LE MATEMATICHE
Vol. LXII (2007) – Fasc. II, pp. 37–52

# WIENER CRITERION AT THE BOUNDARY RELATED TO P-HOMOGENEOUS STRONGLY LOCAL DIRICHLET FORMS

MARCO BIROLI - SILVANA MARCHI

We state a Wiener criterion at the boundary related to p-homogeneous strongly local Riemannian type Dirichlet forms.

## 1. INTRODUCTION

In this paper we prove a Wiener criterion at the boundary for the solutions of a Dirichlet problem for a Riemannian $p$-homogeneous ($p > 1$) Dirichlet form.

For quasilinear elliptic equations with a growth and coercivity condition of order $p$ the sufficient part of the Wiener criterion has been proved in [13]. The necessary part of the Wiener criterion at the boundary for quasilinear elliptic equations with a growth and coercivity condition of order $p$ has been proved in [14] using an estimate on nonnegative subsolutions of the equation.

The estimate has been generalized in [8] and used in [9] to prove the necessary part of a Wiener criterion for relaxed Dirichlet problems relative to the subelliptic $p$-Laplacian. The sufficient part of the criterion has been also proved using the methods of [13]. A Wiener type criterion at the boundary follows in the case of boundary data corresponding to functions which have an extension to $\mathbf{R}^N$ in a suitable Sobolev space related to the vector fields appearing in the subelliptic $p$-Laplacian. A general Wiener criterion at the boundary can be

Entrato in redazione 1 gennaio 2007

*AMS 2000 Subject Classification:* C0D1C3, PR0V4
*Keywords:* Dirichlet forms, Wiener criterion, Boundary behavior.

proved by similar methods. We remark that the sufficient part of the Wiener criterion for the subeliptic $p$-Laplacian has been previously proved in [12].
The notions of $p$-homogeneous strongly local Dirichlet functionals and forms are introduced in [10], [4] and in [11] an Harnack inequality for a positive harmonic function relative to a Riemannian $p$-homogeneous Dirichlet form is proved.

In [5] we have proved the estimate of [14] in the general framework of the Riemannian $p$-homogeneous ($p > 1$) Dirichlet forms . The estimate enables us to prove in this paper the necessary part of the Wiener criterion at the boundary. The sufficient part of the criterion is proved using a refinement of the methods in [13], [9].

As an example of possible applications we remark that the form on $\mathbf{R}^N$

$$\int \sum_{i=1}^{m} |X_i u|^{p-2} X_i u X_i v \; w dx \quad u,v \in H_0^{1,p;X}$$

where the fields $X_i$ are Hörmander's type vector fields with $C^\infty$ coefficients or Grushin-type vector fields, $w$ is a weight in the $A_2$ Muckenhoupt class with respect to the intrinsic distance and $H_0^{1,p;X}$ is the Sobolev space of order 1 and power $p$ relative to the fields $X_i$, is a Riemannian $p$-homogeneous Dirichlet form, if we choose as distance the intrinsic distance defined by the vector fields and $m(dx) = wdx$ as measure on $\mathbf{R}^N$.

## 2. ASSUMPTIONS AND PRELIMINARIES RESULTS

Let $X$ be a locally compact separable Hausdorff space $X$ with a metrizable topology and a positive Radon measure $m$ on $X$ such that supp[$m$]= $X$. We consider a strongly local Dirichlet form of domain $D_0$

$$\Psi(u,v) = \int_X \mu(u,v)(dx)$$

relative to a strongly local $p$-homogeneous Dirichlet functional ($p > 1$) with the same domain $D_0$

$$\Phi(u) = \int_X \alpha(v)(dx)$$

as defined in [10] or [4]. A notion of capacity relative to the functional $\Phi$ (and to the measure space$(X,m)$) can be defined in the usual variational way. The capacity of an open set $O$ is defined as

$$p-cap(O) = inf\{\Phi_1(v);\ v \in D_0,\ v \geq 1 \text{ a.e. on } O\}$$

if the set $\{v \in D_0 \,, v \geq 1$ a.e. on $O\}$ is not empty and

$$p - cap(O) = +\infty$$

otherwise, where $\Phi_1(v) = \Phi(v) + \int_X |v|^p dm$ . Let $E$ be a subset of $X$, we define

$$p - cap(E) = inf\{p - cap(O); O \text{ open set with } E \subset O\}.$$

We recall that the above defined capacity is a Choquet capacity [10]. Moreover we can prove that every function in $D_0$ is quasi-continuous and is defined quasi-everywhere [10].

