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ON THE FORMULATION OF EXTENDED THERMODYNAMICS IN THE CASE OF FRACTIONAL EXCLUSION STATISTICS.

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We consider the non-equilibrium theory for the fractional exclusion statistics (FES) by using the Maximum Entropy Principle and the Entropy Principle. The entropy balance equation is determined and the statistical consequence of theory are discussed. Both the entropy and its flux are computed explicitly in terms of the non-equilibrium Lagrange multipliers while, by using a general expression for the energy dispersion relation, some thermodynamic properties connected with the convexity conditions of the entropy are explicitly analyzed. Finally, for an ideal gas subject to FES, the construction of an arbitrary set of closed hydrodynamic equations, in the context of Extended Thermodynamics, is briefly illustrated.

1. Introduction

It is well known that fermions and bosons can exist in all dimensions, while some low-dimensional systems exhibit elementary excitations that obey quantum statistics that interpolate the behavior of fermions and bosons. In particular,

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the concept of anyons [1, 2] is specific to two dimensions (D=2) being connected to the braid group structure of particle trajectories [1, 2], and the fractional statistics are described by means of a phase factor that describes the particle exchange procedure [2]. A different notion of fractional statistics was introduced by Haldane [3] without specific reference to spatial dimensions. Quasiparticles that obey the fractional exclusion statistics (FES) are called *exclusons*. Therefore, Haldane's approach was developed using an arbitrary dimension of a Hilbert space and by introducing a statistics (for a single species) parameter $\kappa = \delta G/\delta N$, where δG describes the change in size of the subset of available single-particle states corresponding to a variation of δN particles. It is known that FES is, in general, different from anyon statistics. Indeed, the exclusion statistics is assigned to elementary excitations of condensed-matter systems, which are not necessarily connected with braiding considerations [3, 4]. However, there are some systems where a thermodynamics coincidence of the two statistics has been shown [3, 5–8]. Many thermodynamic properties of an ideal gas of *exclusons* have been studied in literature under equilibrium conditions [9–18] and in particular the expressions for the distribution function have been determined using the Maximum Entropy Principle (MEP) both in the framework of a semiclassical local theory [9] than of a non-local quantum Wigner theory (Quantum MEP) [19].

The aim of this work is to develop and apply a general theory for the fractional exclusion statistics in non-equilibrium conditions by connecting the theory, in a semiclassical framework, with the Boltzmann transport equation (BTE) and with the method of moments, in terms of the Lagrange multipliers. For this purpose we include, in the non-equilibrium collisional production of BTE, the terms describing the effect of the fractional exclusion statistics on the accessibility of the final states reached during collisional processes. The entropy balance equation is determined and the H-Theorem is discussed. Both the entropy and its flux are computed explicitly in terms of the non-equilibrium Lagrange multipliers while, by using a general expression for the energy dispersion relation, some thermodynamic relations for the system are determined, by discussing the general conditions for entropy convexity under local equilibrium conditions. Finally, the possible construction of an arbitrary set of closed hydrodynamic equations, in the context of Extended Thermodynamics, is briefly indicated.

2. General theory

We can evaluate the entropy S, for a noninteracting system under nonequilibrium conditions, using the celebrated equation $S = k_B \ln \Gamma$, were k_B the Boltzmann constant and Γ is the number of microscopic states compatible with the macroscopic state of the system.

2.1. FES and the non-equilibrium MEP

To take into account *ab initio* the FES, Wu [9] proposed, an interpolation formula to describe the entropy in equilibrium conditions. Using this approach, in non-equilibrium conditions and in the continuous case, the entropy density can be expressed in the following form

$$S(\mathbf{r},t) = \int h(\mathcal{F})\, d^D p \tag{1}$$

being

$$\begin{aligned} h(\mathcal{F}) = k_B y \Bigg\{ & \left[1+(1-\kappa)\frac{\mathcal{F}}{y}\right] \ln\left[1+(1-\kappa)\frac{\mathcal{F}}{y}\right] - \\ & \left[1-\kappa\frac{\mathcal{F}}{y}\right] \ln\left[1-\kappa\frac{\mathcal{F}}{y}\right] - \frac{\mathcal{F}}{y}\ln\left[\frac{\mathcal{F}}{y}\right]\Bigg\} \end{aligned} \tag{2}$$

where the constant $y = g_s/(2\pi\hbar)^D$ (being g_s the spin degeneracy) and $\kappa \in [0,1]$ is the statistical parameter that for $\kappa = 0,1$ corresponds to Bosons and for Fermions, respectively [9]. Accordingly, we search the non-equilibrium distribution function that maximizes S under the constraints that the expectation values for some set of macroscopic variables (moments of distribution function) can be expressed by means of the relations

$$F_A = \int \psi_A(\mathbf{p})\, \mathcal{F}(\mathbf{p},\mathbf{r},t)\, d^D p\,, \qquad A = 1,\ldots \mathcal{N} \tag{3}$$

