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A DEFORMATION THEORY APPROACH TO LINEAR SYSTEMS WITH GENERAL TRIPLE POINTS

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

1. Introduction: The Segre Conjecture.

Let $p_1, \dots, p_n$ be general points in the complex projective plane $\mathbb{P}^2$ and let $m_1, \dots, m_n$ be positive integers. We let $\mathcal{L}_d(p_1^{m_1}, \dots, p_n^{m_n})$ be the linear system of plane curves of degree d having multiplicity at least m_i at the point p_i, $i = 1, \dots, n$. If $m_i = 1$ we suppress the superscript m_i for p_i in $\mathcal{L}_d(p_1^{m_1}, \dots, p_n^{m_n})$.

Let $\pi : S \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points $p_1, \dots, p_n$. Let $\mathcal{L}$ be a line bundle on S, or, by abusing notation, the corresponding complete linear system. One defines the *virtual dimension* of $\mathcal{L}$ to be:

$$v(\mathcal{L}) := \chi(\mathcal{L}) - 1 = \frac{\mathcal{L} \cdot (\mathcal{L} - K_S)}{2}$$

where K_S is the canonical class on S.

If C is any divisor on S, we similarly define $v(C) := \chi(\mathcal{O}_S(C)) - 1$. The Riemann-Roch Theorem says that if $\mathcal{L}$ is effective, then

$$\dim(\mathcal{L}) = v(\mathcal{L}) + h^1(S, \mathcal{L}) \tag{1.1}$$

since $h^2(\mathcal{L}) = 0$. One also defines the *expected dimension* of $\mathcal{L}$ to be

$$\epsilon(\mathcal{L}) := \max\{\nu(\mathcal{L}), -1\}.$$

If C is any divisor on S we can accordingly define $\epsilon(C) := \max\{\nu(C), -1\}$.

One says that a linear system $\mathcal{L}$ on S is *non-special* if its dimension equals the expected dimension. This is equivalent to saying that $\mathcal{L}$ is non-special if and only if either it is empty or it is *regular*, namely not empty and with $h^1(S, \mathcal{L}) = 0$.

Let H be the pull-back via π of a general line of the plane and let $E_1, \ldots, E_n$ be the exceptional divisors contracted by π to the points $p_1, \ldots, p_n$. The *proper transform* of $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ on S is the complete linear system $\mathcal{L} := |dH - m_1E_1 - \ldots - m_iE_i|$. By abusing notation, we will denote by $\mathcal{L}$ also the line bundle associated to this linear system.

We apply the language of virtual and expected dimension to the system $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ on the plane also, by using the corresponding notions of the proper transform. In particular, the *virtual dimension* of $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ is

$$\nu(\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})) := \nu(\mathcal{L}) = \frac{d(d+3)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}$$

and the *expected dimension* of $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ is

$$\epsilon(\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})) := \epsilon(\mathcal{L}).$$

One says that a system $\mathcal{L}_d(p_1^{m_1}, \ldots, p_n^{m_n})$ of plane curves is *non-special* if the proper transform $\mathcal{L}$ on S is such.

A linear system $\mathcal{L}$ on S, which is not empty, is called *reducible* [resp. *reduced*] if its general curve C is reducible [resp. reduced]. Bertini's theorem tells us that, if $\mathcal{L}$ is reducible, then either it has some fixed components or it is composed with a rational pencil $\mathcal{P}$, i.e. the movable part of $\mathcal{L}$ consists of the sum of $h \geq 2$ curves of $\mathcal{P}$. The following conjecture is due to B. Segre:

Conjecture 1.2 (Segre's Conjecture). *Suppose that $\mathcal{L}$ as above is nonempty and reduced. Then $\mathcal{L}$ is non-special.*

Since a plane curve is reduced if and only if it has isolated singularities, another way of phrasing Segre's Conjecture is: if the general member of $\mathcal{L}$ has isolated singularities, then $H^1(\mathcal{L}) = 0$. In this form it may generalize to higher dimensions.

This conjecture is related to more recent conjectures of Harbourne and Hirschowitz, (see [3] and [4]). It has attracted much attention recently, and although we will not give a full history here, the authors have been able to check the above conjecture for multiplicities at most 13 in [1] and [2]. These articles used a degeneration technique and standard uppersemicontinuity arguments, as has most of the work on this problem. This article presents a technique based on deformation theory, which has not been fully exploited for multiplicities greater than two as far as the authors are aware.