The strong locality property allow us to define the domain of the form with respect to an open set $O$, denoted by $D_0[O]$ and the local domain of the form with respect to an open set $O$, denoted by $D_{loc}[O]$. We recall that, given an open set $O$ in $X$ for a set $E \subset \overline{E} \subset O$ we can define a Choquet capacity $p - cap(E;O)$ with respect to the open set $O$. Moreover the sets of zero capacity are the same with respect to $O$ and to $X$. The following properties can be proved [10], [4]:
(a) $\mu(u,v)$, $u,v \in D_0$ is homogeneous of degree $p-1$ in $u$ and linear in $v$; we have also $\mu(u,u) = p\alpha(u)$.
(b) Chain rule : if $u,v \in D_0 \cap L^\infty(X,m)$ and $\beta \in C^1(\mathbf{R})$ with $\beta(0) = 0$ and $\beta^{'}$ bounded on $\mathbf{R}$, then $\beta(u), \beta(v)$ belong to $D_0$ and

$$\mu(\beta(u),v) = |\beta^{'}(u)|^{p-2}\beta^{'}(u)\mu(u,v) \tag{2.1}$$

$$\mu(u,\beta(v)) = \beta^{'}(v)\mu(u,v) \tag{2.2}$$

We observe that we have also a chain rule for $\alpha$

$$\alpha(\beta(u)) = |\beta^{'}(u)|^p \alpha(u) \tag{2.3}$$

where the above relations make sense, since $u$ is defined quasi-everywhere.
(c) Truncation property: for every $u,v \in D_0$

$$\mu(u^+,v) = \mathbf{1}_{\{u>0\}}\mu(u,v) \tag{2.4}$$

$$\mu(u,v^+) = \mathbf{1}_{\{v>0\}}\mu(u,v) \tag{2.5}$$

where such relations make sense, since $u$ and $v$ are defined quasi-everywhere.
(d) Leibniz rule with respect to the second argument: for every $u \in D_0$, $v,w \in D_0 \cap L^\infty(X,m)$

$$\mu(u,vw) = v\mu(u,w) + w\mu(u,v) \tag{2.6}$$

(e) Leibniz inequality: for every $u,v \in D_0 \cap L^\infty(X,m)$

$$\alpha(uv) \leq |v|^p \alpha(u,w) + |u|^p \alpha(u) \tag{2.7}$$

where $u,v\in D_0\cap L^\infty(X,m)$.
(f) For every $u,v\in D_0$, any $f\in L^{p'}(X,\alpha(u))$ and $g\in L^p(X,\alpha(v))$ with $1/p+1/p'=1$, $fg$ is integrable with respect to $|\mu(u,v)|$ and $\forall a\in\mathbf{R}^+$

$$|fg||\mu(u,v)|(dx)\le 2^{p-1}a^{-p}|f|^{p'}\alpha(u)(dx)+2^{p-1}a^{p(p-1)}|g|^p\alpha(v)(dx) \quad (2.8)$$

Taking into account the strong locality property we can replace $D_0$ by $D_{loc}[X]$ in the above properties (a)-(f).

Assume that a distance $d$ is defined on $X$, such that $\alpha(d)\le m$ in the sense of the measures and
(i) The metric topology induced by $d$ is equivalent to the original topology of $X$ and $X$ is complete with respect to $d$.
(ii) For every fixed compact set $K$ there exist positive constants $c_0$ and $r_0$ such that

$$m(B(x,r))\le c_0 m(B(x,s))\left(\frac{r}{s}\right)^\nu \qquad \forall x\in K \quad\text{and}\quad 0<s<r<r_0, \quad (2.9)$$

where we denote by $B(x,r)$ the open ball of center $x$ and radius $r$ (for the distance $d$). We can assume without loss of generality $p<\nu$.

From the properties of $d$ it follows that there exists a cut-off function of $B(x,r)$ with respect to $B(x,2r)$, i.e. a function $\phi\in D_0[B(x,2r)]$ with $0\le\phi\le 1$, $\phi=1$ on $B(x,r)$ and

$$\alpha(\phi)\le\frac{2}{r^p}m$$

in the sense of the measures.

We assume also that the following scaled *Poincaré inequality* holds: for every fixed compact set $K$ there exist positive constants $c_2$, $r_1$ and $k\ge 1$ such that for every $x\in K$ and every $0<r<r_1$

$$\int_{B(x,r)}|u-\overline{u}_{x,r}|^p m(dx)\le c_2 r^p\int_{B(x,kr)}\alpha(u)(dx) \quad (2.10)$$

for every $u\in D_{loc}[B(x,kr)]$, where $\overline{u}_{x,r}=\frac{1}{m(B(x,r))}\int_{B(x,r)}u\,m(dx)$.

A strongly local $p$-homogeneous Dirichlet form, such that the above assumptions hold, is called a *Riemannian Dirichlet form*.
As proved in [15] the Poincaré inequality imply the following *Sobolev inequality*: for every fixed compact set $K$ there exist positive constants $c_3$, $r_2$ and $k\ge 1$ such that for every $x\in K$ and every $0<r<r_2$

$$\left(\frac{1}{m(B(x,r))}\int_{B(x,r)}|u|^{p^*}m(dx)\right)^{\frac{1}{p^*}}\le \quad (2.11)$$

$$\leq c_3 \left( \frac{r^p}{m(B(x,r))} \int_{B(x,kr)} \alpha(u)(dx) + \frac{r^p}{m(B(x,r))} \int_{B(x,r)} |u|^p m(dx) \right)^{\frac{1}{p}}$$

with $p^* = \frac{p\nu}{\nu - p}$ and $c_3, r_2$ depending only on $c_0$, $c_2$, $r_0$, $r_1$. We observe that we can assume without loss of generality $r_0 = r_1 = r_2$.