being $\{\psi_A(\mathbf{p})\}$ a suitable set of kinetic fields corresponding to the macroscopic moments (3). The method of Lagrange multipliers [20–27] proves to be the most efficient technique to include the constraints and solve this variational problem. A short calculation yields the relation

$$\frac{y}{\mathcal{F}}\left[1-\kappa\frac{\mathcal{F}}{y}\right]^{\kappa}\left[1+(1-\kappa)\frac{\mathcal{F}}{y}\right]^{1-\kappa} = \xi = e^{\Pi} \tag{4}$$

and, consequently, to the nonequilibrium FES distribution function

$$\mathcal{F} = \frac{y}{\mathcal{W}(e^{\Pi})+\kappa} \qquad \text{with} \qquad \Pi = \frac{1}{k_B}\sum_{A=1}^{\mathcal{N}} \Lambda_A\, \psi_A \tag{5}$$

where the function $\mathcal{W}(\xi)$ satisfies the usual functional equation

$$[\mathcal{W}(\xi)]^{\kappa}\,[\mathcal{W}(\xi)+1]^{1-\kappa} = \xi \tag{6}$$

and Λ_A are the *Lagrange multipliers* to be determined.
Obviously, from equation (6) for $\kappa = 0$ we obtain $\mathcal{W}(\xi) = \xi - 1$ while for $\kappa = 1$

we have $\mathcal{W}(\xi) = \xi$ and consequently the two distribution functions for Bosons and Fermions respectively (which in the case $\xi = e^{\Pi} >> 1$ trivially converge to the usual non-equilibrium Boltzman distribution function $\mathcal{F} = y e^{-\Pi}$).
We remark that, in general, the following set of kinetic fields can be considered

$$\psi_A = \{\varepsilon^s, \varepsilon^s u_{i_1}, \varepsilon^s u_{i_1} u_{i_2}, \ldots, \varepsilon^s u_{i_1} u_{i_2} \cdots u_{i_r}\} \tag{7}$$

where ε is the microscopic energy and u_i is the group velocity, with $s = 0, 1, \ldots N$ and $r = 0, 1, \ldots M$. Accordingly, it is possible to consider the general expression for the dispersion relation and for the group velocity

$$\varepsilon = a p^{\sigma}, \qquad u_i = \frac{\partial \varepsilon}{\partial p_i} = a \sigma p^{\sigma - 2} p_i \tag{8}$$

where a is a constant, σ an integer [1] and $\mathbf{p}$ is the momentum of single particle. Finally, it is noteworthy that, only for $\sigma = 2$ it is possible to decompose the moments (3) into their objective and convective parts by considering, as in the usual gasdynamics, only the objective parts as constraints in the maximization procedure of entropy [20, 21, 23, 24]. On the contrary, for an arbitrary value of σ only the general moments F_A can be considered as constraints in the MEP approach.
The quantities $(5)_2$ can be written explicitly by decomposing the Lagrange multipliers into a *local equilibrium part* $\Pi_E = \alpha + \beta\,\varepsilon$ and nonequilibrium part $\widehat{\Pi}$, being $\Pi = \Pi_E + \widehat{\Pi}$ where α and β are the Lagrange multipliers of local equilibrium, whereas $\widehat{\Lambda}_A$ denote the nonequilibrium contributions of Λ_A. In this way we re-obtain the results of local equilibrium [9] for the FES being

$$\mathcal{F}|_E = \frac{y}{\mathcal{W}(e^{\alpha + \beta\,\varepsilon}) + \kappa} \quad \text{with} \quad [\mathcal{W}(\xi_E)]^{\kappa}\,[\mathcal{W}(\xi_E) + 1]^{1-\kappa} = \xi_E = e^{\alpha + \beta\,\varepsilon}. \tag{9}$$

Analogously, we can calculate the variables of local equilibrium

$$n = \int \mathcal{F}|_E \, d^D p\,, \qquad \text{and} \qquad W = \int \varepsilon\,\mathcal{F}|_E \, d^D p\,. \tag{10}$$

Consequently, if we assume the usual relation $\beta = (k_B T)^{-1}$, using (9) we obtain

$$n = \frac{T^{D/\sigma}}{\gamma} I_{2D/\sigma - 1}(\alpha), \qquad W = n k_B T \frac{I_{2D/\sigma + 1}(\alpha)}{I_{2D/\sigma - 1}(\alpha)} \tag{11}$$

where

$$\frac{1}{\gamma} = \frac{4}{\sigma} \frac{\pi^{D/2}}{\Gamma(D/2)} \left(\frac{k_B}{a}\right)^{D/\sigma} \frac{g_s}{(2\pi\hbar)^D}, \quad I_n(\alpha) = \int_0^{+\infty} \frac{x^n}{\mathcal{W}(e^{x^2 + \alpha}) + \kappa} dx \tag{12}$$

