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CONSTRUCTION OF 3-HELIX SYSTEMS OF ANY INDEX

A. CAUSA - M. GIONFRIDDO - E.M. GUARDO

Let $H^{(3)}$ be a uniform hypergraph of rank 3. A 3-*helix* $S^{(3)}(1,3)$ of *centre* $\{c\}$ is a 3-uniform hypergraph, with 3 hyperedges, all having in common exactly the *centre* $\{c\}$, with c of degree 3 and the remaining vertices of degree 1. In this paper we determine the spectrum of $S^{(3)}(1,3)$-*designs*, for every index λ.

1. Introduction

Let $K_v^{(h)} = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank h, defined on the vertex set $X = \{x_1, x_2, \dots, x_v\}$, that is, $\mathcal{E}$ is the collection of all the subsets of X whose cardinality is h. In the sequel, we will call a set of cardinality h an h-subset.

Let $H^{(h)}$ be a subhypergraph of $K_v^{(h)}$. An $H^{(h)}$-*design*, or also *a design of type* $H^{(h)}$ or a *system* of $H^{(h)}$, having *order* v and index λ, is a pair $\Sigma = (X, \mathcal{B})$, where X is a finite set of cardinality v, whose elements are called *vertices*, and $\mathcal{B}$ is a collection of hypergraphs over X, called *blocks*, all isomorphic to $H^{(h)}$, under the condition that every h-subset of X is an hyperedge of exactly λ hypergraphs of the collection $\mathcal{B}$. An $H^{(h)}$-*design*, of order v and index λ, is also called an $H^{(h)}$-*decomposition* of $\lambda K_v^{(h)}$, (see, for example, [1–5, 7]).

It is important to note that in the definition of index λ we don't require that the

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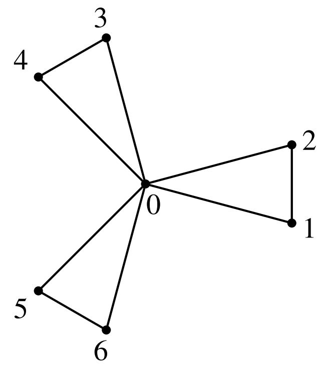

Figure 1: 3-helix system

blocks be distinct. That is, in a given example, a block may be repeated as many times as λ times.

The study of hypergraph designs has become an important research area of combinatorial design. In this field, the focus has always been stressed on construction techniques. In general, in the literature $\mathcal{B}$ is a multiset (see [6]).

We define a *hyperstar* $S^{(h)}(r,s)$ with $h > r$, as the h-uniform hypergraph having s hyperedges and order $(h-r)s+r$, such that all the edges have in common exactly the same r vertices, which form its *centre*; and all the vertices of the centre have degree s.

We denote by $S^{(3)}(1,3)$ the hypergraph whose hyperedges are

$$\{\{0,1,2\},\{0,3,4\},\{0,5,6\}\}$$

as in Figure 1 and we call it a 3-*helix system*. In this paper we determine the spectrum of a 3-helix system for any index λ and v admissible.

2. Preliminary results

Let us consider a 3-helix system and let X be a finite set of cardinality v. The number of triples of X is $\binom{v}{3}$, and since a helix with vertices in X contains 3 triples of X, it follows that the number of blocks $\mathcal{B}$ of a 3-helix system must satisfy $3|\mathcal{B}| = \binom{v}{3}$. Hence, a necessary condition for the cardinality of X is $v \equiv 0,1,2 \pmod 9$.

Notation 2.1. Let $X = \{x_1, x_2, \ldots, x_v\}$ be a finite set of cardinality v and k an integer such that $0 \le k \le v$. We denote by $\mathrm{Binom}(X,k) := \binom{X}{k} = \{A \subseteq X \mid |A| = k\}$ the collection of k-subsets of X, i.e., $\mathrm{Binom}(X,k)$ is the set of the subsets of X of cardinality k.

Lemma 2.2. *Let* $v = |X|$ *and* $v \equiv 0,1 \pmod 9$*, the set* $\mathrm{Binom}(X,2)$ *can be partitioned into triples* $\{x,y,z\}$ *such that* $x \cap y = y \cap z = z \cap x = \emptyset$.