The authors would like to thank Prof. Herb Clemens for suggesting this line of attack.

2. Higher-Order Deformations of Fat Points.

In this section we begin to describe a method to attack Segre's Conjecture via a deformation theory argument. Suppose that for *general* points p_i there exists a divisor $C \in |\mathcal{L}|$ with isolated singularities. Then Segre's Conjecture states that $H^1(X, \mathcal{L}) = 0$. We seek to interpret the H^1 as a vector space which carries obstructions to deforming the divisor C as the points p_i vary. The intention is then to show that every element of H^1 occurs as an obstruction, and also to remark that since the divisor C exists for general points, there can be no obstructions. The conclusion that H^1 must be zero would follow.

Crudely, there is a mapping

$$\left\{ \begin{array}{c} \text{deformations} \\ \text{of the points} \\ {\scriptstyle \{p_i\}} \end{array} \right\} \xrightarrow[\text{to moving } C]{\text{obstruction}} H^1(\mathcal{L}).$$

The interpretation of the H^1 as a space carrying obstructions is essentially the construction of this mapping. The statement that every element of H^1 occurs as an obstruction is the surjectivity of this mapping. The statement that there are no obstructions (because of the general existence of C) is the zero-ness of this mapping. Since a surjective zero map must have target 0, we conclude $H^1 = 0$ as required.

In what follows we will describe the construction of the mapping, and prove the surjectivity for double and triple points (when all $m_i = 2$ and when all $m_i = 3$). We will actually only consider the deformation of the divisor C upon varying a single one of the points $p = p_i$, having multiplicity $m = m_i$. We will work on the plane $\mathbb{P}^2$ instead of on X, and choose affine coordinates (x, y) near p such that $p = (0, 0)$.

The original curve $C = C_0$ is then defined by a polynomial of degree at most d:

$$C_0 : F^{(0)}(x, y) = 0.$$

Fix a direction vector (a, b), and deform the point p to $p_t = (at, bt)$. Now try to deform C_0 to a divisor C_t (defined by $F_t(x, y) = 0$) which will have multiplicity m at p_t. We may assume that the polynomial F_t in (x, y) has coefficients varying formally analytically with t.

If we expand the desired polynomial F_t in a power series in t, we find

$$F_t(x, y) = \sum_{p \geq 0} F^{(p)}(x, y) t^p \tag{2.1}$$

where each term $F^{(p)}$ is a polynomial in (x, y) with constant coefficients.

Change coordinates to (u, v), where $(u, v) = (0, 0)$ at the varying point p_t; this is done by setting

$$x = u + at \quad \text{and} \quad y = v + bt$$

Plug this into (2.1), expand via Taylor's Theorem, and collect terms in t, to obtain

$$\begin{aligned} F_t &= \sum_{p \geq 0} [\sum_{q \geq 0} \frac{1}{q!} R^q F^{(p)}(u, v) t^q] t^p \\ &= \sum_{n \geq 0} [\sum_{p+q=n} \frac{1}{q!} R^q F^{(p)}(u, v)] t^n \end{aligned}$$

where $R = a\partial/\partial x + b\partial/\partial y$.

In order for this to have multiplicity at least m at p_t, we must have that the multiplicity of the t^n coefficient is at least m at $(u, v) = (0, 0)$ for every n. If we call this coefficient A_n, we have then that

$$A_n = \sum_{p+q=n} \frac{1}{q!} R^q F^{(p)}(u, v).$$

The requirement that the multiplicity of A_n at $(u, v) = (0, 0)$ be at least m is a series of conditions, one for each n. We will refer to the condition that $\text{mult}_{(0,0)}(A_n) \geq m$ as the $n - th$ *order multiplicity condition.*

The 0-th order multiplicity condition is that

$$\text{mult}_{(0,0)} F^{(0,0)}(u, v) \geq m$$

which is in fact the hypothesis on the original curve $C = C_0$ in the linear system, and is therefore automatic.