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[1] In literature [17] the case $\sigma = 2$ is called *non-relativistic*, while the case $\sigma = 1$ is called *ultrarelativistic* (in particular, for $D = 2$ we have exactly the dispersion relation of hot carriers in Graphene, being $a = v_F$ the Fermi velocity [28, 29])

being $I_n(\alpha)$ the usual *FES integral functions* that satisfy the following differentiation property with the recurrence relation

$$\frac{\partial^r I_n(\alpha)}{\partial\,\alpha^r} = (-1)^r \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}-r\right)} I_{n-2r}(\alpha) \tag{13}$$

where, the recurrence relation (13) cannot be necessarily restricted to positive values of n, and for $\kappa = 0,1$ these integral functions allow us to reobtain the standard expression for the *Bose and Fermi integral functions* [20, 21, 23, 24].

2.2. FES and non-equilibrium Entropy Principle (EP)

We consider at a kinetic level the microscopic description governed by the Boltzmann transport equation (BTE) for the distribution function $\mathcal{F}(\mathbf{r},\mathbf{p},t)$

$$\frac{\partial\,\mathcal{F}}{\partial\,t} + u_i \frac{\partial\,\mathcal{F}}{\partial\,x_i} + \mathfrak{F}_i \frac{\partial\,\mathcal{F}}{\partial\,p_i} = \mathcal{Q}(\mathcal{F}) \tag{14}$$

where the right-hand side of the above equation describes the collisional processes the gas particles are subjected to, and the $\mathfrak{F}_i = \dot{p}_i$ are the components of the resultant of the specific external forces. The collisional term $\mathcal{Q}(\mathcal{F})$ depends on the particular kind of active scattering processes. In this case, we shall consider only the case in which the relaxation towards thermodynamics equilibrium is brought about by the mutual collisions of the gas particles. Thus, if we denote with $\mathbf{p}$ and $\mathbf{p}_*$ the momentum before the collision and with $\mathbf{p}'$ and $\mathbf{p}'_*$ the new momentum after the collision then, as usual, $\mathcal{F}_*$, $\mathcal{F}'$ and $\mathcal{F}'_*$ denote the values of the function $\mathcal{F}(\mathbf{r},\mathbf{p},t)$ when $\mathbf{p}$ is replaced by $\mathbf{p}_*$, $\mathbf{p}'$ and $\mathbf{p}'_*$ respectively. Thus for ideal gases, with no inner degrees of freedom, we can write $\mathcal{Q}(\mathcal{F})$ in general form

$$\mathcal{Q}(\mathcal{F}) = \int\int\int d^D p_*\, d^D p'\, d^D p'_*\, \mathrm{W}(\mathbf{p}',\mathbf{p}'_*;\mathbf{p},\mathbf{p}_*) \times \\ \left[\mathcal{F}'\,\mathcal{F}'_*\,\mathcal{G}(\mathcal{F})\,\mathcal{G}(\mathcal{F}_*) - \mathcal{F}\,\mathcal{F}_*\,\mathcal{G}(\mathcal{F}')\,\mathcal{G}(\mathcal{F}'_*)\right] \tag{15}$$

where $\mathrm{W}(\mathbf{p}',\mathbf{p}'_*;\mathbf{p},\mathbf{p}_*)$ describes intrinsically the particle collisions in terms of the underlying interaction potential between them, with the symmetries property [23, 30]

$$\mathrm{W}(\mathbf{p}',\mathbf{p}'_*;\mathbf{p},\mathbf{p}_*) = \mathrm{W}(\mathbf{p}'_*,\mathbf{p}';\mathbf{p}_*,\mathbf{p}) = \mathrm{W}(\mathbf{p}_*,\mathbf{p};\mathbf{p}'_*,\mathbf{p}') = \mathrm{W}(\mathbf{p},\mathbf{p}_*;\mathbf{p}',\mathbf{p}'_*) \tag{16}$$

while the functions $\mathcal{G}(\mathcal{F})$, $\mathcal{G}(\mathcal{F}_*)$, $\mathcal{G}(\mathcal{F}')$, and $\mathcal{G}(\mathcal{F}'_*)$ describe the effect of FES on the accessibility of the final states reached during collisional processes. To determine these functions we assume that, if the MEP is valid for the physical system considered, then as a direct consequence, an Entropy Principle (EP) must also be verified (*H-theorem* [23]) . Therefore, when the collisions are treated statistically by means of BTE, then the total entropy of the system cannot be reduced, and, consequently, the non-equilibrium production of entropy must always be non-negative. In other words, if the distribution function, describing an ideal gas of exclusons, solves the entropy maximization of the system under the moments constraint (3), then it is natural to expect that the entropy is an increasing function along the trajectories, and that the collisions cannot reduce the

