LE MATEMATICHE
Vol. LXXIX (2024) – Issue II, pp. 621–639
doi: 10.4418/2024.79.2.19

$W_0^{1,1}(\Omega)-$SOLUTIONS FOR A DEGENERATE DOUBLE PHASE TYPE OPERATOR IN SOME BORDERLINE CASES

F. ACHHOUD - S. D'ASERO

We study the existence of $W_0^{1,1}(\Omega)-$solutions of nonlinear anisotropic problems whose simplest model is

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) - \operatorname{div}\left(|u|^{(r-1)q+1}|\nabla u|^{q-2}\nabla u\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N (N > 2)$, $1 < q \leq p < N$, $r > \frac{q-1}{q}$ and f is a function with poor summability.

1. Introduction and statement of the main results

We consider the following boundary value problem:

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) - \operatorname{div}\left(g(u)|\nabla u|^{q-2}\nabla u\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open, bounded subset of $\mathbb{R}^N (N > 2)$, and p, q are parameters satisfying the condition:

$$1 < q \leq p < 2 - \frac{1}{N}. \tag{2}$$

Received on September 30, 2024

AMS 2010 Subject Classification: 35B65; 35J60; 35J66

Keywords: nonlinear elliptic equations; $W_0^{1,1}(\Omega)-$solutions; double phase problems; $L^m(\Omega)-$data

$a : \Omega \to \mathbb{R}$ is a measurable function such that

$$0 < \alpha \le a(x) \le \beta, \quad \text{a.e. } x \in \Omega, \tag{3}$$

$g : \mathbb{R} \to \mathbb{R}$ is a function defined as

$$g(t) = |t|^{(r-1)q+1}, \quad \text{with } (r-1)q+1 > 0, \tag{4}$$

and

$$f \in L^m(\Omega), \quad \text{with } m > 1. \tag{5}$$

Problem (1) falls within the so called "double phase" problem, a class of problems which exhibit an unbalanced growth, the (p,λ)-*growth* according to Marcellini's definition given in [14].

A simple example of nonlinear double phase elliptic equation is

$$-\text{div}\left(|\nabla u|^{p-2}\nabla u\right) - \text{div}\left(g(x)|\nabla u|^{\lambda-2}\nabla u\right) = f. \tag{6}$$

The left-hand side of this equation is the derivative of the double phase integral functional

$$J(v) = \int_\Omega \left(\frac{|\nabla v|^p}{p} + \frac{g(x)}{\lambda}|\nabla v|^\lambda\right) dx, \quad \text{with } 1 < p < \lambda, \tag{7}$$

which is basically characterized by the fact of having the energy density switching between two different types of degenerate behaviours, according to the size of the set where $g(x) = 0$.

Nowadays, the double phase elliptic problems are widely studied. Some fundamental papers are [2, 3, 10, 11].

We point out that, respect to the double phase equation (6), in (1) numbers p, q play the role of λ, p, and function $g(u)$ depends on a power of u.

Problem (1) has been considered in the papers [1] and [4], where the authors studied the improved regularity of ∇u both depending on the presence of the additional term

$$-\text{div}\left(g(u)|\nabla u|^{q-2}\nabla u\right)$$

and on the summability of the datum f. In particular, for a datum f having a summability exponent m below the duality one, that is

$$\max\left\{1, \frac{N}{N(p-1)+1}\right\} < m < \left(p^*\right)',$$

the existence of a distributional solution u can be summarized as follows

$$\begin{cases} \dfrac{1}{q'} < r \le \bar{r}(m) \Rightarrow u \in W_0^{1,(p-1)m^*}(\Omega) \\ \bar{r}(m) < r < \tilde{r}(m) \Rightarrow u \in W_0^{1,\zeta}(\Omega) \cap L^{(mq)^* r}(\Omega) \end{cases}$$

where

$$\bar{r}(m) = \frac{(mp)^*}{p'(mq)^*}, \qquad \tilde{r}(m) = \frac{N - mq}{Nq(m-1)} \quad \text{and} \quad \zeta = m\frac{(mq)^* rp}{m + (mq)^* r}.$$

Note that,

$$(p-1)m^* = 1 \iff m = \bar{m} = \frac{N}{N(p-1)+1}$$

and

$$\bar{m} > 1 \iff 1 < p < 2 - \frac{1}{N}.$$

Moreover,

$$\zeta = m\frac{(mq)^* rp}{m + (mq)^* r} = 1 \quad \Leftrightarrow \quad m = m_1 = \frac{N(1+qr)}{q(Nrp+1)}$$

and

$$m_1 > 1 \iff r < \frac{N-q}{Nq(p-1)}$$

Thus, in the previous two borderline cases one might expect to find solution u of the problem (1) with gradient merely summable, as firstly done in [6].

Namely, in this paper, under an appropriate balance of the parameters p, q, r and the summability of f, we prove the existence of $W_0^{1,1}(\Omega)$–distributional solutions of the problem (1) in the limiting cases drawn above.

Here, by a *distributional solution* of the Problem (1) we mean a function $u \in W_0^{1,1}(\Omega)$ such that

$$\begin{cases} g(u)|\nabla u|^{q-1} \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla \varphi dx + \int_\Omega g(u)|\nabla u|^{q-2}\nabla u \nabla \varphi dx = \int_\Omega f\varphi dx, \end{cases} \tag{8}$$

for every $\varphi \in C_0^\infty(\Omega)$.

Theorem 1.1. Assume that hypotheses (2), (3), (4) hold true. Let $f \in L^m(\Omega)$ with $m = \frac{N}{N(p-1)+1}$ and

$$\frac{1}{q'} < r \le \bar{r}, \tag{9}$$

where

$$\bar{r} = \frac{N(p-1)+1-q}{q(N-1)}.$$

Then, there exists $u \in W_0^{1,1}(\Omega)$ which solves Problem (1) in the sense of the definition (8).

