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LIE PRODUCT AND LOCAL SPECTRAL SUBSPACE PRESERVERS

S. SABER - M. ELHODAIBI - S. ELOUAZZANI

Consider a complex Banach space, denoted by X with $\dim X \geq 4$, and $\mathcal{B}(X)$ represents the algebra of all bounded linear operators on X. For a fixed complex scalar λ_0, define $X_S(\{\lambda_0\})$ as the local spectral subspace of an operator $S \in \mathcal{B}(X)$ associated with $\{\lambda_0\}$. We provide a characterization of all maps ϕ on $\mathcal{B}(X)$ whose range includes operators of rank at most four, and that satisfy

$$X_{\phi(S)\phi(T)-\phi(T)\phi(S)}(\{\lambda_0\}) = X_{ST-TS}(\{\lambda_0\})$$

for all $S, T \in \mathcal{B}(X)$.

1. Introduction

In this paper, let X represent a complex Banach space, and $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators on X. The local resolvent set of an operator $S \in \mathcal{B}(X)$ at a vector $x \in X$, $\rho_S(x)$, is the union of all open $U \subset \mathbb{C}$ for which there exists an analytic function $\psi : U \longrightarrow X$ such that $(S-\alpha)\psi(\alpha) = x$ for all $\alpha \in U$. The local spectrum of S at x is defined by

$$\sigma_S(x) := \mathbb{C} \backslash \rho_S(x).$$

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If for every open set $V \subset \mathbb{C}$, the only analytic solution $\psi : V \to X$ of the equation $(S-\alpha)\,\psi(\alpha) = 0$ for all $\alpha \in V$, is the null function on V, then S is said to have the single valued extension property (SVEP).

The local spectral subspace of $S \in \mathcal{B}(X)$ associated with $F \subseteq \mathbb{C}$, represented as $X_S(F)$, is defined as follows

$$X_S(F) := \{x \in X : \sigma_S(x) \subseteq F\}.$$

Evidently, when $F_1 \subseteq F_2$, it follows that $X_S(F_1) \subseteq X_S(F_2)$.

In recent years, there has been considerable interest among authors in exploring problems related to preserving additive and linear local spectra; refer, for example, to [1, 3–8, 11, 12] and their references. The class of preserver problems within the framework of local spectral subspaces was initially investigated by M. Elhodaibi and A. Jaatit in [6]. They demonstrated that if the additive map ϕ on $\mathcal{B}(X)$ satisfies

$$X_{\phi(T)}(\lambda) = X_T(\lambda)$$

for every $T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$, then ϕ is the identity map on $\mathcal{B}(X)$.
Let λ_0 be a fixed scalar in $\mathbb{C}$. In the work presented in [8], the author provided a characterization of maps on $\mathcal{B}(X)$ that preserve the local spectral subspace of the sum and the difference of operators associated with $\{\lambda_0\}$. Additionally, in [4], the authors described maps that preserve the local spectral subspace of the generalized product of operators associated with $\{\lambda_0\}$.

The Lie product of $S,T \in \mathcal{B}(X)$ is defined by $[S,T] = ST - TS$. In this context, we consider the result obtained in [13] concerning the set of fixed points of an operator, the authors characterized the forms of surjective maps on $\mathcal{B}(X)$ preserving the Lie product. More precisely, it was shown that if $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ is a surjective map which satisfies

$$F(\phi(A)\phi(T) - \phi(T)\phi(A)) = F(AT - TA) \text{ for every } A,T \in \mathcal{B}(X),$$

if and only if there exist a nonzero scalar $\gamma \in \mathbb{C}$ with $\gamma^2 = 1$ and a scalar function $\tau : \mathcal{B}(X) \to \mathbb{C}$ such that

$$\phi(A) = \gamma A + \tau(A)I \quad \text{for all } A \in \mathcal{B}(X).$$

Now, considering λ_0 as a fixed complex scalar with $\lambda_0 \in \mathbb{C}\backslash\{0\}$, given that the case where λ_0 equals zero has already been investigated in [12]. The aim of this paper is to establish that a map ϕ on $\mathcal{B}(X)$, whose range includes operators of rank at most four and satisfies

$$X_{\phi(A)\phi(T)-\phi(T)\phi(A)}(\{\lambda_0\}) = X_{AT-TA}(\{\lambda_0\})$$

for all $A, T \in \mathcal{B}(X)$, if and only if there exist a scalar $\alpha \in \mathbb{C}$ and a map $\beta : \mathcal{B}(X) \longrightarrow \mathbb{C}$ such that $\alpha^2 = 1$ and

$$\phi(T) = \alpha T + \beta(T)I \text{ for all } T \in \mathcal{B}(X).$$

2. Preliminaries

The purpose of this section is to introduce the identities and tools needed in the following sections of this paper. For any operator $S \in \mathcal{B}(X)$, let $\ker(S)$ denotes the kernel of S, and $\mathrm{Ran}(S)$ represent its range. Let y be a non-zero vector in X and g be a non-zero linear functional in X^*. As customary, we represent the rank-one operator, denoted by $y \otimes g$, and defined by $(y \otimes g)z = g(z)y$ for all $z \in X$. Consider the set $\mathcal{F}_n(X)$, comprising operators with a rank no greater than n, where $n \in \mathbb{N}\backslash\{0\}$. Set

$$\mathcal{P}_1(X) = \{x \otimes f : x \in X,\ f \in X^* \text{ and } f(x) = 1\}.$$

The set of rank-one idempotent operators in $\mathcal{B}(X)$.

The first lemma enumerates some fundamental characteristics of the local spectrum.