**Remark 2.1.** (a) From (1.10) we can easily deduce by standard methods that

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |u|^p m(dx) \leq c_2' \frac{r^p}{m(B(x,r) \cap \{u=0\})} \int_{B(x,kr)} \alpha(u)(dx)$$

where $c_2'$ is a positive constant depending only on $c_2$.
(b) From (a) it follows that for every fixed compact set $K$, such that the closed neighborhood $K'$ of $K$ of radius $r_0(K)$ is compact and strictly contained in $X$,

$$\int_{B(x,r)} |u|^p m(dx) \leq c_2^\star r^p \int_{B(x,r)} \alpha(u)(dx)$$

for every $x \in K$ and $0 < r < \frac{r_0(K')}{2}$, where $u \in D_0[B(x,r)]$ and $c_2^\star$ depends only on $c_2'(K')$ and $c_0(K')$.

As a consequence of the assumptions on $X$ and $d$ and of the Poincaré inequality we have the following estimate on the capacity of a ball [10]

**Proposition 2.2.** For every fixed compact set $K$ there exists positive constants $c_4$ and $c_5$ such that

$$c_4 \frac{m(B(x,r))}{r^p} \leq p - cap(B(x,r), B(x,2r)) \leq c_5 \frac{m(B(x,r))}{r^p}$$

where $x \in K$ and $0 < 2r < r_0$.

The left-hand-side inequality is consequence of Remark 2.1 applied to the potential of the ball $B(x,r)$ with respect to the ball $B(x,2r)$ (the existence of such a potential has been proved in [10], [4]).The right-hand-side inequality is a consequence of the existence of a cut-off function of $B(x,r)$ with respect to $B(x,2r)$.

## 3. THE RESULTS

Let $\Omega$ be an open set in $X$ such that the closed neighborhood $\overline{\Omega}'$ of radius $r_0(\overline{\Omega})$ of $\overline{\Omega}$ is compact and strictly contained in $X$. In the following we denote $r_0 = r_0(\overline{\Omega}')$. Denote by $D[\Omega]$ the space of the function $v$ in $D_{loc}[\Omega]$ such that $\int_\Omega \alpha(v)(dx) < +\infty$.

A function $g$ in $D[\Omega]$ is continuous on $\partial\Omega$ at $x_0 \in \partial\Omega$ with value $g(x_0)$ if there exists an increasing function $k(r)$, $0 < r < \overline{R}$ with

$$lim_{r\to 0}k(r) = 0$$

such that for $\eta \in D_0[B(x_0,r)]$ with $\alpha(\eta)(dx)$ having an $L^\infty(B(x_0,r),m)$ density with respect to $m(dx)$, then $\eta(g-(k(r)+g(x_0)))^+$ and $\eta(g+k(r)-g(x_0))^-$ are in $D_0[B(x_0,r)\cap|Omega]$. We assume without loss of generality that $\overline{R} \leq r_0$.

**Definition 3.1.** Let $g$ be a function in $D[\Omega]$. The function $u \in D[\Omega]$ is a solution of the Dirichlet problem relative to $\mu$, $\Omega$, $g$ if $u-g \in D_0[\Omega]$ and

$$\int_\Omega \mu(u,\varphi)(dx) = 0 \tag{3.1}$$

for any $\varphi \in D_0[\Omega]$.

**Definition 3.2.** The function $u \in D_{loc}[\Omega]$ is a local sub-solution of the Dirichlet problem relative to $\mu$, $\Omega$ if

$$\int_\Omega \mu(u,\varphi)(dx) \leq 0 \tag{3.2}$$

for any nonnegative $\varphi \in D_0[\Omega]$ with $supp(\varphi) \subset \Omega$.

**Remark 3.3.** Let $g \in D[\Omega]$ and let $u \in D[\Omega]$ be a solution of the Dirichlet problem relative to $\mu$, $\Omega$, $g$, then

$$||u||^p_{D[\Omega]} \leq C||g||^p_{D[\Omega]} \tag{3.3}$$

If $g \in L^\infty(\Omega,m)$ we have also

$$||u||_{L^\infty(\Omega)} \leq C||g||_{L^\infty(\Omega)}$$

. Moreover we recall that if $u$ is a local nonnegative sub-solution of the Dirichlet problem relative to $\mu$, $\Omega$ then

$$\sup_{B(x,\frac{r}{2})} u \leq C(q)(\frac{1}{m(B(x,r))}\int_{B(x,r} u^q m(dx))^{\frac{1}{q}}$$

for every $q > 0$.[11]

**Definition 3.4.** A point $x_0 \in \partial\Omega$ is a regular point for (3.1) if for every function $g \in D[\Omega]$, which is continuous on $\partial\Omega$ at $x_0 \in partial\Omega$ with value $g(x_0)$ the solution $u$ of (3.1) is continuous at $x_0$ with respect to the value $u(x_0) = g(x_0)$.