Remark 1.1. The above Theorem gives the same result as in [7], where $g(u) = 0$.

Remark 1.2. Note that

$$\lim_{m \to \bar{m}} \bar{r}(m) = \bar{r},$$

hence the result of Theorem 1.1 links up continuously with the result of the Theorem 1.3 in [1].

The next result investigates the regularizing effect due to the term

$$-\mathrm{div}(g(u)|\nabla u|^{q-2}\nabla u).$$

Specifically, assuming that the exponent r in $g(u)$ is sufficiently large, we establish the existence of a distributional solution under a weaker hypotheses on the datum f.

Theorem 1.2. Assume that (2), (3), (4) are fulfilled. Suppose that

$$\bar{r} < r < \tilde{r}, \tag{10}$$

where

$$\tilde{r} = \frac{N-q}{Nq(p-1)} \tag{11}$$

and

$$f \in L^{m_1}(\Omega), \qquad \text{with } m_1 = \frac{N(1+qr)}{q(1+Nrp)}. \tag{12}$$

Then, there exists $u \in W_0^{1,1}(\Omega)$ which solves Problem (1) in the sense of the definition (8). Moreover, $u \in L^{\mu}(\Omega)$ with $\mu = \left(\frac{q}{p}\right)^* \left(r + \frac{1}{q}\right)$.

Remark 1.3. Notice that the condition (10) implies that $1 < m_1 < \frac{N}{N(p-1)+1}$. Thus, the summability assumption on f in Theorem 1.2 is weaker than the one required in Theorem 1.1. Moreover, we point out that $\mu > \frac{N}{N-1}$, i.e. the regularity obtained in our result is better than the one obtained in [5, 7] by means of the embedding Sobolev Theorem.

Remark 1.4. Note that

$$\lim_{m\to m_1}(mq)^* r = \mu,$$

hence the result of Theorem 1.2 links up continuously with the Theorem 1.4 in [1].

The regularizing effect of other lower order terms on the summability of u and ∇u has been studied by various author, see for example [5], [6], [7], [9], [12],

The structure of the paper unfolds as follows: in Section 2 we introduce a suitable sequence of approximating problems and we will establish a priori estimates on the corresponding sequence of solutions. Section 3 is devoted to the proof of Theorem 1.1 and Theorem 1.2.

2. Approximating problems and a priori estimates

To prove our existence results, we begin by approximating the boundary value Problem (1).

Let $f\in L^m(\Omega)$, $m\geq 1$. For $n\in\mathbb{N}$, let us consider the sequence of approximating problems

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u_n|^{p-2}\nabla u_n\right)-\operatorname{div}\left(g(u_n)|\nabla u_n|^{q-2}\nabla u_n\right)=f_n & \text{in } \Omega,\\ u_n=0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where $\{f_n\}$ is a sequence of bounded functions such that

$$f_n\to f \quad \text{strongly in } L^m(\Omega), \tag{14}$$

and

$$\|f_n\|_{L^m(\Omega)}\leq \|f\|_{L^m(\Omega)}, \qquad \forall n\in\mathbb{N}.$$

Thanks to the results of [4], Problem (13) admits a weak solution u_n in the sense that

$$\begin{cases} u_n\in W_0^{1,p}(\Omega) \qquad g(u_n)|\nabla u_n|^{q-1}\in L^{p'}(\Omega),\\ \displaystyle\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi dx+\int_\Omega g(u_n)|\nabla u_n|^{q-2}\nabla u_n\nabla\varphi dx=\int_\Omega f_n\varphi dx, \end{cases} \tag{15}$$

for any $\varphi\in W_0^{1,p}(\Omega)$. Moreover, u_n is also bounded, since $f_n\in L^\infty(\Omega)$.

We are going to prove that, under the assumptions of the Theorem 1.1 and Theorem 1.2, the sequence $\{u_n\}$ converges to a distributional solution $u\in W_0^{1,1}(\Omega)$ of the Problem (1). As already done in [5, 7], the main tool will be to derive the boundedness of the sequence $\{u_n\}$ in the non reflexive space $W_0^{1,1}(\Omega)$.

Let $k>0$ and $T_k:\mathbb{R}\to\mathbb{R}$ the usual truncation function defined by

$$T_k(s)=\max\{-k,\min\{s,k\}\},\quad \forall s\in\mathbb{R}.$$

By using $T_k(u_n)$ as test function in the integral identity in (15) and dropping the positive term in the left-hand side, we obtain the following estimate

$$\int_\Omega |\nabla T_k(u_n)|^p\,dx\le \frac{k}{\alpha}\|f\|_{L^1(\Omega)},\qquad \forall n\in\mathbb{N}, \tag{16}$$

The following lemma will be used in the proof of the Theorem 1.1.

Lemma 2.1. *Let the assumptions* (2) – (4) *hold. Assume that* $f\in L^m(\Omega)$ *with* $m=\frac{N}{N(p-1)+1}$ *and*

$$\frac{1}{q'}<r\le\bar{r}.$$

Then, there exists a positive constant $\mathcal{M}_1$*, independent of n, such that*

$$\|u_n\|_{W_0^{1,1}(\Omega)}\le\mathcal{M}_1,\qquad \forall n\in\mathbb{N}. \tag{17}$$

Moreover, for every measurable subset $E\subset\Omega$*, it holds*

$$\lim_{|E|\to 0}\int_E|\nabla u_n|\,dx=0,\qquad \forall n\in\mathbb{N}. \tag{18}$$

Here and in the sequel, for any measurable set $E\subset\mathbb{R}^N$, $|E|$ denotes its N - dimensional measure.