total entropy.
Thus, multiplying Eq.(14) by the kinetic quantity $\psi(\mathbf{r},\mathbf{p},t)=h(\mathcal{F})/\mathcal{F}$ and integrating in momentum space we obtain the general balance equation of entropy

$$\frac{\partial S}{\partial t}+\frac{\partial \varphi_k}{\partial x_k}=\mathcal{P}_s\,, \tag{17}$$

where φ_k and $\mathcal{P}_s$ are, respectively, the entropy flux and the entropy production due to collisions, defined as

$$\varphi_k=\int h(\mathcal{F})\,u_k\,d^Dp\,,\quad \mathcal{P}_s=\int \Phi(\mathcal{F})\,Q(\mathcal{F})\,d^Dp \quad \text{with} \quad \Phi(\mathcal{F})=h'(\mathcal{F}). \tag{18}$$

In particular, by considering for the collisional term $Q(\mathcal{F})$ the expression (15) and by using the symmetry properties (16), we can write Eq. $(18)_2$ in the form

$$\mathcal{P}_s=\frac{1}{4}\int\!\!\int\!\!\int\!\!\int d^Dp\,d^Dp_*\,d^Dp'\,d^Dp'_*\,\left[\Phi(\mathcal{F})+\Phi(\mathcal{F}_*)-\Phi(\mathcal{F}')-\Phi(\mathcal{F}'_*)\right]\times$$
$$W(\mathbf{p}',\mathbf{p}'_*;\mathbf{p},\mathbf{p}_*)\,\left[\mathcal{F}'\,\mathcal{F}'_*\,\mathcal{G}(\mathcal{F})\,\mathcal{G}(\mathcal{F}_*)-\mathcal{F}\,\mathcal{F}_*\,\mathcal{G}(\mathcal{F}')\,\mathcal{G}(\mathcal{F}'_*)\right]$$

and, by using the relations (2), (4) and $(18)_3$ we obtain

$$\mathcal{P}_s=\frac{k_B}{4}\int\!\!\int\!\!\int\!\!\int d^Dp\,d^Dp_*\,d^Dp'\,d^Dp'_*\,W(\mathbf{p}',\mathbf{p}'_*;\mathbf{p},\mathbf{p}_*)\times$$
$$\ln\left[\frac{\mathcal{F}\,\mathcal{F}_*\left(\mathcal{F}'\,e^{\Pi'}/y\right)\left(\mathcal{F}'_*\,e^{\Pi'_*}/y\right)}{\mathcal{F}'\,\mathcal{F}'_*\,(\mathcal{F}\,e^{\Pi}/y)\,(\mathcal{F}_*\,e^{\Pi_*}/y)}\right]\,\left[\mathcal{F}\,\mathcal{F}_*\,\mathcal{G}(\mathcal{F}')\,\mathcal{G}(\mathcal{F}'_*)-\mathcal{F}'\,\mathcal{F}'_*\,\mathcal{G}(\mathcal{F})\,\mathcal{G}(\mathcal{F}_*)\right]$$

therefore to obtain a non-negative production of entropy it is sufficient to define

$$\mathcal{G}(\mathcal{F})=\frac{\mathcal{F}}{y}\,e^{\Pi}=\frac{e^{\Pi}}{\mathcal{W}(e^{\Pi})+\kappa} \qquad \text{with} \qquad \Pi=\frac{1}{k_B}\sum_{A=1}^{\mathcal{N}}\Lambda_A\;\psi_A \tag{19}$$

by rewriting $\mathcal{P}_s$ in the usual form [23]

$$\mathcal{P}_s=\frac{k_B}{4}\int\!\!\int\!\!\int\!\!\int d^Dpd^Dp_*d^Dp'd^Dp'_*\,W(\mathbf{p}',\mathbf{p}'_*;\mathbf{p},\mathbf{p}_*)\left\{\ln\left(\frac{K}{H}\right)(K-H)\right\}$$

where the functions K and H are expressed by the quantities $K=\mathcal{F}\,\mathcal{F}_*\,\left(\mathcal{F}'/y\,e^{\Pi'}\right)\left(\mathcal{F}'_*/y\,e^{\Pi'_*}\right)$ and $H=\mathcal{F}'\,\mathcal{F}'_*\,\left(\mathcal{F}/y\,e^{\Pi}\right)\left(\mathcal{F}_*/y\,e^{\Pi_*}\right)$. In this way being $W(\mathbf{p}',\mathbf{p}'_*;\mathbf{p},\mathbf{p}_*)\geq 0$ and analogously $\ln(K/H)\,(K-H)\geq 0$, we obtain