**Definition 3.5.** A point $x_0$ in $\partial\Omega$ is a Wiener point if

$$\int_0^1 \delta(\rho)^{\frac{1}{p-1}} \frac{d\rho}{\rho} = +\infty \tag{3.4}$$

where

$$\delta(\rho) = \frac{p-cap(B(x_0,\frac{\rho}{2})\backslash\Omega, B(x_0,\rho))}{p-cap(B(x_0,\frac{\rho}{2}), B(x_0,\rho))}$$

We are now in position to state the main result of this paper

**Theorem 3.6.** Let $x_0 \in \partial\Omega$. Then the point $x_0$ is regular for (3.1) iff $x_0$ is a Wiener point of $\partial\Omega$. Moreover there exist some constants $C_1$ $C_1'$ and $C_2$ such that for any solution $u$ of (3.1) with $g$ continuous on $\partial\Omega$ at $x_0$ with value $g(x_0)$, we have

$$sup_{B(x_0,s)}|u-g(x_0)| \leq \tag{3.5}$$

$$\leq C_1 exp\left[-C_2\int_s^r \delta(\rho)\frac{d\rho}{\rho}\right] sup_{B(x_0,r)}|u-g(x_0)| + 4k(R) \leq$$

$$\leq C_1' exp\left[-C_2\int_s^r \delta(\rho)\frac{d\rho}{\rho}\right]\left(\left(\frac{1}{m(B(x_0,\overline{R}))}\int_{B(x_0,\overline{R})} u^p\, m(dx)\right)^{\frac{1}{p}} + g(x_0)+\right.$$

$$\left. k(\overline{R})\right) + 4k(R)$$

for $0 < 2s \leq r$, $2r \leq R$, $8R \leq \overline{R}$.

In the section 4 we prove the sufficient part of Theorem 3.1. The section 5 contains the proof of the necessary part of Theorem 3.1.

## 4. PROOF OF THE SUFFICIENT PART OF TH. 3.6

Let $x_0 \in \partial\Omega$. Assume that $u \in D[\Omega]$ is a weak solution of (3.1). We may assume without loss of generality that $g(x_0) = 0$. Let $u_k := (u-k)^+$ where $k = k(R) + g(x_0)$ and define

$$M(r) = sup_{B(x_0,r)} u_k$$

$$M_\varepsilon(r) = M(r) + \varepsilon$$

where $\varepsilon \in (0,\frac{1}{2})$, $0 < r < \frac{R}{2} < R < \frac{\overline{R}}{8}$.

**Proposition 4.1.** Define $v^{-1} = M_\varepsilon(r) - u_k$. Let $p \in (1, \nu]$ and $\eta \in D_0[B(x_0, \frac{3r}{4}]$ with $0 \le \eta \le 1$ and $\eta = 1$ on $B(x_0, \frac{r}{2}$ and $\alpha(\eta) \le 2(\frac{4}{r})^p m$ in $\Omega$. Then there exists a constant dependent only on $\Omega$, $p$ and the structure but independent of $\varepsilon$, $r$ such that

$$\frac{r^p}{m(B(x_0,r))}\int_\Omega \alpha(\eta v^{-1})(dx) \le \tag{4.1}$$

$$\le CM_\varepsilon(r)\left[M_\varepsilon(r) - M_\varepsilon(\frac{r}{2}) + \varepsilon\right]^{p-1}$$

where $2r \le R \le \frac{\overline{R}}{8}$ and $C$ is a structural constant.

We assume the Proposition 4.1 and we prove the sufficient part of Theorem 3.6. Let $r \le R$ $k = sup_{B(x_0,R)} g$ and let $\eta = 1$ on $B(x_0, \frac{r}{2})$. Multiplying (4.1) by $M_\varepsilon^{-1}$ we obtain

$$M_\varepsilon^{p-1}\frac{r^p}{m(B(x_0,r))}\int_\Omega \alpha(\eta\widetilde{v}^{-1})(dx) \tag{4.2}$$

$$\le C\left[M_\varepsilon(r) - M_\varepsilon(\frac{r}{2}) + \varepsilon\right]^{p-1}$$

where $\widetilde{v} = 1 - \frac{u_k}{M_\varepsilon(r)}$. Taking into account the definition of the $p$-capacity we obtain

$$M_\varepsilon(r)\left[\frac{p - cap(B(x_0,\frac{r}{2})\backslash\Omega, B(x_0,r))}{p - cap(B(x_0,\frac{r}{2}), B(x_0,r))}\right]^{\frac{1}{p-1}} \le$$

$$\le (2C)^{\frac{1}{p-1}}\left[M_\varepsilon(r) - M_\varepsilon(\frac{r}{2}) + \varepsilon\right]$$

where here and in the following $C$ denotes a possibly different structural constant. Here we assume $C \ge 1$. Taking the limit $\varepsilon \to 0$ in the above inequality gives

$$M(\frac{r}{2}) \le \left[1 - (2C)^{-\frac{1}{p-1}}\delta(r)^{\frac{1}{p-1}}\right]M(r) \tag{4.3}$$