Proof. Let $\delta>0$. For any $t\in\mathbb{R}$, define the function

$$v_\delta(t)=\left[(\delta+|t|)^{\tilde{s}+1}-\delta^{\tilde{s}+1}\right]sign(t),$$

where

$$\tilde{s}=\frac{N(1-p)}{N-1}. \tag{19}$$

In the sequel, we denote by $\mathcal{C}_i$, $i=1,2,3...$, any positive constant depending only on the known data and independent of n.

Substituting $v_\delta(u_n)$ as the test function into the integral identity (15), we obtain

$$\begin{aligned}(\tilde{s}+1)\int_\Omega a(x)|\nabla u_n|^p(\delta+|u_n|)^{\tilde{s}}dx+(\tilde{s}+1)\int_\Omega g(u_n)|\nabla u_n|^q(\delta+|u_n|)^{\tilde{s}}dx\\ =\int_\Omega f_n v_\delta(u_n)dx.\end{aligned}\tag{20}$$

Thanks to (3) and dropping the second (positive) term in the left-hand side, we get

$$\alpha(\tilde{s}+1)\int_\Omega |\nabla u_n|^p(\delta+|u_n|)^{\tilde{s}}dx \le \|f_n\|_{L^m(\Omega)}\left(\int_\Omega |v_\delta(u_n)|^{m'}\right)^{\frac{1}{m'}}, \tag{21}$$

Moreover, dropping the first term in the left-hand side of (20), we have

$$k^{q(r-1)+1}\int_{D_n(k)} |\nabla u_n|^q(\delta+|u_n|)^{\tilde{s}}dx \le \|f\|_{L^m(\Omega)}\left(\int_\Omega |v_\delta(u_n)|^{m'}\right)^{\frac{1}{m'}}, \tag{22}$$

where

$$D_n(k)=\{x\in\Omega : |u_n(x)|>k\}.$$

Thus, by (21) and Sobolev's inequality, we deduce

$$\begin{aligned}C(\alpha,\tilde{s})\left(\int_\Omega \left|(\delta+|u_n|)^{\frac{N}{(N-1)p^*}}-\delta^{\frac{N}{(N-1)p^*}}\right|^{p^*}\right)^{\frac{p}{p^*}}\\ \le \|f\|_{L^m(\Omega)}\left(\int_\Omega (\delta+|u_n|)^{m'(\tilde{s}+1)}\right)^{\frac{1}{m'}}\end{aligned}$$

with

$$C(\alpha,\tilde{s})=\frac{\alpha(\tilde{s}+1)\left(\frac{p^*(N-1)}{N}\right)^p}{\mathcal{S}^p}.$$

Thanks to the Lebesgue Theorem and the Fatou Lemma, we can pass to the limit as δ tends to 0 and we deduce

$$C(\alpha,\tilde{s})\left(\int_\Omega |u_n|^{\frac{N}{N-1}}\right)^{\frac{p}{p^*}} \le \|f\|_{L^m(\Omega)}\left(\int_\Omega |u_n|^{m'(\tilde{s}+1)}\right)^{\frac{1}{m'}}.$$

Since $1^*=\dfrac{N}{N-1}=m'(\tilde{s}+1)$ and $\frac{p}{p^*}>\frac{1}{m'}$, we deduce

$$C(\alpha,\tilde{s})\left(\int_\Omega |u_n|^{\frac{N}{N-1}}\right)^{\frac{p}{p^*}-\frac{1}{m'}} \le \|f\|_{L^m(\Omega)}. \tag{23}$$

In addition, from (21) and (22) we obtain

$$\int_{D_n(k)} |u_n|^{\tilde{s}} |\nabla u_n|^p dx \le \mathcal{C}_1, \tag{24}$$

$$\int_{D_n(k)} |u_n|^{\tilde{s}} |\nabla u_n|^q dx \le \frac{\mathcal{C}_2}{k^{q(r-1)+1}} \tag{25}$$

and the following estimate

$$\big|D_n(k)\big| \le \frac{C(\alpha,\tilde{s})}{k^{\frac{N}{N-1}}}, \quad \forall k > 0. \tag{26}$$

Therefore, if we fix $\varepsilon > 0$, then there exists $k_\varepsilon > 0$ such that for any $k \ge k_\varepsilon$

$$\big|D_n(k)\big| \le \varepsilon, \quad \text{uniformly with respect to } n. \tag{27}$$

Now, we prove that $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$. Exploiting (21) with $\delta = 1$, we get

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\frac{N(p-1)}{N-1}}} dx \le \mathcal{C}_3. \tag{28}$$

By the Hölder's inequality, we have

$$\begin{aligned}\int_\Omega |\nabla u_n|\, dx &= \int_\Omega \frac{|\nabla u_n|}{(1+|u_n|)^{\frac{N}{p'(N-1)}}} (1+|u_n|)^{\frac{N}{p'(N-1)}}\, dx \\ &\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\frac{N(p-1)}{N-1}}}\, dx\right]^{\frac{1}{p}} \left[\int_\Omega (1+|u_n|)^{\frac{N}{N-1}}\, dx\right]^{\frac{1}{p'}}.\end{aligned}$$

Thus, by (23) and (28), we conclude that (17) holds true.

Let $k > 0$, and for any $t \in \mathbb{R}$, we define the function

$$\psi_k(t) = \left[|t|^{\tilde{s}+1} - k^{\tilde{s}+1}\right]^+ \operatorname{sign}(t), \quad (^1)$$

where $\tilde{s}$ is the constant defined in (19).

By using $\psi_k(u_n)$ as a test function in the integral identity (15), we obtain

$$\begin{aligned}\int_\Omega a(x)\,|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_k(u_n)dx + \int_\Omega g(u_n)\,|\nabla u_n|^{q-2}\nabla u_n\nabla\psi_k(u_n)dx \\ = \int_\Omega f_n\psi_k(u_n)dx.\end{aligned}$$

[1]Here, as usual $v^+ = \max\{v,0\}$, $\forall v \in \mathbb{R}$.