It is convenient to expand each $F^{(\ell)}$ into homogeneous parts; let us denote by

$$F_k^{(\ell)}$$

the homogeneous piece of degree k of the polynomial $F^{(\ell)}$. The hypothesis that C_0 has multiplicity at least m at $(0, 0)$ is therefore that

$$F_k^{(0)} = 0 \text{ for each } k = 0, \ldots, m-1. \tag{2.2}$$

3. The First-Order Multiplicity Condition.

The first-order multiplicity condition is that A_1 has multiplicity at least m at the origin; since

$$A_1 = F^{(1)} + RF^{(0)}$$

we see that this has multiplicity at least m if and only if $F_k^{(1)} + RF_{k+1}^{(0)} = 0$ for each $k = 0, \ldots m-1$. Since $F^{(0)}$ is already assumed to have multiplicity at least m, using (2.2) we see that this is equivalent to the following:

(a) $F_k^{(1)} = 0$ for $k = 0, \ldots, m-2$ (i.e., $\mathrm{mult}_{(0,0)}(F^{(1)}) \geq m-1$);

(b) $F_{m-1}^{(1)} = -RF_m^{(0)}$

The sheaf interpretation of this is as follows. Let E denote the exceptional divisor above the point p which is the origin for this coordinate system. Then we have the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}(E) \to \mathcal{L}(E)|_E \cong \mathcal{O}_E(m-1) \to 0$$

of sheaves on the blowup X. Taking cohomology, this gives

$$0 \to H^0(\mathcal{L}) \to H^0(\mathcal{L}(E)) \to H^0(\mathcal{O}_E(m-1)) \xrightarrow{\Delta} H^1(\mathcal{L}).$$

The requirement of (a) in the first-order multiplicity condition is that $F^{(1)}$ lie in the space $H^0(\mathcal{L}(E))$. The restriction map from this space to $H^0(\mathcal{O}_E(m-1))$ is simply the map taking $F^{(1)}$ to its lowest-order part, the piece of homogeneous degree $m-1$, which is $F_{m-1}^{(1)}$.

Now the homogeneous polynomial $-RF_m^{(0)}$ of degree $m-1$ can be considered also to lie in the space $H^0(\mathcal{O}_E(m-1))$. Therefore the requirement of (b) is that the polynomial $F^{(1)}$ in $H^0(\mathcal{L}(E))$ must map to this element $-RF_m^{(0)} \in H^0(\mathcal{O}_E(m-1))$.

By the exactness of the sequence, such a polynomial $F^{(1)}$ exists if and only if this element $-RF_m^{(0)}$ maps to zero in $H^1(\mathcal{L})$ under the coboundary map Δ. We therefore obtain the obstruction element

$$\Delta(-RF_m^{(0)}) \in H^1(\mathcal{L})$$

which must be zero for the original curve C_0 to deform to first order.

At this point we want to make an important remark. The first-order curve (defined by $F^{(1)} = 0$) depends only on the lowest-order term $F_m^{(0)}$ of the original curve. In particular if one locally makes an analytic change of coordinates which does not affect this lowest-order term $F_m^{(0)}$, then the computations will not produce any change at all in the first-order curve $F^{(1)}$.

4. Double Points.

Let us show how the case when $m = 2$ in Segre's Conjecture can be handled using the first-order multiplicity conditions. Fix $r + s$ general points $\{p_i\}$, and denote by $\mathcal{L}_d(1^r, 2^s)$ the invertible sheaf on the blowup X of the plane at the p_i's associated to the divisor $dH - \sum_{i=1}^r E_i - \sum_{i=r+1}^{r+s} 2E_i$. This corresponds to the linear system of plane curves of degree d having r simple base points and s double points.

With this notation, the precise statement would then be the following.

Theorem 4.1. *Suppose that for* general *points p_i there exists a divisor $C \in |\mathcal{L}_d(1^r, 2^s)|$ with isolated singularities. Then $H^1(X, \mathcal{L}_d(1^r, 2^s)) = 0$.*

Proof. We work by induction on s, the number of double points. If $s = 0$, we are imposing only simple base points, and the vanishing of the H^1 in this case is a triviality, for all r, as long as the system is non-empty. Suppose then the theorem is true for $s - 1 \geq 0$ double points (and all r); let us prove it for s double points.