where $\delta(r) = \frac{p-cap(B(x_0,\frac{r}{2})\backslash\Omega, B(x_0,r))}{p-cap(B(x_0,\frac{r}{2}), B(x_0,r))}$. It follows

$$sup_{B(x_0,\frac{r}{2})\cap\Omega}u^+ \le \left[1 - (2C)^{-\frac{1}{p-1}}\delta(r)^{\frac{1}{p-1}}\right]sup_{B(x_0,r)\cap\Omega}u^+ + 2k(R)$$

where $0 < r < R$. Taking into account that $-u$ is a local solution of (3.1) relative to $-g$, we obtain

$$sup_{B(x_0,\frac{r}{2})\cap\Omega}u^- \le \left[1 - (2C)^{-\frac{1}{p-1}}\delta(r)^{\frac{1}{p-1}}\right]sup_{B(x_0,r)\cap\Omega}u^- + 2k(R)$$

Then

$$osc_{B(x_0,\frac{r}{2})\cap\Omega}|u| \le \left[1-(2C)^{-\frac{1}{p-1}}\delta\,(r)^{\frac{1}{p-1}}\right] osc_{B(x_0,r)\cap\Omega}|u| + 4k(R) \qquad (4.4)$$

where $0 < r < R$. From (4.4) by iteration [16] we obtain

$$sup_{B(x_0,s)\cap\Omega}|u| \le C_1 exp\left[-C_2\int_s^r \delta(\rho)^{\frac{1}{p-1}}\frac{d\rho}{\rho}\right] sup_{B(x_0,r)\cap\Omega}|u| + 4k(r)g$$

where $0 < s < \frac{r}{2} < r < R$. The first inequality in Theorem 3.6 is so proved. The second inequality follows observing that $(u \mp (k(\overline{R} \pm g(x_0))^{\pm}$ are positive subsolutions in $B(x_0,\overline{R})$ relative to our form ( we can use the methods in [9]).

**Remark 4.2.** Let us observe that (3.4) gives an estimate on the velocity of convergence of $u$ to $g(x_0)$ as $x \to x_0$. In particular if $\delta(\rho) \ge c > 0$ $\alpha = C_2 \wedge 1$ we have

$$exp\left(-C_2\int_s^r \delta(\rho)\frac{d\rho}{\rho}\right) \sim \left(\frac{s}{r}\right)^{\alpha}$$

If $osc_{B(x_0,r)\cap\partial\Omega}g \le C_3 r^{\beta}$ for $0 < r < \frac{\overline{R}}{2}$, then we obtain

$$sup_{B(x_0,r)\cap\Omega}|u-g(x_0)| \le C_4 r^{\gamma}$$

for $r < \frac{\overline{R}}{2}^2$ where $\gamma = \left(\frac{\alpha}{2} \wedge \frac{\beta}{2}\right)$.

**Proof of Proposition 4.1** In the proof $C$ will denote possibly different structural constants. At first we observe that $u_k$ is locally bounded in $B(x_0,R)$. By the same methods used in [9] we can prove that $u_k$ is a positive subsolution in $B(x_0,R)$ (relative to our form). We prove now that $v$ is again a positive subsolution in $B(x_0,r)$ (relative to our form). Let $\phi$ be a positive function in $D_0[B(x_0,r)]$. We have

$$\int_{B(x_0,r)} \alpha(v,\phi)(dx) = \int_{B(x_0,r)} (M_\varepsilon(r)-u_k)^{-2(p-1)}\alpha(u_k,\phi)(dx) =$$

$$= \int_{B(x_0,r)} \alpha(u_k,(M_\varepsilon(r)-u_k)^{-2(p-1)}\phi)(dx) -$$

$$-2(p-1)\int_{B(x_0,r)} (M_\varepsilon(r)-u_k)^{(-2p+1)}\phi\,\alpha(u_k,(M_\varepsilon(r)-u_k))(dx) \le$$

$$\le -\int_{B(x_0,r)} (M_\varepsilon(r)-u_k)^{-4(p-1)}\phi\,\alpha(u_k,u_k)(dx) \le 0$$

and the result follows. Let now $\eta$ be a positive function in $D_0[B(x,s)]$ where $B(x,s) \subset B(x_0,r)$. We have

$$\int_{B(x_0,s)} \alpha(u_k, v^{p-1}\eta^p)(dx) \leq 0 \tag{4.5}$$

Then

$$\int_{B(x_0,s)} (p-1)v^{p-2}\eta\alpha(u_k,v)\eta^p(dx) =$$

$$= (p-1)\int_{B(x_0,s)} v^{p-2}v^{-2(p-1)}\eta\alpha(v,v)\eta^p(dx) =$$

$$= (p-1) \leq \int_{B(x,s)} v^{-p}\eta\alpha(v,v)\eta^p(dx) =$$

$$= (p-1)\int_{B(x,s)} \eta\alpha(log(v),log(v))\eta^p(dx)$$

From (4.5) we obtain

$$\int_{B(x,s)} \eta^p\alpha(log(v),log(v))(dx) \leq \int_{B(x,s)} v^{p-1}\eta^{p-1}\alpha(u_k,\eta)(dx) \leq$$

$$\leq \frac{1}{2}\int_{B(x,s)} \eta^p v^p\alpha(u_k,u_k)(dx) + 4\int_{B(x,s)} \alpha(\eta,\eta)(dx) =$$

$$= \frac{1}{2}\int_{B(x,s)} \eta^p v^{-p}\alpha(v,v)(dx) + 4\int_{B(x,s)} \alpha(\eta,\eta)(dx) =$$

$$= \frac{1}{2}\int_{B(x,s)} \eta^p\alpha(log(v),log(v))(dx) + 4\int_{B(x,s)} \alpha(\eta,\eta)(dx)$$

