v$, *with* $v \equiv 1,5 \mod 12$, *with no vertex in common, then there exist balanced* $P^{(3)}(1,5)$-*designs of order* $2v$, *embedding* $\Sigma_1, \Sigma_2$.

**Theorem 7.4.** *For every* $v \equiv 1$ *or* 5, *mod* 12, $v \geq 13$, *there exist balanced non-cyclic* $P^{(3)}(1,5)$-*designs of order* $2v$.

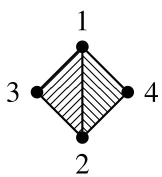
In [5] the hypergraph $P^{(3)}(2,5)$ has been considered. It is the hypergraph with vertices $\{a,b,c,d,e\}$ and edges $\{a,b,c\}$, $\{b,c,d\}$ and $\{c,d,e\}$. In this case, it has 3 automorphism classes, $A_1 = \{a,e\}$, $A_2 = \{b,d\}$ and $A_3 = \{c\}$. In [5] it has been proved that:

**Theorem 7.5.** *There exists a strongly balanced* $P^{(3)}(2,5)$-*design of order* $v$ *if and only if* $v \equiv 1,2 \mod 9$, $v \geq 10$.

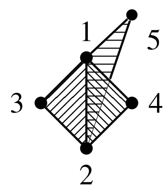
Note that the above designs are all cyclic. It has also been proved that:

**Theorem 7.6.** *There exists a simply balanced* $P^{(3)}(2,5)$-*design of order* $v$ *if and only if* $v \equiv 1,2 \mod 9$, $v \geq 10$.

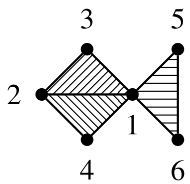
In [19] Gionfriddo started the study of edge balanced hypergraph designs by considering $P^{(3)}(2,4)$-designs. Later, in [9] the study of edge-balanced $P^{(3)}(2,4)$-designs has been completed, considering also other hypergraphs as reported here:



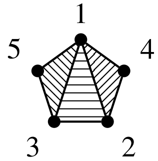

$P^{(3)}(2,4)$



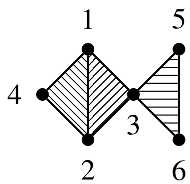

$S^{(3)}(2,5)$




$S^{(3)}(2,4) + E$




$P^{(3)}(2,5)$




$P^{(3)}(2,4) + E$

First, note that in $P^{(3)}(2,4)$ and in $S^{(3)}(2,5)$ we have two automorphism classes for 2-edges and they have different degrees, so that any edge balanced $P^{(3)}(2,4)$ and $S^{(3)}(2,5)$-design is also strongly edge balanced. In [9] it is proved that:

**Theorem 7.7.** *There exists an edge balanced $\mathcal{H}$-design of order $v$ if and only if either $v \equiv 2 \mod 6$, $v \geq 8$, if $\mathcal{H} = P^{(3)}(2,4)$ or $v \equiv 2 \mod 9$, $v \geq 11$, if $\mathcal{H} \in \{S^{(3)}(2,5), S^{(3)}(2,4)+E, P^{(3)}(2,5), P^{(3)}(2,4)+E\}$.*

Note that the construction provided in [9] are not cyclic, with the exception of $v = 11$ and $\mathcal{H} \in \{S^{(3)}(2,5), S^{(3)}(2,4)+E, P^{(3)}(2,5), P^{(3)}(2,4)+E\}$. Note also that in [15] the spectrum of $S^{(3)}(2,5)$-designs of any index has been determined.

In [11] the results in the cases $P^{(3)}(2,4)$ and $S^{(3)}(2,5)$ have been generalized by considering hypergraphs $S^{(3)}(2,m+2)$, where for any $m \geq 2$ $S^{(3)}(2,m+2)$ is the hypergraph on $m+2$ vertices and $m$ edges, all having the same 2-edge in common. So, this hypergraph has two automorphism classes, both for vertices and edges, with different degrees. So in [11] it was proved that:

**Theorem 7.8.** *For any $v \in \mathbb{N}$, $v = 3m+2$, $m \geq 2$ there exists a cyclic edge balanced $S^{(3)}(2,m+2)$-design of order $v$.*

It was also proved that for star hypergraph designs there exists a connection between vertex balanced and edge balanced designs:

**Theorem 7.9.** *Let $\Sigma = (V, \mathcal{B})$ be an edge balanced $S^{(3)}(2,m+2)$-design of order $v$. Then $\Sigma$ is vertex balanced.*

So of course we have:

**Theorem 7.10.** *There exists a vertex balanced $S^{(3)}(2,m+2)$-design of order $v$ for any $m \geq 2$ and any $v \equiv 2 \mod 3m$, $v \geq 3m+2$.*

Note that in [11] it was also shown the existence of a vertex balanced $P^{(3)}(2,4)$-design that is not edge balanced.

## REFERENCES


[1] B. Alspach and H. Gavlas, *Cycle decompositions of $K_n$ and $K_n - I$*, J. Combin. Theory Ser. B 81 (1) (2001), 77–99.

[2] L. Berardi, M. Gionfriddo, R. Rota, *Balanced and strongly balanced $P_k$-designs*, Discrete Math. 312 (3) (2012), 633–636.

[3] J. C. Bermond, C. Huang, A. Rosa, and D. Sotteau, *Decomposition of complete graphs into isomorphic subgraphs with five vertices*, Ars Combin. 10 (1980), 211–254.

[4] J. C. Bermond and J. Schönheim, *G-decomposition of $K_n$, where G has four vertices or less*, Discrete Math. 19 (2) (1977), 113–120.