$$\mathcal{P}_s\geq 0\,,$$

besides, using Eq. (19), if we assume the conservation of energy $\varepsilon+\varepsilon_*=\varepsilon'+\varepsilon'_*$, then the collisional term (15) vanishes in equilibrium conditions, since $\mathcal{F}'\,\mathcal{F}'_*\,\mathcal{G}(\mathcal{F})\,\mathcal{G}(\mathcal{F}_*)-\mathcal{F}\,\mathcal{F}_*\,\mathcal{G}(\mathcal{F}')\,\mathcal{G}(\mathcal{F}'_*)|_E=0$. Analogously, using the relation (19) with $\kappa=0,1$ we reobtain the usual terms introduced to describe the accessibility of the final states, reached during the non-equilibrium collisional processes, for Bosons and Fermions respectively [23, 31, 32], being

$$\mathcal{G}(\mathcal{F})=\frac{e^{\Pi}}{\mathcal{W}(e^{\Pi})+\kappa}=\begin{cases}1+\mathcal{F}_B/y & \kappa=0 \quad \text{(Bose)}\\[2ex] 1-\mathcal{F}_F/y & \kappa=1 \quad \text{(Fermi)}\end{cases}$$

2.3. Entropy, Entropy flux and convexity conditions

From the entropy maximization procedure described in section (2.1) we can derive the following relation

$$\ln\left[\frac{\mathcal{F}}{y}\right] - \kappa \ln\left[1-\kappa\frac{\mathcal{F}}{y}\right] - (1-\kappa)\ln\left[1+(1-\kappa)\frac{\mathcal{F}}{y}\right] = -\frac{1}{k_B}\sum_{A=1}^{\mathcal{N}} \Lambda_A\ \psi_A \tag{20}$$

if we insert this relation into eqs.(1),(2) and $(18)_1$, then using the divergence theorem and imposing that on the asymptotic surface for $p \rightarrow +\infty$ the distribution function vanishes, after some calculations, we can explicitly obtain, in the framework of FES, the following exact expressions for entropy and for the entropy flux

$$S(\mathbf{r},t) = \sum_{A=1}^{\mathcal{N}} \Lambda_A \int \psi_A\, \mathcal{F}\, d^D p + \frac{1}{D}\sum_{A=1}^{\mathcal{N}} \Lambda_A \int p_k\, \frac{\partial \Psi_A}{\partial p_k}\, \mathcal{F}\, d^D p$$

$$\varphi_k(\mathbf{r},t) = \sum_{A=1}^{\mathcal{N}} \Lambda_A \int \psi_A\, u_k\, \mathcal{F}\, d^D p + \sum_{A=1}^{\mathcal{N}} \Lambda_A \int \varepsilon\, \frac{\partial \Psi_A}{\partial p_k}\, \mathcal{F}\, d^D p\,.$$

In particular, it is easy to verify that in local equilibrium conditions (i.e. when we consider only the local equilibrium Lagrange multipliers α and β and the kinetic fields $\psi_A = \{1,\, \varepsilon\}$) and considering the general relations (8), we obtain

$$\widetilde{S}(\mathbf{r},t)|_E = \frac{S(\mathbf{r},t)|_E}{k_B} = \alpha\, n + \frac{D+\sigma}{D}\,\beta\, W, \qquad \varphi_k(\mathbf{r},t)|_E = 0. \tag{21}$$

Analogously, using the usual relation $\beta = (k_B T)^{-1}$, we can introduce the quantities

$$z = \frac{n}{T^{D/\sigma}}, \qquad F(z) = \frac{\sigma}{D}\, k_B\, z\, \frac{I_{2D/\sigma+1}}{I_{2D/\sigma-1}}, \tag{22}$$

and, in substitution of relations (11) we obtain

$$z = \frac{1}{\gamma} I_{2D/\sigma-1}(\alpha), \qquad W = \frac{D}{\sigma}\, T^{D/\sigma+1}\, F(z), \qquad \text{with} \qquad \alpha = \alpha(z), \tag{23}$$

with the differential relations of local equilibrium

$$\frac{\partial \alpha}{\partial n} = -\frac{1}{n}\,\frac{\sigma}{D-\sigma}\,\frac{I_{2D/\sigma-1}}{I_{2D/\sigma-3}}, \qquad \frac{\partial \alpha}{\partial T} = \frac{D}{D-\sigma}\,\frac{1}{T}\,\frac{I_{2D/\sigma-1}}{I_{2D/\sigma-3}}, \tag{24}$$

$$\frac{\partial W}{\partial n} = \frac{D}{D-\sigma}\, k_B T\, \frac{I_{2D/\sigma-1}}{I_{2D/\sigma-3}}, \quad \frac{\partial W}{\partial T} = \frac{D+\sigma}{\sigma}\,\frac{W}{T} - \frac{D^2/\sigma}{D-\sigma}\, k_B\, n\, \frac{I_{2D/\sigma-1}}{I_{2D/\sigma-3}} \tag{25}$$

$$\alpha'(z) = -\frac{\sigma}{D-\sigma}\,\frac{1}{z}\,\frac{I_{2D/\sigma-1}}{I_{2D/\sigma-3}}, \quad F'(z) = \frac{\sigma}{D-\sigma}\, k_B\, \frac{I_{2D/\sigma-1}}{I_{2D/\sigma-3}}, \tag{26}$$