Let $\eta$ be the cut-off function between $B(x,\frac{1}{2}s)$ and $B(x,s)$, we obtain

$$\int_{B(x,s)} \eta^p\alpha(log(v),log(v))(dx) \leq Cs^p m(B(x,s))$$

so we obtain that $v \in BMO_{loc}(B(x_0,r))$. As in [6] we obtain that there exists $\sigma_0$ such that for $\sigma \leq \sigma_0$

$$\left(\frac{1}{m(B(x_0,\frac{3r}{4}))}\int_{B(x_0,\frac{3r}{4})} v^\sigma m(dx)\right)\left(\frac{1}{m(B(x_0,\frac{3r}{4}))}\int_{B(x_0,\frac{3r}{4})} v^{-\sigma}m(dx)\right) \leq C$$

Since $v$ is a positive subsolution, we obtain that

$$sup_{B(x_0,\frac{r}{2})} v \leq C\frac{1}{m(B(x_0,\frac{5r}{8}))}\int_{B(x_0,\frac{5r}{8})} v^\sigma m(dx)^{\frac{1}{\sigma}} \leq$$

$$\leq C(\frac{1}{m(B(x_0,\frac{5r}{8}))}\int_{B(x_0,\frac{5r}{8})} v^{-\sigma}m(dx))^{-\frac{1}{\sigma}}$$

(see [11]). Taking into account the definition of $v$ we obtain

$$(\frac{1}{m(B(x_0,\frac{3r}{4}))}\int_{B(x_0,\frac{5r}{8})} v^{-\sigma}m(dx))^{\frac{1}{\sigma}} \leq$$

$$\leq C(M_\varepsilon(r)-M_\varepsilon(\frac{r}{2})+\varepsilon)$$

We choose now as test-function in (3.1)

$$\varphi = \eta^p\psi$$

where $\eta \in D_0[B(x_0,r)]$ with $\alpha(\eta)(dx)$ with a bounded density and

$$\psi = \left(v^\beta - \left(\frac{1}{M_\varepsilon(r)}\right)^\beta\right)$$

(we observe that $\psi \in L^\infty(B(x_0,r),m)$), [13]. Take $\eta \geq 0$, so $\varphi \geq 0$. We obtain

$$\beta\int_{B(x_0,r)} \eta^p v^{\beta+1}\alpha(u_k)(dx) \leq p\int_{B(x_0,r)} \eta^{p-1}\psi\mu(u_k,\eta)(dx)$$

Since $\psi \leq v^\beta$ we have

$$\beta\int_{B(x_0,r)} \eta^p v^{\beta+1}\alpha(u_k)(dx) \leq p\int_{B(x_0,r)} \eta^{p-1}v^\beta\mu(u_k,\eta)(dx)$$

The Young's inequality gives

$$\int_{B(x_0,r)} \eta^{p-1}v^\beta\mu(u_k,\eta)(dx)$$

$$\leq \theta^{\frac{p}{p-1}}\frac{p}{p-1}\int_{B(x_0,r)} \eta^p v^{\beta+1}\alpha(u_k)(dx) + \theta^{-p}\frac{1}{p}\int_{B(x_0,r)} v^{\beta-p+1}\alpha(\eta)(dx)$$

If $\theta = \beta^{\frac{p-1}{p}}$, we have

$$\int_{B(x_0,r)} \eta^p v^{\beta+1}\alpha(u_k)(dx) \leq C\beta^{-p}\int_{B(x_0,r)} v^{\beta-p+1}\alpha(\eta)(dx) \tag{4.6}$$

From (4.6) choosing $0 < \beta \neq p-1$, $\beta = \tau p + p - 1$, $\tau < 0$ (then $\frac{1-p}{p} < \tau < 0$) we obtain

$$\int_{B(x_0,r)} \eta^p\alpha(v^\tau)(dx) \leq K(\tau)\int_{B(x_0,r)} v^{p\tau}\alpha(\eta)(dx) \tag{4.7}$$

where $K(\tau) \simeq |\tau|^p + \beta^{-p}$. Given any $0 < s < t \leq 1$, let us take $\eta$ such that $\eta \in D_0[B(x_0,tr)]$, $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x_0,sr)$, $\alpha(\eta) \leq \frac{C}{r^p(t-s)^p}$. We obtain