As noted above we have the long exact sequence

$$0 \to H^0(\mathcal{L}_d(1^r, 2^s)) \to H^0(\mathcal{L}_d(1^{r+1}, 2^{s-1})) \to$$

$$\to H^0(\mathcal{O}_E(1)) \xrightarrow{\Delta} H^1(\mathcal{L}_d(1^r, 2^s)) \to 0$$

where E is the exceptional divisor on the blowup X over the last point. The last term is actually $H^1(\mathcal{L}_d(1^{r+1}, 2^{s-1}))$, which by induction we may assume is zero. Hence the coboundary map Δ is onto.

First assume that the general divisor C has an ordinary double point at the last point. Then we may assume that $F_2^{(0)}(x, y)$ has the form xy in suitable coordinates. If $R = a\partial/\partial x + b\partial/\partial y$, then the element $-RF_2^{(0)} \in H^0(\mathcal{O}_E(1))$ is $ay + bx$; as a and b vary, these elements span the space $H^0(\mathcal{O}_E(1))$. We conclude (since Δ is onto, and is zero on these elements), that Δ itself is zero, and hence that $H^1(\mathcal{L}(d, 1^r, 2^s))$ vanishes as required.

If the general divisor C has a non-ordinary double point or a point of higher multiplicity, we argue differently. By imposing additional simple points (that is, increasing r if necessary), we may assume that $\dim H^0(\mathcal{L}_d(1^r, 2^s)) = 1$, so that the general C does not move in its linear system on X. By induction the system $\mathcal{L}_d(1^r, 2^{s-1})$ is non-special, and hence has affine dimension at most 4 (since imposing the additional double point can impose at most 3 linear conditions).

If the dimension is in fact 4, then the original system $\mathcal{L}_d(1^r, 2^s)$ has the expected dimension, and we are done. If the dimension is 3, then by imposing a simple base point plus a tangent direction, we will reduce the dimension to one; and therefore we will have the general non-ordinary double point. This is a contradiction unless the system is composed with a pencil, in which case the general member will be non-reduced. Similarly if the dimension is two, imposing a general base point leads to a non-ordinary double point, which is again a contradiction. The dimension cannot be one by assumption. **Q.E.D.**

5. The Second-Order Multiplicity Condition.

The second-order multiplicity condition is that A_2 has multiplicity at least m at the origin; since

$$A_2 = F^{(2)} + RF^{(1)} + \frac{1}{2}R^2F^{(0)}$$

we see that this has multiplicity at least m if and only if $F_k^{(2)} + RF_{k+1}^{(1)} + \frac{1}{2}R^2F_{k+2}^{(0)} = 0$ for each $k = 0, \dots m-1$. This is equivalent to the following, assuming the first-order multiplicity condition:

(a) $F_k^{(2)} = 0$ for $k = 0, \dots, m-3$ (i.e., $\mathrm{mult}_{(0,0)}(F^{(2)}) \geq m-2$);

(b) $F_{m-2}^{(2)} = -RF_{m-1}^{(1)} - \frac{1}{2}R^2F_m^{(0)} = \frac{1}{2}R^2F_m^{(0)}$;

(c) $F_{m-1}^{(2)} = -RF_m^{(1)} - \frac{1}{2}R^2F_{m+1}^{(0)}$.

The sheaf interpretation of this is as follows. Using the notation above we have the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{L} & \to & \mathcal{L}(E) & \to & \mathcal{O}_E(m-1) & \to & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{L} & \to & \mathcal{L}(2E) & \to & \mathcal{L}(2E)|_{2E} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_E(m-2) & \cong & \mathcal{O}_E(m-2) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Taking cohomology, this gives

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathcal{L}) & \to & H^0(\mathcal{L}(E)) & \to & \mathbb{C}^m & \to & H^1(\mathcal{L}) \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \cong\downarrow \\
0 & \to & H^0(\mathcal{L}) & \to & H^0(\mathcal{L}(2E)) & \to & H^0(\mathcal{L}(2E)|_{2E}) & \to & H^1(\mathcal{L}) \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{C}^{m-1} & \cong & \mathbb{C}^{m-1} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

of sheaves on the blowup X.

The requirement of (a) in the second-order multiplicity condition is that $F^{(2)}$ lie in the space $H^0(\mathcal{L}(2E))$. The restriction map from this space to $H^0(\mathcal{L}(2E)|_{2E})$ is simply the map taking $F^{(2)}$ to its two lowest-order parts (the parts of homogeneous degrees $m-2$ and $m-1$); this target space is, as the diagram shows, naturally filtered by the $\mathbb{C}^m$ piece of homogeneous degree $m-1$ and the $\mathbb{C}^{m-1}$ piece of homogeneous degree $m-2$.