In general, by introducing the entropy density, we assume that the function $-S$ is a strictly convex function with respect to the variables on which it depends [2] [20, 21,

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[2] Usually in Extended Thermodynamics ($\sigma = 2$ and $a = 1/2m$) entropy depends only on *objective moments*, so the entropy at equilibrium can only depend on the density n and the pressure

24, 33]. In particular, in the equilibrium state we assume that the matrix of the second derivatives of $-\widetilde{S}_E$ with respect to $F_A = \{n, W\}$ must be positive definite. Thus, using eq.$(21)_1$ and the differential relations (22)-(26), after some calculations, we obtain the matrix $-[\partial^2 \widetilde{S}_E]/[\partial F_A \, \partial F_B]$ in the form

$$\begin{bmatrix} \dfrac{(D+\sigma)/D\left[F(z)/z^2\right]F'(z)}{k_B T^{D/\sigma}\left\{(D+\sigma)/D\left[F(z)/z\right]-F'(z)\right\}} & \dfrac{-F'(z)}{(D/\sigma)\,n k_B T\left\{(D+\sigma)/D\left[F(z)/z\right]-F'(z)\right\}} \\[2ex] \dfrac{-F'(z)}{(D/\sigma)\,n k_B T\left\{(D+\sigma)/D\left[F(z)/z\right]-F'(z)\right\}} & \dfrac{1}{(D/\sigma)^2\,n k_B T^2\left\{(D+\sigma)/D\left[F(z)/z\right]-F'(z)\right\}} \end{bmatrix}$$

assuming that this matrix is positive definite, we obtain the following conditions that guarantees the hyperbolicity of the system, in the equilibrium state

$$0 < F'(z) < \frac{D+\sigma}{D}\,\frac{F(z)}{z}, \tag{27}$$

where for $\kappa = 1$, $\sigma = 1, D = 2$ we find the convexity conditions $0 < F'(z) < 3/2\,F(z)/z$ determined in the case of electrons in graphene [34], where in this case

$$z = \frac{n}{T^2}, \quad F(z) = \frac{k_B}{2}\,z\,\frac{I_5^+}{I_3^+}, \quad F'(z) = k_B\,\frac{I_3^+}{I_1^+}, \quad I_n^+ = \int_0^{+\infty} \frac{x^n}{\exp(\alpha + x^2) + 1}\,dx.$$

Analogously, for $\kappa = 0, 1$, $\sigma = 2, D = 3$ we reobtain the convexity conditions ($0 < F'(z) < 5/3\,F(z)/z$) in the case of bosons and fermions in Extended thermodynamics [20, 21, 24, 33] being

$$z = \frac{n}{T^{3/2}}, \quad F(z) = \frac{2}{3} k_B z \frac{I_4^\pm}{I_2^\pm}, \quad F'(z) = 2k_B \frac{I_2^\pm}{I_0^\pm}, \quad I_n^\pm = \int_0^{+\infty} \frac{x^n}{\exp(\alpha + x^2) \pm 1} dx$$

where the upper and lower signs correspond to cases of Fermi and Bose, respectively. As is well know the convexity conditions imply that the differential system of hydrodynamic equations (HD) is symmetric hyperbolic and the local Cauchy problem is well posed [35, 36]. The conditions (27) guarantee the hyperbolicity only in the equilibrium state and therefore it is of great importance to check the region, in the neighborhood of equilibrium, in which the hyperbolicity remains valid. In nonequilibrium conditions, the hyperbolicity domain is determined through a computation of the roots of the characteristic polynomial associated to the jacobian matrix of fluxes for the HD system considered [20, 21, 23, 24]. In the next section we will briefly describe the strategy that can be adopted to obtain closed hydrodynamic models, containing the statistical information derived from FES, in the context of the usual Extended Thermodynamics.

P (or equivalently on the internal energy). But, in general for $\sigma \neq 2$ the decomposition into *objective* and *convective* parts cannot be done, and the average energy W (which can no longer be decomposed into an objective part and a convective part) will assume a role similar to that which the internal energy assumes in gasdynamics. Furthermore, even in the case $\sigma = 2$ there are some physical systems in which a "total energy scheme" must always be adopted [23, 25], due to collisional processes (for example, in the transport of hot carriers in semiconductors, when the electron-phonon collision must be described using specifically the band structure for electron and phonon defined in the crystal.)