$$\frac{1}{m(B(x_0,sr))}\int_{B(x_0,sr)} v^{\gamma p\tau} m(dx) \leq \tag{4.8}$$

$$\leq \frac{CK(\tau)}{(t-s)^p m(B(x_0,tr))}\int_{B(x_0,tr)} v^{p\tau} m(dx)$$

where $\gamma = \frac{\nu}{\nu-p}$. Using a Moser type iteration method as in [11] we obtain

$$\frac{1}{m(B(x_0,3r/4))}\int_{B(x_0,3r/4)} v^{-q} m(dx) \leq \tag{4.9}$$

$$\leq C(q)(\frac{1}{m(B(x_0,r))}\int_{B(x_0,r)} v^{-\sigma} m(dx))^{\frac{q}{\sigma}} \leq$$

$$\leq CC(q)\left(M_\varepsilon(r) - M_\varepsilon(\frac{r}{2}) + \varepsilon\right)^q$$

where $C(q)$ is a finite valued increasing function of $q$ for any $0 < q < (p-1)\frac{\nu}{\nu-p}$ We are finally able to conclude the proof of Proposition 4.1. Let $\eta \in D_0[B(x_0,\frac{3r}{4})]$ with $0 \leq \eta \leq 1$, $\eta = 1$ on $B(x_0,\frac{r}{2})$ and $\alpha(\eta) \leq 2(\frac{4}{r})^p$ and choose as test function in (3.1) the function $\varphi = \eta^p u_k$. We have

$$\int_{B(x_0,r/2)} \eta^p \alpha(u_k)(dx) + \int_{B(x_0,r/2)} u_k p\eta^{p-1}\mu(u_k,\eta)(dx) = 0$$

Let us observe that

$$\frac{1}{m(B(x_0,3r/4))}\int_{B(x_0,3r/4)} u_k\eta^{p-1}\mu(u_k,\eta) =$$

$$= \frac{1}{m(B(x_0,3r/4))}\int_{B(x_0,3r/4)} u_k\eta^{p-1}v^{-(\tau+1)(p-1)}\mu(v^\tau,\eta)$$

$$\leq CM(r)\left(\frac{1}{m(B(x_0,r/2))}\int_{B(x_0,r/2)} \eta^p\alpha(v^\tau)(dx)\right)^{\frac{p-1}{p}}$$

$$\left(\frac{1}{m(B(x_0,3r/4))}\int_{B(x_0,3r/4)} v^{-(\tau+1)(p-1)p}\alpha(\eta)(dx)\right)^{\frac{1}{p}} \leq$$

$$\leq CM(r)\left(\frac{1}{m(B(x_0,3r/4))}\int_{B(x_0,3r/4)} \alpha(\eta v^\tau)(dx) +\right.$$

$$+\frac{1}{m(B(x_0,r/2))}\int_{B(x_0,3r/4)} v^{\tau p}\alpha(\eta)(dx)\Big)^{\frac{p-1}{p}} \cdot$$

$$\cdot\left(\frac{1}{m(B(x_0,3r/4))}\int_{B(x_0,3r/4)} v^{-(\tau+1)(p-1)p}\alpha(\eta)(dx)\right)^{\frac{1}{p}}$$

$$\leq CM(r)\left[\left(M_\varepsilon(r)-M_\varepsilon(\frac{r}{2})+\varepsilon\right)^{\sigma p} r^{-p}\right]^{\frac{p-1}{p}} \cdot$$

$$\left[\left(M_\varepsilon(r)-M_\varepsilon(\frac{r}{2})+\varepsilon\right)^{(\tau+1)(p-1)p} r^{-p}\right]^{\frac{1}{p}} =$$

$$= CM(r)\left(M_\varepsilon(r)-M_\varepsilon(\frac{r}{2})+\varepsilon\right)^{p-1} r^{-p}$$

where we have chosen $\tau$ suitable near enough to $(1-p)$. Then

$$\int_{B(x_0,3r/4)} \eta^p\alpha(u_k)(dx)$$

$$\leq CM_\varepsilon(r)\left(M_\varepsilon(r)-M_\varepsilon(\frac{r}{2})+\varepsilon\right)^{p-1} r^{-p}m(B(x_0,r))$$

Hence we obtain

$$\int_{B(x_0,3r/4)} \alpha(\eta v^{-1})(dx) \leq$$

$$\leq CM_\varepsilon(r)\left(M_\varepsilon(r)-M_\varepsilon(\frac{r}{2})+\varepsilon\right)^{p-1} r^{-p}m(B(x_0,r))$$

for every $0<r<\frac{R}{2}$ where we use the estimate

$$\int_{B(x_0,3r/4)} v^{-p}\alpha(\eta)(dx) \leq Cr^{-p}M_\varepsilon(r)\int_{B(x_0,3r/4)} v^{(1-p)}m(dx)$$

We have so completed the proof of Proposition 4.1.