Now the homogeneous polynomials $\frac{1}{2}R^2\ F_m^{(0)}$ of degree $m-2$ and $-RF_m^{(1)}-\frac{1}{2}R^2F_{m+1}^{(0)}$ of degree $m-1$ can be therefore considered, as an ordered pair, to lie in this space $H^0(\mathcal{L}(2E)|_{2E})$. Therefore the requirement of (b) and (c) is that the polynomial $F^{(2)}$ in $H^0(\mathcal{L}(2E))$ must map to this ordered pair in $H^0(\mathcal{L}(2E)|_{2E})$.

By the exactness of the horizontal sequences, such a polynomial $F^{(2)}$ exists if and only if this ordered pair maps to zero in $H^1(\mathcal{L})$ under the coboundary map Δ. We therefore obtain the obstruction element

$$\Delta(\frac{1}{2}R^2F_m^{(0)}, -RF_m^{(1)} - \frac{1}{2}R^2F_{m+1}^{(0)}) \in H^1(\mathcal{L})$$

which must be zero for the original curve C_0 to deform to second order.

6. Triple Points.

Let us apply these considerations to the analysis of the case of triple points. We again use the notation $\mathcal{L}_d(1^r, 2^s, 3^t)$ to denote the invertible sheaf on the blowup of the plane at $r + s + t$ points, corresponding to the linear system of plane curves of degree d with r simple points, s double points, and t triple points prescribed.

Firstly, let us remark that we can assume that the general triple point is either ordinary (with three distinct tangents) or has at least two distinct tangents. For this we may suppose (by increasing the number r of simple points) that the dimension of the linear system is exactly zero. Now suppose that the general member has a non-ordinary triple point with a triple tangent. Removing this triple base point from the conditions on the system, we see that we arrive at a linear system $|D|$ of dimension at most six. If the dimension is exactly six, we are done. If the dimension is at most two, already by induction and the double point case, imposing a double point will give a non-reduced general curve, which is a contradiction. If the dimension is three, then imposing a double point will either make the general member non-reduced, or will give a triple point; in either case we have a contradiction.

If the dimension is four, consider the induced map $\phi : X \to \mathbb{P}^4$, where X is the blowup of the plane at the base points. If the image of ϕ is a curve, then imposing a double point is equivalent to imposing a tangent line to the curve, in which case again we get a non-reduced component. Hence we may assume that the image of ϕ is a surface. Look at the tangent hyperplanes at the point in question, which form a pencil. In this pencil there is assumed to be a triple point intersection; therefore the general element of this pencil has at least two fixed tangent directions (a pencil generated by a double point curve and a triple point curve will have every member having the two tangents of the double point curve). Therefore the second fundamental form of the surface must be zero-dimensinal, since this is happening at a general point. Hence the surface is developable, and therefore already imposing a general double point gives a non-reduced component.

Finally if the dimension is five, again consider the map to $\mathbb{P}^5$ given by the linear system. The image is again a surface by the same arguments. Assume that the general triple point has a single tangent. Look at the tangent space to this family of hyperplanes: these correspond to curves with a double point and a single tangent. However there is a two-dimensional family of such curves, and therefore these are all of the tangents. Again this implies that the surface is developable, and we have a non-reduced component to the double point system already.

Theorem 6.1. *Suppose that for* $r+s+t$ general *points there exists a divisor* $C \in |\mathcal{L}_d(1^r, 2^s, 3^t)|$ *with isolated singularities. Then* $H^1(X, \mathcal{L}_d(1^r, 2^s, 3^t)) = 0$.

Proof. Again we work by induction, on t; for $t = 0$ the result is the double point case. By the above discussion we may assume that the general triple point is ordinary, or has at least two tangents. Hence we may assume that coordinates have been chosen so that $F_3^{(0)} = x^3 + y^3$ or $F_3^{(0)} = x^2 y$. We will start with the first case.