2.4. Construction of closed hydrodynamic models using the MEP strategy for the FES.

To pass from the kinetic level of the BTE to the Hydrodynamic (HD) level of the balance equations in the general framework of the moment theory, we use the set of generalized kinetic fields $\psi_A(\mathbf{p})$ obtaining, as corresponding average expectation values, the quantities (3). Multiplying the BTE by the quantities $\psi_A(\mathbf{p})$ and integrating over the $\mathbf{p}$ space we obtain the following set of generalized balance equations [20, 21, 23] for the moments F_A

$$\frac{\partial F_A}{\partial t}+\frac{\partial F_{Ak}}{\partial x_k}=\mathfrak{F}_k R_{Ak}+P_A\ , \qquad A=1,\ldots\mathcal{N} \tag{28}$$

being $\mathcal{N}$ the number of moments used, and F_{Ak}, R_{Ak}, P_A, respectively, indicate the fluxes, the external field productions, and the collisional productions

$$F_{Ak}=\int \psi_A\, u_k\, \mathcal{F}\, d^D p, \quad R_{Ak}=\int \frac{\partial \psi_A}{\partial p_k}\,\mathcal{F}\, d^D p, \quad P_A=\int \psi_A\, Q(\mathcal{F})\, d^D p. \tag{29}$$

where, in particular, using the MEP strategy and considering only the mutual collisions of the gas particles, in the case of an ideal gases (with no inner degrees of freedom) with the symmetry properties (16), we obtain

$$\begin{aligned}P_A=\frac{1}{4}\int\int\int\int d^D p\, d^D p_*\, d^D p'\, d^D p'_*\, \mathrm{W}(\mathbf{p}',\mathbf{p}'_*;\mathbf{p},\mathbf{p}_*)\,\left(\psi_A+\psi_A^*-\psi'_A-\psi'^*_A\right)\\ \times\left[\mathcal{F}'\mathcal{F}'_*\left(\frac{\mathcal{F}}{y}e^{\Pi}\right)\left(\frac{\mathcal{F}_*}{y}e^{\Pi_*}\right)-\mathcal{F}\mathcal{F}_*\left(\frac{\mathcal{F}'}{y}e^{\Pi'}\right)\left(\frac{\mathcal{F}'_*}{y}e^{\Pi'_*}\right)\right]\end{aligned}$$

The structure of this system of equations shows that there are some unknown constitutive functions $H_A=\{\widehat{F}_{Ak},R_{Ak},P_A\}$ represented by some unknown fluxes $\{\widehat{F}_{Ak}\}$ (which are not present in the list of the moments F_A), the external field productions $\{R_{Ak}\}$, and the collisional productions $\{P_A\}$. The above system should be closed in a self-consistent way with the determination of the H_A expressed by means of the F_A. In this way we obtain a closed system of balance equations for the expectation values F_A, and each solution of this set is called a *thermodynamic process* [20, 21, 23] for the ideal exclusons gases, in the context of Extended Thermodynamics.

Once again, it is possible to determine the closure searched for the constitutive functions self-consistently by using the MEP strategy. Therefore, expanding the distribution function and the constitutive functions around the local equilibrium distribution function $\mathcal{F}|_E$,these can be expressed as polynomials in the nonequilibrium variables whose coefficients will depend on the local equilibrium quantities $\{n, W\}$. In particular, by considering only a linear expansion of the distribution function $(5)_1$ around the local equilibrium distribution function $\mathcal{F}|_E$, we obtain

$$\mathcal{F}=\mathcal{F}|_E+\mathcal{L}\,\frac{1}{k_B}\sum_{B=1}^{\mathcal{N}}\widehat{\Lambda}_B\,\psi_B \qquad \text{with} \qquad \mathcal{L}=\left.-y\,\frac{\mathcal{W}(\mathcal{W}+1)}{[\mathcal{W}+\kappa]^3}\right|_E$$

inserting this expansion in the moment expressions (3)

$$F_A-F_A|_E=\frac{1}{k_B}\sum_{B=1}^{\mathcal{N}}\widehat{\Lambda}_B\int\psi_A\,\psi_B\,\mathcal{L}\,d^D p$$

we can determine analytically the nonequilibrium part of the Lagrange multipliers $\widehat{\Lambda}_B$ as explicit functions of the moments [20, 21, 23]. Consequently, both the distribution function and the constitutive functions can be written in the explicit form, by means of their kinetic expressions (29).