From Hölder's inequality and the fact that the second term on the left-hand side is positive, it follows that

$$\mathcal{C}_5 \int_{D_n(k)} \frac{|\nabla u_n|^p}{|u_n|^{\frac{N(p-1)}{N-1}}} dx \leq \left(\int_{D_n(k)} |f_n|^m dx \right)^{\frac{1}{m}} \left(\int_{D_n(k)} |u_n|^{\frac{N}{N-1}} dx \right)^{\frac{1}{m'}}. \tag{29}$$

Applying again the Hölder's inequality and taking into account inequalities (23) and (29), we get

$$\begin{aligned} \int_{D_n(k)} |\nabla u_n| \, dx &= \int_{D_n(k)} \frac{|\nabla u_n|}{|u_n|^{\frac{N}{p'(N-1)}}} |u_n|^{\frac{N}{p'(N-1)}} \, dx \\ &\leq \left[\int_{D_n(k)} \frac{|\nabla u_n|^p}{|u_n|^{\frac{N(p-1)}{N-1}}} dx \right]^{\frac{1}{p}} \left[\int_{D_n(k)} |u_n|^{\frac{N}{N-1}} \, dx \right]^{\frac{1}{p'}} \\ &\leq \mathcal{C}_6 \left(\int_{D_n(k)} |f_n|^m dx \right)^{\frac{1}{pm}}. \end{aligned}$$

Thanks to the previous inequality, for every measurable subset $E \subset \Omega$ we have

$$\begin{aligned} \int_E |\nabla u_n| \, dx &\leq \int_E |\nabla T_k(u_n)| \, dx + \int_{D_n(k)} |\nabla u_n| \, dx \\ &\leq \left[\int_\Omega |\nabla T_k(u_n)|^p \, dx \right]^{\frac{1}{p}} |E|^{1-\frac{1}{p}} + \mathcal{C}_6 \left(\int_{D_n(k)} |f_n|^m \, dx \right)^{\frac{1}{pm}} \\ &\leq \left[\frac{k}{\alpha} \|f\|_{L^1(\Omega)} \right]^{\frac{1}{p}} |E|^{1-\frac{1}{p}} + \mathcal{C}_6 \left(\int_{D_n(k)} |f|^m \, dx \right)^{\frac{1}{pm}}, \end{aligned} \tag{30}$$

which, by (27) and the absolute continuity of the integral, gives (18). □

The following lemma will be used in the proof of the Theorem 1.2.

Lemma 2.2. *Let the assumptions* (2) – (4) *hold. Assume that*

$$\bar{r} < r < \tilde{r}. \tag{31}$$

and $f \in L^{m_1}(\Omega)$ *with* $m_1 = \frac{N(1+qr)}{q(1+Nrp)}$.

Then, there exist two positive constants $\mathcal{M}_2$ *and* $\mathcal{M}_3$, *independent of n, such that*

$$\|u_n\|_{L^\mu(\Omega)} \leq \mathcal{M}_2, \qquad \forall n \in \mathbb{N} \tag{32}$$

and

$$\|u_n\|_{W_0^{1,1}(\Omega)} \leq \mathcal{M}_3, \qquad \forall n \in \mathbb{N}. \tag{33}$$

Moreover, for every measurable subset $E \subset \Omega$, it holds

$$\lim_{|E|\to 0} \int_E |\nabla u_n|\, dx = 0, \qquad \forall n \in \mathbb{N}. \tag{34}$$

Proof. Let us define for any $t \in \mathbb{R}$, the function

$$w(t) = \left[|t|^{s+1} - 1\right]^+ \operatorname{sign}(t),$$

with

$$s = \frac{N(1-p)(1+qr)}{Np-q}. \tag{35}$$

Notice that

$$r < \tilde{r} \Rightarrow s+1 > 0.$$

We choose $w(u_n)$ as test function in (15). Removing the first positive term in the left-hand side we get

$$\int_{D_n(1)} |\nabla u_n|^q |u_n|^{q(r-1)+s+1} dx \le \frac{\|f\|_{L^{m_1}(\Omega)}}{(s+1)} \left(\int_{D_n(1)} |u_n|^{(s+1)m_1'} dx\right)^{\frac{1}{m_1'}}, \tag{36}$$

which implies

$$\int_\Omega \left|\nabla\left[|u_n|^{\frac{q(r-1)+1+s}{q}+1} - 1\right]^+\right|^q dx \le \frac{\left(\frac{s+1}{q}+r\right)^q \|f\|_{L^{m_1}(\Omega)}}{s+1} \left(\int_{D_n(1)} |u_n|^{(s+1)m_1'} dx\right)^{\frac{1}{m_1'}}.$$

By employing the Sobolev's inequality, from (36) we obtain

$$\left(\int_\Omega \left|\left[|u_n|^{r+\frac{s+1}{q}} - 1\right]^+\right|^{q^*} dx\right)^{\frac{q}{q^*}} \le \mathcal{C}_7 \left(\int_{D_n(1)} |u_n|^{(s+1)m_1'} dx\right)^{\frac{1}{m_1'}}.$$

We recall that by the definition of s, it follows

$$rq^* + \frac{(s+1)q^*}{q} = (s+1)m_1' = \mu$$

and by easy calculations, from the last inequality, we get

$$\left(\int_{D_n(1)} |u_n|^\mu\, dx\right)^{\frac{q}{q^*}} \le \mathcal{C}_8 \left(\int_{D_n(1)} |u_n|^{(s+1)m_1'} dx\right)^{\frac{1}{m_1'}} + \mathcal{C}_9.$$

Since $\dfrac{q}{q^*} > \dfrac{1}{m_1'}$, we deduce that

$$\int_{D_n(1)} |u_n|^\mu\, dx \le \mathcal{C}_{10}.$$

On the other hand, we have

$$\begin{aligned}\int_\Omega |u_n|^\mu dx &= \int_{D_n(1)} |u_n|^\mu dx + \int_{D_n^c(1)} |u_n|^\mu dx\\ &\le \int_{D_n(1)} |u_n|^\mu dx + |\Omega|,\end{aligned}$$

hence

$$\int_\Omega |u_n|^\mu\, dx \le \mathcal{C}_{11},$$

which implies that (32) holds.