[5] P. Bonacini, *On a 3-Uniform Path-Hypergraph on 5 Vertices*, Applied Mathematical Sciences, 10 (29–32) (2016), 1489–1500.

[6] P. Bonacini, M. Di Giovanni, M. Gionfriddo, L. Marino, A. Tripodi, *The spectrum of balanced $P^{(3)}(1,5)$-designs*, Contrib. Discrete Math. 12 (1) (2017), 133–142.

[7] P. Bonacini, M. Gionfriddo, L. Marino, *Balanced House-systems and Nestings*, Ars Combin. 121 (2015), 429–436.

[8] P. Bonacini, M. Gionfriddo, L. Marino, *Construction of non-cyclic balanced $P^{(3)}(1,5)$-designs*, Applied Mathematical Sciences 9 (126) (2015), 6273–6282.

[9] P. Bonacini, M. Gionfriddo, L. Marino, *Edge balanced* 3*-uniform hypergraph designs*, Mathematics 8 (8) (2020), 1353.

[10] P. Bonacini P, M. Gionfriddo, L. Marino, *Locally Balanced G-Designs*, Mathematics 11 (2) (2023), 408.

[11] P. Bonacini, L. Marino, *Edge balanced star-hypergraph designs and vertex colorings of path designs*, J. Combin. Des. 30 (7) (2022), 497–514.

[12] A. Bonisoli, S. Bonvicini, G. Rinaldi, *A hierarchy of balanced graph-designs*, Quad. Mat. 28 (2012), 151–163.

[13] A. Bonisoli, B. Ruini, *Tree-designs with balanced-type conditions*, Discrete Math. 313 (11) (2013), 1197–1205.

[14] S. Bonvicini, *Degree- and orbit-balanced $\Gamma$-designs when $\Gamma$ has five vertices*, J. Combin. Des. 21 (9) (2013), 359–389.

[15] A. Causa, M. Gionfriddo, E. Guardo, *Construction of $S^{(3)}(2,3)$-Designs of Any Index*, Mathematics 12 (2024), 1968.

[16] Y. Chang, G. Lo Faro, A. Tripodi, *Determining the spectrum of Meta ($K_4 + e > K_4$, $\lambda$) for any $\lambda$*, Discrete Math., 308 (2008), 3007–3017.

[17] C.J. Colbourn, J.H. Dinitz, eds, *CRC Handbook of Combinatorial Designs*, 2nd edition, Chapman and Hall, CRC, Boca Raton, FL, 2007.

[18] M.Gionfriddo, *Construction of Cyclic $H^{(3)}$-Designs*, Applied Math. Sciences 9 (2015) n.70, 3485–3503.

[19] M. Gionfriddo, *Edge balanced $H^{(3)}$-designs*, Applied Mathematical Sciences 9 (73–76) (2015), 3629–3641.

[20] M. Gionfriddo, S. Küçükçifçi, L. Milazzo, *Balanced and strongly balanced 4-kite designs*, Util. Math. 91 (2013), 121–129.

[21] M. Gionfriddo, G. Lo Faro, S. Milici, *Some techniques for the construction of hyperpath-designs - a survey*, Lect. Notes Semin. Interdiscip. Mat. 14, Seminario Interdisciplinare di Matematica (S.I.M.), Potenza, 2017, 71–84.

[22] M. Gionfriddo, S. Milici, *Balanced* $P^{(3)}(2,4)$*-designs*, Util. Math. 99 (2016), 81–88.

[23] P. Hell, A. Rosa, *Graph decompositions, handcuffed prisoners and balanced P-designs*, Discrete Math. 2 (3) (1972), 229–252.

[24] C. Huang, *On the existence of balanced bipartite designs, II*, Discrete Math. 9 (1974), 147–159.

[25] C. Huang, *On handcuffed designs*, Dept. of C. and O. Research Report CORR75 - 10, University of Waterloo.

[26] C. Huang, *On some classes of balanced graph designs*, Ph.D. Thesis, McMaster University, Hamilton, Ontario, 1974. Open Access Dissertations and Theses. Paper 969. Available at `https://macsphere.mcmaster.ca/bitstream/11375/14330/1/fulltext.pdf`

[27] C. Huang, A. Rosa, *On the existence of balanced bipartite designs*, Util. Math. 4 (1973), 55–75.

[28] S.H.Y. Hung, N.S. Mendelsohn, *Handcuffed designs*, Aequationes Math. 11 (1974), 256–266.

[29] S.H.Y. Hung, N.S. Mendelsohn, *Handcuffed designs*, Discrete Math. 18 (1) (1977), 23–33.

[30] J.F. Lawless, *On the construction of handcuffed designs*, J. Combin. Theory Ser. A 16 (1974), 76–86.

[31] J.F. Lawless, *Further results concerning the existence of handcuffed designs*, Aequationes Math. 11 (1974), 97–106.

[32] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967), 349–355.

[33] M. Šajna, *Cycle decompositions III: Complete graphs and fixed length cycles*, J. Combin. Des., 10 (1) (2002), 27–78.

*P. BONACINI*
*Department of Mathematics and Computer Science*
*University of Catania*
*Viale A. Doria 6, 95125 Catania, Italy*
*e-mail:* `paola.bonacini@unict.it`

*M. GIONFRIDDO*
*Department of Mathematics and Computer Science*
*University of Catania*
*Viale A. Doria 6, 95125 Catania, Italy*
*e-mail:* gionfriddo@dmi.unict.it

*L. MARINO*
*Department of Mathematics and Computer Science*
*University of Catania*
*Viale A. Doria 6, 95125 Catania, Italy*
*e-mail:* lucia.marino@unict.it