Proof. Let $|X| = v$ and consider the elements of X indexed by $\mathbb{Z}/(v)$, i.e., the group of integers modulus v. The group $\mathbb{Z}/(v)$ acts freely on X by translation, i.e., given $a \in \mathbb{Z}/(v)$ then $a(x_i) = x_{a+i}$ and, obviously, it acts on $\mathrm{Binom}(X,2)$ elementwise. The orbits can be indexed by the positive integer δ between 1 and $\lfloor \frac{v}{2} \rfloor$, where $O_\delta = \{\{a,b\} \mid a-b = \delta \vee b-a = \delta\}$; we have that $\mathrm{Binom}(X,2) = \cup_{1 \le \delta \le \lfloor \frac{v}{2} \rfloor} O_\delta$.

The proof will be divided into three cases: $v = 9h$, $v = 9h+1$ even and $v = 9h+1$ odd.

1. $v = 9h$. We have that $|O_i|$ is divisible by 3 for all $i = 1, \ldots, \lfloor \frac{v}{2} \rfloor$. The action of $3\mathbb{Z}/(v)$ on each O_i gives 3 classes of equivalence of cardinality $\frac{|O_i|}{3}$. Choose x,y,z in each class, respectively, such that $x \cap y = y \cap z = z \cap x = \emptyset$. Let us denote by $(X,\mathcal{P})$ the set of all the triples $\{a(x),a(y),a(z)\}$ for $a \in 3\mathbb{Z}/(v)$. We have that $(X,\mathcal{P})$ is a partition of $\mathrm{Binom}(X,2)$ into triples $\{x,y,z\}$ such that $x \cap y = y \cap z = z \cap x = \emptyset$.

2. $v = 9h+1$ *even*. The number of $\mathbb{Z}/(v)$-orbits is $\lfloor \frac{v}{2} \rfloor \equiv 2 \pmod 3$ and $|O_\delta| = v$ for $\delta \neq \frac{v}{2}$ and $|O_{\frac{v}{2}}| = \frac{v}{2}$. Choose $x = \{0,1\}$ and $y = \{\frac{v}{2}, 1+\frac{v}{2}\} \in O_1$ and $z = \{2, 2+\frac{v}{2}\} \in O_{\frac{v}{2}}$ and consider the triples $\{a(x),a(y),a(z)\}$ for $a \in \mathbb{Z}/(v)$. We collect the remaining orbits into triples $\{O_i,O_j,O_k\}$. Choose $x_i \in O_i$, $y_j \in O_j$ and $z_k \in O_k$ such that $x_i \cap y_j = y_j \cap z_k = z_k \cap x_i = \emptyset$ and consider the triples $\{a(x_i),a(y_j),a(z_k)\}$ for $a \in \mathbb{Z}/(v)$. This construction gives a partition $(X,\mathcal{P})$ of $\mathrm{Binom}(X,2)$ in triples $\{x,y,z\}$ such that $x \cap y = y \cap z = z \cap x = \emptyset$.

3. $v = 9h+1$ *odd*. The number of orbits is a multiple of 3 and their cardinality is v. We collect the orbits into triples $\{O_i,O_j,O_k\}$. Choose $x_i \in O_i$, $y_j \in O_j$ and $z_k \in O_k$ such that $x_i \cap y_j = y_j \cap z_k = z_k \cap x_i = \emptyset$ and consider the triples $\{a(x_i),a(y_j),a(z_k)\}$ for $a \in \mathbb{Z}/(v)$. This construction gives a partition $(X,\mathcal{P})$ of $\mathrm{Binom}(X,2)$ in triples $\{x,y,z\}$ such that $x \cap y = y \cap z = z \cap x = \emptyset$. □

3. The spectrum of 3-helix systems

In this section we determine the spectrum of a 3-helix system of index $\lambda = 1$, that is, we are treating exclusively the case of systems of index 1 and by the word *block* we mean a block of type $S^{(3)}(1,3)$.

We will prove the result by induction on v, i.e., the order of the block design.

Theorem 3.1. *There exists a 3-helix system of order* $v = 9$.