Here and in the sequel, for any set $A \subset \mathbb{R}^N$, A^c denotes the complement of A, i.e. $A^c = \Omega \setminus A$.

From (36), we derive

$$\int_{D_n(1)} |\nabla u_n|^q |u_n|^{q(r-1)+s+1} dx \le \mathcal{C}_{12}. \tag{37}$$

Thanks to the Hölder's inequality with exponents q and $q' = \frac{q}{q-1}$, we can write

$$\begin{aligned}\int_\Omega |\nabla u_n| dx &= \int_{D_n^c(1)} |\nabla u_n| dx + \int_{D_n(1)} |\nabla u_n| dx\\ &\le \int_\Omega |\nabla T_1(u_n)| dx + \int_{D_n(1)} |u_n|^{r-1+\frac{s+1}{q}} |\nabla u_n| dx\\ &\le \left(\frac{\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} + \left(\int_{D_n(1)} |\nabla u_n|^q |u_n|^{q(r-1)+s+1} dx\right)^{\frac{1}{q}} |D_n(1)|^{\frac{1}{q'}}\\ &\le \left(\frac{\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} + \mathcal{C}_{12}^{\frac{1}{q}} |D_n(1)|^{\frac{1}{q'}}.\end{aligned}$$

We note that, thanks to (26) with $k = 1$, the following estimate holds

$$|D_n(1)| \le \mathcal{C}_{13}.$$

Then

$$\begin{aligned}\int_\Omega |\nabla u_n| dx &\le \left(\frac{\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} + \mathcal{C}_{12}^{\frac{1}{q}}\\ &\le \mathcal{C}_{14},\end{aligned}$$

hence (33) is proved.

Now, we prove that the sequence $\{\nabla u_n\}$ is equiintegrable. To this aim, let $E \subset \Omega$ be measurable subset.

Thanks to Hölder's inequality, (16) and the estimate (37), we get

$$\begin{aligned}\int_E |\nabla u_n|\,dx &\le \left[\int_\Omega |\nabla T_1(u_n)|^p\,dx\right]^{\frac{1}{p}} |E|^{1-\frac{1}{p}} + \int_{D_n(1)\cap E} |u_n|^{r-1+\frac{s+1}{q}} |\nabla u_n| dx \\ &\le \left[\frac{\|f_n\|_{L^1(\Omega)}}{\alpha}\right]^{\frac{1}{p}} |E|^{1-\frac{1}{p}} + \left(\int_{D_n(1)\cap E} |\nabla u_n|^q |u_n|^{q(r-1)+s+1} dx\right)^{\frac{1}{q}} |E|^{\frac{1}{q'}} \\ &\le \left[\frac{\|f_n\|_{L^1(\Omega)}}{\alpha}\right]^{\frac{1}{p}} |E|^{1-\frac{1}{p}} + \mathcal{C}_{12}^{\frac{1}{q}} |E|^{\frac{1}{q'}}.\end{aligned} \tag{38}$$

Therefore (34) holds. □

Remark 2.1. By following the same proof as before, the previous result gives

$$\begin{aligned}\int_{D_n(1)} |u_n|^s |\nabla u_n|^p dx &\le \frac{\|f\|_{L^{m_1}(\Omega)}}{\alpha(s+1)} \left(\int_{D_n(1)} |u_n|^\mu dx\right)^{\frac{1}{m_1'}} \\ &\le \frac{\|f\|_{L^{m_1}(\Omega)}}{\alpha(s+1)} \mathcal{M}_2^{\frac{1}{m_1'}}.\end{aligned} \tag{39}$$

3. Proof of Main Results

In this section, we give the proof of the Theorems 1.1 and 1.2.

Thanks to the lemmas 2.1 and 2.2 as well as the estimate (16), we have

$$\begin{cases} \|u_n\|_{W_0^{1,1}(\Omega)} \le \mathcal{M}, \quad \forall n \in \mathbb{N}, \\ u_n \to u \text{ strongly in } L^\gamma(\Omega) \text{ for } 1 \le \gamma < \frac{N}{N-1}, \\ u_n(x) \to u(x) \text{ a.e. } x \in \Omega, \\ T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega). \end{cases} \tag{40}$$

Let us start proving that under the hypotheses of Theorem 1.1 or Theorem 1.2

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,1}(\Omega).$$

As a matter of fact, from inequalities (30) or (38), we deduce that, for any $i = 1,\dots,N$, the sequence $\left\{\frac{\partial u_n}{\partial x_i}\right\}$ is equi–integrable. Thus, by the Dunford-Pettis theorem, and up to subsequences, there exists v_i in $L^1(\Omega)$ such that

$$\frac{\partial u_n}{\partial x_i} \rightharpoonup v_i \quad \text{weakly in } L^1(\Omega).$$

Since $\frac{\partial u_n}{\partial x_i}$ is the distributional partial derivative of u_n, for every $n \in \mathbb{N}$, we have

$$\int_\Omega \frac{\partial u_n}{\partial x_i} \varphi \,\mathrm{d}x = -\int_\Omega u_n \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x, \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

Passing to the limit in the above integral identity, using that $\frac{\partial u_n}{\partial x_i}$ converges weakly to v_i in $L^1(\Omega)$ and u_n strongly converges to u in $L^1(\Omega)$, we obtain

$$\int_\Omega v_i \varphi \,\mathrm{d}x = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x, \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

Then $v_i = \frac{\partial u}{\partial x_i}$, for $i = 1\ldots,N$. Therefore $u \in W_0^{1,1}(\Omega)$ and

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,1}(\Omega).$$

Next, we aim to prove the strong convergence of ∇u_n to ∇u in $L^1(\Omega)$, which will allows us to pass to limit as $n \to +\infty$ in the approximating problem (13).