## 5. PROOF OF THE NECESSARY PART OF TH. 3.1

This proof follows by the methods of [14]. Let $x_0$ be a regular point in the boundary of $\Omega$. Let $\int_0^1 \delta(\rho)\frac{d\rho}{\rho}$ be finite. Then the singleton $\{x_0\}$ has to be of capacity zero with respect to $X$. Choose $\varepsilon>0$ and $r>0$ to be specified later. We can find a function $g\in D_0\cap C_0(X)$, such that $g(x)\leq 1$, $g(x_0)=1$ and $||g||_{D_0}<\varepsilon$. Let $u$ be a solution of (3.1) relative to $g$, we have that $u$ is positive in $\Omega$. From

(3.3) we have $||u||_{D[\Omega]} \leq C\varepsilon.$ Since $g$ is bounded we have that $u$ is also bounded and $|||u||_{L^\infty(\Omega,m)} \leq |||g||_{L^\infty(\Omega,m)}$ (see again Remark 3.3). From [5] we have

$$p - fine - limsup_{x\to x_0} u(x) \leq C_1 \left( \frac{1}{m(B(x_0,r))} \int_{B(x_0,r)\cap\Omega} |u|^p m(dx) \right)^{1/p} +$$

$$+C_2 \int_0^{4r} \delta(\rho)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \leq$$

$$\leq C_3 \left( \frac{1}{m(B(x_0,r))} \right)^{1/p} \varepsilon + C_2 \int_0^{4r} \delta(\rho)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$$

There exists $r > 0$ such that

$$C_2 \int_0^{4r} \delta(\rho)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < \frac{1}{3}$$

In this case choosing $\varepsilon$ such that

$$C_1 C_3 \left( \frac{1}{m(B(x_0,r))} \right)^{1/p} \varepsilon < \frac{1}{3}$$

Then

$$p - fine - limsup_{x\to x_0} u(x) < \frac{2}{3} = g(x_0)$$

and a contradiction follows.

## REFERENCES


[1] H. Attouch, *Variational convergence for functions and operators*, Pitman, Applicable Mathematics Series, London-Marshfield 1984.

[2] M. Biroli, *Nonlinear Kato measures and nonlinear subelliptic Schröedinger problems*, Rend. Acc. Naz. Sc. detta dei XL, Memorie di Matematica e Appl., 115 Vol. XXI (1997), 235-252.

[3] M. Biroli, *Weak Kato measures and Shröedinger problems for a Dirichlet form*, Rend. Acc. Naz. Sc. detta dei XL, Memorie di Matematica e Appl., 118 Vol. XXIV (2000), 197-217.

[4] M.Biroli, *Nonlinear p-homogeneous Dirichlet forms on nonreflexive Banach spaces*, Rend. Acc. Naz. Sc. detta dei XL, Memorie di Matematica e Appl., 123 Vol. XXIX (2005), 55-78.

[5] M. Biroli - S. Marchi, *Oscillation estimates relative to p-homogeneous forms and Kato measures data*, Le Matematiche (2007), preprint.

[6] M. Biroli - U. Mosco, *A Saint Venant type principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura Appl., 169 (IV) (1995), 125-181.

[7] M. Biroli - U. Mosco, *Sobolev inequalities on homogeneous spaces*, Potential Anal. (1995), 311-324.

[8] M. Biroli - N. Tchou, *Nonlinear subelliptic problems with measure data*, Rend. Acc. Naz. Scienze detta dei XL, Memorie di Matematica e Applicazioni, XXIII (1999), 57-82.

[9] M. Biroli - N. Tchou, *Relaxed Dirichlet problem for the subelliptic p-Laplacian*, Ann. Mat. Pura Appl. (IV), CLXXIX (2001), 39-64.

[10] M. Biroli - P. Vernole, *Strongly local nonlinear Dirichlet functionals and forms*, Advances in Mathematical Sciences and Applications, 15 (2005), 655-682.

[11] M. Biroli - P. Vernole, *Harnack inequality for harmonic functions relative to a nonlinear p-homogeneous Riemannian Dirichlet form*, Nonlinear Analysis, 64 (2006), 51-68.

[12] D. Danielli, *Regularity at the boundary for solutions of nonlinear subelliptic equations*, Indiana Un. Math. J. 44 (1955), 269-286.

[13] R. Gariepy - W. Ziemer, *A regularity condition at the boundary for solutions of quasilinear elliptic equations*, Arch. Rat. Mech. Anal. 67 (1977), 25-39.

[14] J. Maly, *Pointwise estimates of nonnegative subsolutions of quasilinear elliptic equations at irregular points*, Comm. Math. Univ. Carolinae, 37 (1996), 23-42.

[15] J. Maly - U. Mosco, *Remarks on measure-valued Lagrangians on homogeneous spaces*, Ricerche Mat. 48 (1999), Supplemento, 217-231.

[16] U. Mosco, *Wiener criterion and potential estimates for the obstacle problem* Indiana Un. Math. J., 36 (1987), 455-494.

*MARCO BIROLI*
*Dipartimento di Matematica F. Brioschi,*
*Politecnico di Milano, Milano, Italy.*
*e-mail:* marbir@mate.polimi.it.

*SILVANA MARCHI*
*Dipartimento di Matematica, Universitá di Parma,*
*Viale Usberti, 53/A, Parma, Italy.*
*e-mail:* silvana.marchi@unipr.it