Both the first-order and second-order conditions come into play in the triple point analysis. The first-order analysis gives the long exact sequence

$$0 \to H^0(\mathcal{L}_d(1^r, 2^s, 3^t)) \to H^0(\mathcal{L}_d(1^r, 2^{s+1}, 3^{t-1})) \to$$

$$\to H^0(\mathcal{O}_E(2)) \xrightarrow{\Delta} H^1(\mathcal{L}_d(1^r, 2^s, 3^t)) \to 0$$

where E is the exceptional divisor on the blowup X over the last triple point. The last term is actually $H^1(\mathcal{L}_d(1^r, 2^{s+1}, 3^{t-1}))$, which by induction we may assume is zero. Hence the coboundary map Δ is onto, and the dimension of the H^1 in question is at most three.

The first-order condition gives that the element $-RF_3^{(0)} = -3(ax^2+by^2)$, as a homogeneous quadratic in the coordinates of E, must go to zero for all choices of a and b. Unfortunately this cannot prove the zeroness of the H^1, since for this (or any) fixed $F_3^{(0)}$, we cannot span the three-dimensional space $H^0(\mathcal{O}_E(2))$ by varying a and b.

The second-order condition gives the sequence

$$0 \to H^0(\mathcal{L}_d(1^r, 2^s, 3^t)) \to H^0(\mathcal{L}_d(1^{r+1}, 2^s, 3^{t-1})) \to$$

$$\to \mathbb{C}^5 \xrightarrow{\Delta} H^1(\mathcal{L}_d(1^r, 2^s, 3^t)) \to 0$$

where the last term is zero by the induction hypothesis. The map on the right of the first row sends the second-order term $F^{(2)}$ to its two homogeneous lower-order pieces, of degree one and two. The second-order analysis says that this must go to the pair of homogeneous polynomials

$$(\frac{1}{2}R^2 F_3^{(0)}, -RF_3^{(1)} - \frac{1}{2}R^2 F_4^{(0)});$$

note that the linear term here is $3(a^2x + b^2y)$.

As we remarked earlier, the first-order polynomial $F^{(1)}$, and hence its cubic term $F_3^{(1)}$, depends only on $F_3^{(0)}$, by the first-order analysis. Hence we can treat the term $-RF_3^{(1)}$ as fixed, and not subject to change at the second-order step.

By making a nonlinear analytic change of coordinates at the given point (essentially replacing (x, y) with $(x + P, y + Q)$ for polynomials P and Q of higher degree) we can arrange that the degree four part $F_4^{(0)}$ of the original equation is general. (Note that this has no effect on the $F_3^{(1)}$ term as remarked above.) In this case the contribution of $R^2F_4^{(0)}$ to the second component of the map above will be general enough so that this second component will vary as a general quadratic in the parameters a and b, with quadratic expressions in x and y. This is sufficient to prove that we will span the full 5-dimensional space in the restriction map to $\mathbb{C}^5$, and hence the coboundary map Δ will be zero.

To be specific, this change of coordinates has the following effect on the third and fourth order terms of $F^{(0)}$:

$$(x + P)^3 + (y + Q)^3 + F_4^{(0)}(x + P, y + Q) = \tag{5}$$

$$= x^3 + y^3 + [3x^2P_2 + 3y^2Q_2 + F_4^{(0)}(x, y)] + O$$

where P_2 and Q_2 are the quadratic terms of P and Q, respectively; the bracketed terms above form the "new" $F_4^{(0)}$ term. Applying R^2 to this, we see that we have effected a change of $3R^2(x^2P_2 + y^2Q_2)$ in the mapping. Collected in terms of x and y, these are three independent quadratic expressions for general P_2 and Q_2. Specifically, if $P_2 = c_0x^2 + c_1xy + c_2y^2$ and $Q_2 = d_0x^2 + d_1xy + d_2y^2$, then the above quadratic part is

$$x^2(12c_0a^2 + 6c_1ab + (2c_2 + 2d_0)b^2) +$$

$$+ xy(6c_1a^2 + (8c_2 + 8d_0)ab + 6d_1b^2) +$$

$$+ y^2((2c_2 + 2d_0)a^2 + 6d_1ab + 12d_2b^2)$$

and the determinant of the above 3×3 matrix of coefficients is not identically zero.

The nonlinear change of local coordinates is equivalent to making a nonlinear deformation arc (instead of the linear deformation arc $p_t = (at, bt)$).

The same considerations (and similar computations) apply in the case when $F_3^{(0)} = x^2y$. $\square$

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