Firstly, we will prove that the sequence $\{\nabla u_n(x)\}$ converges to $\nabla u(x)$ almost everywhere in Ω. To this aim we will follow the proof of Lemma A.1 in [7] (see also [8]).

Let $0 < \theta < \frac{1}{p}$ and $0 < j < k$. Let us consider the following integral

$$I_n = \int_\Omega \left\{ a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right] \nabla (u_n - u) \right\}^\theta \mathrm{d}x.$$

We will prove that $I_n \to 0$ as n tend to $+\infty$.

We set

$$C_k = \{x \in \Omega : |u(x)| \le k\}, \quad A_k = \{x \in \Omega : |u(x)| > k\},$$

and

$$D_{n,k}(j) = \{x \in \Omega : |u_n(x) - T_k(u(x))| \ge j\}.$$

Moreover, by $\omega_i(k)$ we denote some quantities such that

$$\lim_{k \to \infty} \omega_i(k) = 0.$$

Using twice Hölder inequality with exponent $\frac{1}{p\theta}$, $\frac{1}{1-p\theta}$ and $\frac{1}{\theta}$, $\frac{1}{1-\theta}$ and the estimate $\|u_n\|_{W_0^{1,1}(\Omega)} \le \mathcal{M}$, we get

$$
\begin{aligned}
I_n = &\int_{C_k} \left\{ a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right] \nabla (u_n - T_k(u)) \right\}^{\theta} dx \\
&+ \int_{A_k} \left\{ a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right] \nabla (u_n - u) \right\}^{\theta} dx \\
\leq &\int\limits_{C_k \cap D^c_{n,k}(j)} \left\{ a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right] \nabla T_j (u_n - T_k(u)) \right\}^{\theta} dx \\
&+ \Bigg(\int\limits_{D_{n,k}(j)} \left(a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right] \nabla (u_n - T_k(u)) \, \mathrm{d}x \right)^{\frac{1}{p}} \Bigg)^{p\theta} \\
&\times |D_{n,k}(j)|^{1-p\theta} \\
&+ \left(\int_{\Omega} \left(a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right] \nabla (u_n - u) \right)^{\frac{1}{p}} dx \right)^{p\theta} |A_k|^{1-p\theta} \\
\leq &\left(\int_{\Omega} a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right] \nabla T_j (u_n - T_k(u)) \, dx \right)^{\theta} |\Omega|^{1-\theta} \\
&+ \left(\int_{\Omega} \left(a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right] \nabla (u_n - u) \right)^{\frac{1}{p}} dx \right)^{p\theta} |A_k|^{1-p\theta} \\
&+ \mathcal{C}_{15} \, |D_{n,k}(j)|^{1-p\theta} .
\end{aligned}
$$

Note that, for every fixed j, we have

$$
\limsup_{n \to +\infty} |D_{n,k}(j)|^{1-p\theta} = |\{x \in \Omega : |u(x) - T_k(u(x))| > j\}|^{1-p\theta} = \omega_1(k),
$$

therefore, we obtain

$$
\begin{aligned}
I_n \leq |\Omega|^{1-\theta} &\left(\int_{\Omega} a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right] \nabla T_j (u_n - T_k(u)) \, dx \right)^{\theta} \\
&+ \mathcal{C}_{15} \omega_1(k) + \mathcal{C}_{16} \omega_2(k).
\end{aligned} \tag{41}
$$

Let us estimate the first integral in the previous inequality.
By using $T_j(u_n - T_k(u))$ as test function in (15), it yields

$$
\begin{aligned}
&\int_{\Omega} a(x) \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right] \nabla T_j(u_n - T_k(u)) dx \\
&+ \int_{\Omega} g(u_n) \left[|\nabla u_n|^{q-2} \nabla u_n - |\nabla T_k(u)|^{q-2} \nabla T_k(u) \right] \nabla T_j(u_n - T_k(u)) dx \\
&\leq \int_{\Omega} f_n T_j(u_n - T_k(u)) dx - \int_{\Omega} a(x) \, |\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla T_j(u_n - T_k(u)) dx \\
&- \int_{\Omega} g(u_n) \, |\nabla T_k(u)|^{q-2} \nabla T_k(u) \nabla T_j(u_n - T_k(u)) dx
\end{aligned}
$$

which implies, taking into account the positivity of the second term in the left-hand side of the above inequality, that

$$I_n \le |\Omega|^{1-\theta}\left(j\int_\Omega |f_n|dx - \mathcal{J}_{k,j,n} - \mathcal{K}_{k,j,n}\right)^{\theta} + \mathcal{C}_{15}\omega_2(k) + \mathcal{C}_{16}\omega_1(k),$$

with

$$\mathcal{J}_{k,j,n} = \int_\Omega a(x)\,|\nabla T_k(u)|^{p-2}\,\nabla T_k(u)\nabla T_j(u_n - T_k(u))dx,$$
$$\mathcal{K}_{k,j,n} = \int_\Omega g(u_n)\,|\nabla T_k(u)|^{q-2}\,\nabla T_k(u)\nabla T_j(u_n - T_k(u))dx.$$