Proof. Let $|X| = 9$, and $\mathbb{Z}/(9)$ acting on $\mathrm{Binom}(X,3)$ by translation. Consider the action of $\mathbb{Z}/(9)$ on the following blocks: $H_1 = \{\{1,2,3\},\{1,5,6\},\{1,4,8\}\}$, $H_2 = \{\{0,1,3\},\{0,5,8\},\{0,2,6\}\}$, $H_3 = \{\{0,1,4\},\{0,5,7\},\{0,6,8\}\}$. Since the set of differences of the hyperedges of the blocks H_i are all distincts, it follows that the hyperedeges of translated blocks are distincts. The triples that do not belong to the set of translated hyperedeges are $\{\{0,6,3\},\{1,4,7\},\{2,5,8\}\}$ which can not form a block (of a helix system).
We proceed by swapping the hyperedge $\{1,4,8\}$ of the block H_1 with $\{1,4,7\}$ and the hyperedge $\{0,6,8\}$ of the block H_3 with $\{0,3,6\}$. At this point the triples, which are not collected in blocks yet, are $\{\{0,6,8\},\{1,4,8\},\{2,5,8\}\}$; these triples form a block with centre 8. We get a helix system over X collecting the blocks $\tau(H_1)$ and $\tau(H_3)$ with $\tau \in \mathbb{Z}/(9)$ and $\tau \neq 0$, the blocks obtained after the swapping, i.e., $\{\{1,2,3\},\{1,5,6\},\{1,4,7\}\}$, $\{\{0,1,4\},\{0,5,7\},\{0,3,6\}\}$ and $\{\{0,6,8\}, \{1,4,8\},\{2,5,8\}\}$, and $\tau(H_2)$ with $\tau \in \mathbb{Z}/(9)$. □

Lemma 3.2. *Given a 3-helix system of order* $v \equiv 0,1 \pmod 9$*, there exists a helix system of order* $v+1$.

Proof. Let $(X,\mathcal{H})$ a 3-helix system over X, and $(X,\mathcal{P})$ a partition of $\mathrm{Binom}(X,2)$ in triples $\{x,y,z\}$ as in Lemma 2.2; we can construct a 3-helix system over $X \cup \{\star\}$ in the following way. Let us observe that, to a given $\alpha = \{x,y,z\} \in \mathcal{P}$ one can associate the 3-helix $\alpha_\star = \{x \cup \{\star\}, y \cup \{\star\}, z \cup \{\star\}\}$; let us denote by $\mathcal{P}_\star$ the set of blocks obtained in such a way. It is straightforward that $(X \cup \{\star\}, \mathcal{H} \cup \mathcal{P}_\star)$ is a 3-helix system of order $|X|+1$. □

Lemma 3.3. *Given a 3-helix system of order* $v \equiv 0,1 \pmod 9$ *there exists a 3-helix system of order* $v+9$.

Proof. Let $(X,\mathcal{H})$ and $(Y,\mathcal{H}')$ be 3-helix systems of order v and 9 over the disjoint sets X and Y, respectively; let $(X,\mathcal{P})$ and $(Y,\mathcal{P}')$ be partitions of $\mathrm{Binom}(X,2)$ and $\mathrm{Binom}(Y,2)$ as in Lemma 2.2. It is straightforward to prove that $\mathcal{H} \cup \mathcal{H}' \cup_{x\in X} \mathcal{P}_x \cup_{y \in Y} \mathcal{P}'_y$ is a 3-helix system over $X \cup Y$ of order $|X|+|Y| = v+9$. □

We can now formulate and prove the main result of this section.

Theorem 3.4. *If* $v \equiv 0,1,2 \pmod 9$*, and* $v \geq 9$*, there exists a 3-helix system of order* v*.*

Proof. The proof is by induction on v. The base case is given for $v = 9$ by Theorem 3.1; if $v \equiv 0,1 \pmod 9$ then a 3-helix system of order $v+1$ exists by Lemma 3.2. By Lemma 3.3 there exists a 3-helix system of order $v+9$. □

4. The spectrum of 3-helix systems of index $\lambda = 2$

Before proving the main result of this section, we note that if X is a set of cardinality v and there exists a 3-helix system of index 2 over X then $v \equiv 0,1,2$ (mod 9).

Theorem 4.1. *There exists a 3-helix design of order 9 and index 2 with pairwise distinct blocks.*

Proof. In order to construct a 3-helix system of order 9 and index 2, it suffices to construct a 3-helix system of index 1 and order 9 which do not have blocks in common with the one obtained as in Lemma 3.1.
So that, we can proceed as in Lemma 3.1 and consider the following base blocks $K_1 = \{\{0,1,7\},\{0,4,8\},\{0,2,6\}\}$, $K_2 = \{\{1,2,4\}, \{1,8,0\},\{1,3,5\}\}$, $K_3 = \{\{0,1,4\},\{0,3,7\},\{0,5,8\}\}$ and swap the hyperedge $\{0,2,6\} \in K_1$ with $\{0,3,6\}$ and $\{1,2,4\} \in K_2$ with $\{1,4,7\}$. After swapping we are left with $\{0,2,6\}$, $\{1,2,4\}$ and $\{2,5,8\}$ which form a block with centre $\{2\}$. Proceeding as in Lemma 3.1 we get another 3-helix system with no blocks in common with that constructed as in Lemma 3.1. □