We remark that to obtain the explicit forms of all constitutive functions it is necessary to know both the specific expression of microscopic energy $\varepsilon(\mathbf{p})$ and the explicit form of the term $Q(f)$ which describes the specific collisional processes [23].
Thus, for example, the specific choice of the independent kinetic fields ψ_A (and therefore of the corresponding moments F_A), to be taken into account in the maximization procedure, will depend on the specific form of the dispersion relation used to describe the microscopic energy. Indeed, it is possible to prove that, using the general form (8) with $\sigma = 1,2$ only the following complete set of kinetic fields (and corresponding independent moments) must be considered

$$\psi_A = \{\varepsilon^s u_{\langle i_1} u_{i_2} \cdots u_{i_r\rangle}\} \quad \text{with} \quad F_{(s)|\langle i_1 \cdots i_r\rangle} = \int \varepsilon^s u_{\langle i_1} \cdots u_{i_r\rangle} \mathcal{F} d^D p$$

where $s = 0, \ldots N$, $r = 0, \ldots M$ with arbitrary values for the integers N and M, and $u_{\langle i_1} u_{i_2} \cdots u_{i_s\rangle}$ is the traceless part [20, 21, 23] of tensor $u_{i_1} u_{i_2} \cdots u_{i_s}$.
On the contrary, for $\sigma \neq 1,2$ the complete set of kinetic fields (and corresponding independent macroscopic variables) will assume the most general form

$$\psi_A = \{\varepsilon^s u^{2l} u_{\langle i_1} u_{i_2} \cdots u_{i_r\rangle}\} \quad \text{with} \quad F_{(s,l)|\langle i_1 \cdots i_r\rangle} = \int \varepsilon^s u^{2l} u_{\langle i_1} \cdots u_{i_r\rangle} \mathcal{F} d^D p$$

where $s = 0, \ldots N$, $l = 0, \ldots L$ and $r = 0, \ldots M$ with arbitrary values for the integers N, L and M.
In this paper, therefore, a general formulation of a closed set of HD equations cannot be determined without specifying the physical system considered. However, with the strategy indicated in the previous sections, the determination of specific closed HD systems, for an ideal gas obeying the FES, will be the subject of future works in the context of usual Extended Thermodynamics.

3. Conclusions

Fractional exclusion statistics describe interaction phenomena that, as an end result, inhibit or favor the occupation of possible elementary excited states of individual particles due to the presence of the remaining N particles in the system (Generalized Exclusion Principle). The fundamental purpose of this paper is to formulate a non-equilibrium theory for FES, within the framework of kinetic theory, connected with the theory of moments in Extended Thermodynamics. The main informations, of a specific physical system, are actually contained in the correct form of band structure and in the correct description of collisional processes between the particles of the system. Consequently:

1. Imposing that the entropy is an increasing function along the motion trajectories satisfying the BTE (H-Theorem), we explicitly determine the *correct functional form* of the terms describing the final states of occupancy (in non-equilibrium conditions) contained in the collisional Boltzmann operator.

2. Entropy and its flux are computed explicitly in terms of non-equilibrium Lagrange multipliers.

3. Given the physical relevance and the possible applications connected with an ideal gas subject to FES [3, 5–18], using the MEP we describe the *correct strategy* for determining closed Hydrodynamic systems within the framework of Extended Thermodynamics in non-equilibrium conditions.

4. Finally, using a general form for the band structure, both some thermodynamic properties connected with FES and the *correct convexity conditions* of the entropy are explicitly obtained in equilibrium conditions.

It is clear that, at present, there are many open problems, in describing fractional exclusion statistics, in the context of Extended Thermodynamics. For example, a relevant problem within the usual Extended Thermodynamics of non-degenerate gases is the convergence of the integrals defining the moments of the distribution function [37] (a problem that, in gas-dynamics, turns out to be connected with the order on the truncation (even or odd) of kinetic fields which are a polynomial in the group velocity $\mathbf{u}$ [37]). However, it is observed that when considering transport phenomena in solid state physics the integrals of moments (associated with the particles or quasi-particles) are certainly convergent. Indeed, when transport occurs in a crystal lattice, the domain of integration (in the $\mathbf{k}$ or in the $\mathbf{p}$ space) of moments of the distribution function is necessarily a domain closed, limited and compact (corresponding to the well-known first Brillouin zone [38]). Thus, from a purely mathematical point of view (even considering the exact MEP distribution function, and not an approximate expansion of it) there seem to be no problems with convergence for these integrals regardless of the tensorial order of the moments and regardless of the type of statistics to which the particles are subjected.

In contrast, the problem of convergence, for the integrals of moments, could perhaps be present in the case where fractional statistics were applied in the usual gas-dynamics ($\sigma = 2, D = 3$ and $a = 1/2m$) with a non-closed and non-limited domain of integration. This analysis could be the subject of future work and, most likely, will have to be done numerically. Indeed, it is noted that the explicit form of the distribution function contained in the integrals of moments, in the case of fractional statistics, is connected with solving the nonlinear functional equation (6) for the function $\mathcal{W}(\xi)$ (which is usually solved numerically as the parameter κ varies). The problem is therefore much more complicated than that described for non-degenerate gases. It is not necessarily the case, that this problem can be addressed analytically; rather, it will almost certainly have to be addressed numerically. However, this analysis is beyond the scope of present paper.

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