Let us analyze the previous terms one by one. First of all, thanks to the properties of f_n we have

$$\lim_{j\to 0}\lim_{n\to\infty} j\int_\Omega |f_n|dx = \lim_{j\to 0} j\int_\Omega |f|dx = 0.$$

Since $a(x)\,|\nabla T_k(u)|^{p-2}\,\nabla T_k(u) \in (L^{p'}(\Omega))^N$ and $\{T_k(u_n)\}$ weakly converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$, we can pass to the limit as n tends to $+\infty$ and we obtain

$$\lim_{j\to 0}\lim_{n\to\infty} \mathcal{J}_{k,j,n} = \lim_{j\to 0}\int_{\{k<|u|<k+j\}} a(x)\,|\nabla T_k(u)|^{p-2}\,\nabla T_k(u)\nabla(u - T_k(u))dx = 0.$$

On the other hand, we have

$$\lim_{j\to 0}\lim_{n\to\infty} \mathcal{K}_{k,j,n} = \lim_{j\to 0}\lim_{n\to\infty}\int_\Omega g(T_{k+j}(u_n))\,|\nabla T_k(u)|^{q-2}\,\nabla T_k(u)\nabla T_j(u_n - T_k(u))dx = 0.$$

Hence we have

$$0 \le \lim_{n\to\infty} I_n \le \mathcal{C}_{16}\omega_2(k) + \mathcal{C}_{17}\omega_1(k) + \mathcal{C}_{18}\omega_3(k).$$

Therefore, letting $k \to +\infty$ we deduce

$$\lim_{n\to+\infty}\int_\Omega \left\{a(x)\left[|\nabla u_n|^{p-2}\,\nabla u_n - |\nabla u|^{p-2}\,\nabla u\right]\nabla(u_n - u)\right\}^{\theta} \mathrm{d}x = 0,$$

which implies (for a suitable subsequence, still denoted by u_n)

$$\left\{a(x)\left[|\nabla u_n|^{p-2}\,\nabla u_n - |\nabla u|^{p-2}\,\nabla u\right]\nabla(u_n - u)\right\}^{\theta} \to 0 \quad \text{a.e. in } \Omega,$$

and also (since θ is positive)

$$\left\{a(x)\left[|\nabla u_n|^{p-2}\,\nabla u_n - |\nabla u|^{p-2}\,\nabla u\right]\nabla(u_n - u)\right\} \to 0 \quad \text{a.e. in } \Omega.$$

Then the previous limit implies that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in} \quad \Omega. \tag{42}$$

as it is proved in [13].

Thanks to (2), (3) and to the estimate $\|u_n\|_{W_0^{1,1}(\Omega)} \le \mathcal{M}$, applying Hölder's inequality we get

$$\int_\Omega a(x)|\nabla u_n|^{p-1}dx \le \beta\left(\int_\Omega |\nabla u_n|dx\right)^{p-1}|\Omega|^{2-p}$$
$$\le \beta\mathcal{M}^{p-1}|\Omega|^{2-p}.$$

Therefore, by (42) and (18) or (34), we can apply Vitali's theorem and we deduce that

$$a(x)|\nabla u_n|^{p-2}\nabla u_n \to a(x)|\nabla u|^{p-2}\nabla u \quad \text{strongly in } (L^1(\Omega))^N. \tag{43}$$

Moreover, thanks to (42) we readily have

$$g(u_n)|\nabla u_n(x)|^{q-2}\nabla u_n(x) \to g(u)|\nabla u(x)|^{q-2}\nabla u(x) \quad \text{a.e. in } \Omega.$$

Now we assume that the hypotheses of the Theorem 1.1 hold.

Let $E \subset \Omega$ be a measurable set. Due to Hölder's inequality and inequality (17), we have

$$\begin{aligned}\int_E g(u_n)|\nabla u_n|^{q-1}dx &= \int_{E\cap D_n(k)} g(u_n)|\nabla u_n|^{q-1}dx + \int_{E\cap D_n^c(k)} g(u_n)|\nabla u_n|^{q-1}dx\\
&\le \int_{E\cap D_n(k)} |u_n|^{q(r-1)+1-\frac{\tilde{s}(q-1)}{p}}|u_n|^{\frac{\tilde{s}(q-1)}{p}}|\nabla u_n|^{q-1}dx\\
&+k^{q(r-1)+1}\mathcal{M}_1^{q-1}|E|^{2-q}\\
&\le \left[\int_{E\cap D_n(k)} |u_n|^{\lambda}dx\right]^{\frac{1}{\tau}}\left[\int_{E\cap D_n(k)} |u_n|^{\tilde{s}}|\nabla u_n|^p dx\right]^{\frac{1}{\tau'}}\\
&+k^{q(r-1)+1}\mathcal{M}_1^{q-1}|E|^{2-q}\\
&\le \left[\int_\Omega |u_n|^{\frac{N}{N-1}}dx\right]^{\frac{\lambda(N-1)}{N\tau}}|E|^{\frac{1}{\tau}-\frac{\lambda(N-1)}{N\tau}}\left[\int_{D_n(k)} |u_n|^{\tilde{s}}|\nabla u_n|^p dx\right]^{\frac{1}{\tau'}}\\
&+k^{q(r-1)+1}\mathcal{M}_1^{q-1}|E|^{2-q},\end{aligned}$$

where $\tilde{s}$ is the number defined in (19),

$$\tau = \frac{p}{p-q+1} \quad \text{and} \quad \lambda = \left(q(r-1)+1+\frac{N(q-1)(p-1)}{p(N-1)}\right)\tau.$$

We note that hypotheses $r < \bar{r}$ and (2) ensure that $\lambda < \frac{N}{N-1}$.