Lemma 4.2. *Let* $v = |X|$ *and* $v \equiv 0,1 \pmod 9$, *the set* $\mathrm{Binom}(X,2) \sqcup \mathrm{Binom}(X,2)$ *can be partitioned into triples* $\{x,y,z\}$ *such that* $x \cap y = y \cap z = z \cap x = \emptyset$ *and the triples* $\{x,y,z\}$ *are pairwise distinct.*

Proof. Let us consider two triples $\{x,y,z\}$ and $\{x',y',z'\}$ as in Lemma 2.2 with the additional property that $x = x'$, $y = y'$ and $z \neq z'$, i.e., $\{x,y,z\}$ and $\{x',y',z'\}$ belongs to different orbits of the action of the cyclic groups $\mathbb{Z}/(v)$ (or $3\mathbb{Z}/(v)$). The triples obtained in this way give a partition of $\mathrm{Binom}(X,2) \sqcup \mathrm{Binom}(X,2)$ with the required property. □

Lemma 4.3. *Given a 3-helix of order* $v \equiv 0,1 \pmod 9$ *and index 2, there exists a 3-helix system of order* $v+1$ *and same index.*

Proof. Let $(X,\mathcal{H})$ a 3-helix system over X of index $\lambda = 2$, and $(X,\mathcal{P})$ a partition of $\mathrm{Binom}(X,2) \sqcup \mathrm{Binom}(X,2)$ in triples as in Lemma 4.2. We can construct a 3-helix system of index 2 over $X \cup \{\star\}$ in the following way. Given $\alpha = \{x,y,z\} \in \mathcal{P}$ associate the 3-helix $\alpha_\star = \{x \cup \{\star\}, y \cup \{\star\}, z \cup \{\star\}\}$ and denote by $\mathcal{P}_\star$ the set of $\alpha_\star$. It is straighforward that $(X \cup \{\star\}, \mathcal{H} \cup \mathcal{P}_\star)$ is a 3-helix system of index 2 over $X \cup \{\star\}$. □

We leave to the reader the proof of the following lemma, which is analogous to that of Lemma 3.3.

Lemma 4.4. *Given a 3-helix system of index 2 and order* $v \equiv 0,1 \pmod 9$ *there exists a 3-helix system of order* $v+9$.

Exploiting the preceding lemmas we are ready to prove the following

Theorem 4.5. *If* $v \equiv 0,1,2 \pmod 9$ *and* $v \geq 9$, *there exist 3-helix systems of order v and index 2 whose blocks are pairwise distinct.*

Proof. The proof is by induction on v. The base case is given for $v = 9$ by Theorem 4.1; if $v \equiv 0,1 \pmod 9$ then a 3-helix system of order $v+1$ and index 2 exists by Lemma 4.3. By Lemma 4.4 there exists a 3-helix system of order $v+9$. □

5. The spectrum of 3-helix systems of index $\lambda = 3$

In this section we prove that if $v \geq 7$ there exists a 3-helix system of order v and index $\lambda = 3$ and we will give a proof using induction on v.
The following result guaranties the existence of a 3-helix system of order 7 and index 3.

Theorem 5.1. *There exists a 3-helix system of order 7 and index 3.*

Proof. The 35 blocks of a system of order 7 and index 3 can be obtained as translated by the action of $\mathbb{Z}/(7)$ of the following base blocks:
$\{\{0,2,6\},\{0,1,3\},\{0,4,5\}\}$, $\{\{0,1,2\},\{0,5,6\},\{0,3,4\}\}$,$\{\{0,1,6\},\{0,2,4\},\{0,3,5\}\}$, $\{\{0,1,4\},\{0,3,6\},\{0,2,5\}\}$, $\{\{0,1,5\},\{0,4,6\},\{0,2,3\}\}$. □

Set $\sqcup^3 \operatorname{Binom}(X,2) := \operatorname{Binom}(X,2) \sqcup \operatorname{Binom}(X,2) \sqcup \operatorname{Binom}(X,2)$. In order to do the induction step we need the following result.