Using (24) and the fact that the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$, we deduce

$$\lim_{|E|\to 0}\int_E g(u_n)|\nabla u_n|^{q-1}dx = 0, \quad \text{uniformly with respect to } n,$$

hence we apply Vitali's Theorem to get

$$g(u_n)|\nabla u_n|^{q-2}\nabla u_n \to g(u)|\nabla u|^{q-2}\nabla u \quad \text{strongly in } (L^1(\Omega))^N. \tag{44}$$

Therefore, thanks to the convergences (43) and (44), it is possible to pass to the limit in the integral identity (15) and we prove that the limit u is a distributional solution of the Problem (1) in the sense of definition (8).

Finally, suppose that the hypotheses of Theorem 1.2 hold.

Applying again the Hölder's inequality and recalling the Lemma 2.2, we have

$$\begin{aligned}
\int_E g(u_n)|\nabla u_n|^{q-1}\,dx &= \int_E |u_n|^{r-\eta(q-1)}\left(|u_n|^{r-1+\eta}|\nabla u_n|\right)^{q-1}dx \\
&\le \left[\int_E |u_n|^{q(r-\eta(q-1))}\,dx\right]^{\frac{1}{q}}\left[\int_E |u_n|^{(r-1+\eta)q}|\nabla u_n|^q\,dx\right]^{\frac{1}{q'}} \\
&\le \left[\int_\Omega |u_n|^{\rho q(r-\eta(q-1))}\,dx\right]^{\frac{1}{\rho q}}|E|^{\frac{1}{\rho' q}}\left[\int_E |u_n|^{(r-1+\eta)q}|\nabla u_n|^q\,dx\right]^{\frac{1}{q'}} \\
&\le \left[\int_\Omega |u_n|^{\mu}\,dx\right]^{\frac{1}{\rho q}}|E|^{\frac{1}{\rho' q}}\left[\int_\Omega |u_n|^{q(r-1)+s+1}|\nabla u_n|^q\,dx\right]^{\frac{1}{q'}} \\
&\le \mathcal{M}_2^{\frac{\mu}{\rho q}}|E|^{\frac{1}{\rho' q}}\mathcal{C}_{12}^{\frac{1}{q'}},
\end{aligned}$$

where μ is the number defined in the statement of the Theorem 1.2, number s is defined in (35) and

$$\eta = \frac{s+1}{q}, \qquad \rho = \frac{\mu}{q(r-\eta(q-1))}.$$

Notice that hypotheses $r < \tilde{r}$ and (2) ensure that $\rho > 1$.

Then, by Lemma 2.2 and Vitali's Theorem we affirm that the convergence (44) holds true. Hence, taking into account also the convergence (43), we can pass to the limit in the integral identity (15) and we prove that u is a distributional solution of the Problem (1) in the sense of definition (8).

Acknowledgment

This work has been supported by Piano della Ricerca di Ateneo 2024-2026-PIACERI EdP.EReMo – "Equazioni differenziali alle derivate parziali: esistenza, regolarità e molteplicità delle soluzioni".

The second author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

REFERENCES

[1] F. Achhoud - G. R. Cirmi, *Regularity results for an anisotropic nonlinear Dirichlet problem*, Complex Variables and Elliptic Equations, 1-22. https://doi.org/10.1080/17476933.2024.2336975

[2] P. Baroni - M. Colombo - G. Mingione, *Harnack inequalities for double phase functionals*, Nonlinear Analysis: Theory, Methods and Applications 121 (2015), 206-222.

[3] P. Baroni - M. Colombo - G. Mingione, Regularity for general functionals with double phase. Calculus of Variations and Partial Differential Equations 57(2) (2018), pp 48.

[4] L. Boccardo - G. R. Cirmi, *Regularizing effect in some Mingione's double phase problems with very singular data*, Mathematics in Engineering 5(3) (2022), 1-15.

[5] L. Boccardo - G. R. Cirmi, *Some elliptic equations with $W_0^{1,1}(\Omega)$ solutions*, Nonlinear Analysis: Theory, Methods and Applications 153 (2017), 130-141.

[6] L. Boccardo - T. Gallouët, *Nonlinear Elliptic Equations with Right Hand Side Measures*, Communications in Partial Differential Equations 17(3-4) (1992), 189-258.

[7] L. Boccardo - T. Gallouët, $W_0^{1,1}(\Omega)$ *solutions in some borderline cases of Calderon-Zygmund theory*, Journal of Differential Equations. 253 (2012), 2698-2714.

[8] L. Boccardo - F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Analysis: Theory, Methods & Applications, 19(6) (1992), 581-597.

[9] G. R. Cirmi - S. D'Asero, *Nonlinear degenerate elliptic equations with a convection term*, Journal of Elliptic and Parabolic Equations, 8(2) (2022), 1041-1066.

[10] M. Colombo - G. Mingione, *Bounded Minimisers of Double Phase Variational Integrals*, Archive for Rational Mechanics and Analysis 218(1) (2015), 219-273.

[11] M. Colombo - G. Mingione, *Regularity for Double Phase Variational Problems*, Archive for Rational Mechanics and Analysis 215(2) (2014), 443-496.

[12] G. Croce, *The regularizing effects of some lower order terms on the solutions in an elliptic equation with degenerate coercivity*, Rendiconti di Matematica Serie VII. 27 (2007), 299-314.

[13] J. Leray - J. L. Lions, *Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France 79 (1965), 97-107.

[14] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, Journal of Differential Equations 90(1) (1991), 1-30.

F. ACHHOUD
Dipartimento di Matematica e Informatica, University of Catania, Viale A. Doria, 6, 95125 Catania, Italy
e-mail: `f.achhoud@uhp.ac.ma`

S. D'ASERO
Dipartimento di Matematica e Informatica, University of Catania, Viale A. Doria, 6, 95125 Catania, Italy
e-mail: `salvatore.dasero@unict.it`