Lemma 5.2. *Suppose* $|X| \geq 7$, *then the multiset* $\sqcup^3 \operatorname{Binom}(X,2)$ *can be partitioned into pairwise distinct triples* $\{x,y,z\}$ *such that* $x \cap y = y \cap z = z \cap x = \emptyset$.

Proof. The cyclic group $\mathbb{Z}/(v)$ acts on $\operatorname{Binom}(X,2)$ and the orbits are parametrized by the 'differences' δ, where $1 \leq \delta \leq \lfloor \frac{v}{2} \rfloor$. For each δ consider the multiset $O_\delta \sqcup O_\delta \sqcup O_\delta$ and choose pairwise disjoint $x,y,z \in O_\delta$, the set of triples $\{a(x),a(y),a(z)\}$ as a varies in $\mathbb{Z}/(v)$ and δ varies from 1 to $\lfloor \frac{v}{2} \rfloor$ gives the requested partition of $\sqcup^3 \operatorname{Binom}(X,2)$. □

Theorem 5.3. *If* $|X| \geq 7$ *then there exists a 3-helix system of index 3 over the set* X.

Proof. The proof is by induction on $v = |X|$. Let $(X, \mathcal{H})$ a 3-helix system of index 3 and $(X, \mathcal{P})$ a partition of the multiset $\sqcup^3 \operatorname{Binom}(X, 2)$ in triples $\{x, y, z\}$ as in Lemma 5.2. Given the triple $\alpha = \{x, y, z\}$ one can associate the block, isomorphic to $S^{(3)}(1,3)$, that is, the 3-helix $\alpha_\star = \{x \cup \{\star\}, y \cup \{\star\}, z \cup \{\star\}\}$ and denote by $\mathcal{P}_\star$ the set of $\alpha_\star$ such that $\alpha \in \mathcal{P}$.
It is straightforward that $(X \cup \{\star\}, \mathcal{H} \cup \mathcal{P}_\star)$ is a 3-helix system of order $v+1$ and index 3. □

6. The spectrum of $S^{(3)}(1,3)$-designs of index $\lambda > 3$

For the sake of completeness, in this section we deal with the problem of the existence of $\lambda S^{(3)}(1,3)$-designs for any index λ, where the blocks can be repeated with multiplicity. It is immediate that,

Theorem 6.1. *A $\lambda S^{(3)}(1,3)$-design of order v and index λ exists if and only if the pair $(v, \lambda) \in \mathbb{N} \times \mathbb{N}$ is of type $v \equiv 0,1,2 \pmod 9$ and $\lambda \not\equiv 0 \pmod 3$ or $v \geq 7$ and $\lambda \equiv 0 \pmod 3$.*

Proof. As noted at the beginning of Sections 2, 3 and 4, if a $\lambda S^{(3)}(1,3)$-design of order v and index λ exists then $v \equiv 0,1,2 \pmod 9$ and $\lambda \not\equiv 0 \pmod 3$ or $v \geq 7$ and $\lambda \equiv 0 \pmod 3$.

If $v \equiv 0,1,2 \pmod 9$ and $\lambda \not\equiv 0 \pmod 3$, by Theorem 3.4, there exists a block system $(X, \mathcal{B})$ of type $S^{(3)}(1,3)$; it follows that the block system $(X, \lambda\mathcal{B})$ is a $\lambda S^{(3)}(1,3)$ of index λ.

If $\lambda \equiv 0 \pmod 3$, by Theorem 5.1, there exists a $3S^{(3)}(1,3)$-design $(X, \mathcal{B})$ of order $v \geq 7$; in order to construct a $\lambda S^{(3)}(1,3)$-design over X it is sufficient to consider the design $(X, \frac{\lambda}{3}\mathcal{B})$ where $\frac{\lambda}{3}\mathcal{B}$ is the uniform multiset with underlying set $\mathcal{B}$ and multiplicity $\frac{\lambda}{3}$. □

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A. CAUSA
Dipartimento di Matematica e Informatica
Viale A. Doria, 6 - 95100 - Catania, Italy
e-mail: causa@dmi.unict.it

M. GIONFRIDDO
Dipartimento di Matematica e Informatica
Viale A. Doria, 6 - 95100 - Catania, Italy
e-mail: gionfriddo@dmi.unict.it

E.M. GUARDO
Dipartimento di Matematica e Informatica
Viale A. Doria, 6 - 95100 - Catania, Italy
e-mail: guardo@dmi.unict.it