Lemma 2.1. *Let $S \in \mathcal{B}(X)$, $x \in X$ and a scalar $\mu \in \mathbb{C}\backslash\{0\}$. The following statements hold.*

1. $\sigma_S(\mu x) = \sigma_S(x)$ *and* $\sigma_{\mu S}(x) = \mu \sigma_S(x)$.
2. *If $Sx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_S(x) \subseteq \{\lambda\}$. Furthermore, if S has SVEP and $x \neq 0$, then $\sigma_S(x) = \{\lambda\}$.*

Proof. See [2, 9]. □

The following lemma presents essential properties of the local spectral subspace of an operator $S \in \mathcal{B}(X)$ associated with a singleton $\{\lambda\}$, where $\lambda \in \mathbb{C}$.

Lemma 2.2. *Let $S \in \mathcal{B}(X)$, $\lambda \in \mathbb{C}$ and $c \in \mathbb{C}\backslash\{0\}$. The following statements hold.*

1. $(S - \mu)X_S(\{\lambda\}) = X_S(\{\lambda\})$ *for every scalar* $\mu \in \mathbb{C}$ *such that* $\mu \neq \lambda$.
2. $X_{S-\lambda}(\{0\}) = X_S(\{\lambda\})$ *and* $X_{cS}(\{\lambda\}) = X_S(\{\frac{\lambda}{c}\})$.
3. $\ker((S - \lambda I)^n) \subseteq X_S(\{\lambda\})$ *for all* $n \in \mathbb{N}$.

Proof. See [2, 9]. □

3. Maps preserving local spectral subspace of Lie product of operators

In the following, we introduce the equivalence relation defined by

$$S \sim T \iff S-T \text{ is a scalar}$$

for all $S,T \in \mathcal{B}(X)$.

Remark 3.1. By noting that $X_{AT-TA}(\{\lambda_0\}) = X_{A\frac{T}{\lambda_0}-\frac{T}{\lambda_0}A}(\{1\})$ for all $A,T \in \mathcal{B}(X)$ and $\lambda_0 \in \mathbb{C}\backslash\{0\}$, we will establish the lemmas specifically for $X_{AT-TA}(\{1\})$.

The following lemma provides necessary and sufficient conditions for two operators in $\mathcal{B}(X)$ to be equivalent.

Lemma 3.1. *Let A,B in $\mathcal{B}(X)$, then the following statements are equivalent.*

1. *$B \sim A$.*
2. *$X_{AT-TA}(\{1\}) = X_{BT-TB}(\{1\})$ for all $T \in \mathcal{P}_1(X)$.*

Proof. We only need to prove $(2) \Longrightarrow (1)$. Let $A,B \in \mathcal{B}(X)$ and $T \in \mathcal{P}_1(X)$.

If x, Ax and Bx are linearly independent for certain vector $x \in X$. We will discuss two cases.

Case 1. If x, Ax and A^2x are linearly independent. There exists $f \in X^*$ such that $f(x)=1,\ f(Ax)=0$ and $f(A^2x)=-1$. Set $T = x\otimes f$, then we get

$$\begin{cases} (AT-TA)x &= Ax \\ (AT-TA)Ax &= x. \end{cases}$$

This implies that $x+Ax \in X_{AT-TA}(\{1\}) = X_{BT-TB}(\{1\}) \subseteq \text{span}\{x,Bx\}$, which is a contradiction.

Case 2. If x, Ax and A^2x are linearly dependent, then there exist $\alpha,\beta \in \mathbb{C}$ such that $A^2x = \alpha Ax + \beta x$. Pick up $a \in \mathbb{C}$ and $f \in X^*$ such that $f(x)=1$, $f(Ax) = 1-a$ and $(a-1)(a+\alpha)-\beta = a$. For $T = x\otimes f$, then

$$(AT-TA)(ax+Ax) = ax+Ax.$$

Hence $ax+Ax \in X_{AT-TA}(\{1\}) = X_{BT-TB}(\{1\}) \subseteq \text{span}\{x,Bx\}$, a contradiction. Finally, by [10, Lemma 2.4], there exists a non-zero scalar $\alpha \in \mathbb{C}$ and a scalar $\mu \in \mathbb{C}$ such that $B = \alpha A + \mu I$.

Now, let $x \in X$ and $f \in X^*$ such that $\{x, Ax, A^2x\}$ is linearly independent, $f(x) = f(Ax) = 1$ and $f(A^2x) = 0$. Then $(Ax\otimes f - x\otimes fA)Ax = Ax$ and so $X_{Ax\otimes f-x\otimes fA}(\{1\}) \neq \{0\}$. On the other hand we have

$$\begin{aligned} X_{Ax\otimes f-x\otimes fA}(\{1\}) &= X_{Bx\otimes f-x\otimes fB}(\{1\}) \\ &= X_{\alpha(Ax\otimes f-x\otimes fA)}(\{1\}) \\ &= X_{Ax\otimes f-x\otimes fA}\left(\left\{\frac{1}{\alpha}\right\}\right). \end{aligned}$$

This yields that $\alpha = 1$, thus $B \sim A$. □

In terms of the local spectral subspace of Lie product of operators, the next lemma characterizes operators $A \in \mathcal{F}_1(X) + \mathbb{C}I$.

Lemma 3.2. *Let $A \in \mathcal{B}(X)$. Then the following statements are equivalent.*

1. $A \in \mathcal{F}_1(X) + \mathbb{C}I$.
2. $\dim X_{AT-TA}(\{1\}) \leq 1$ *for all* $T \in \mathcal{F}_4(X)$.

Proof. Assume that (1) holds and let $x \in X$ and $f \in X^*$. Consider $A = x \otimes f + \beta I$, with $\beta \in \mathbb{C}$, then we obtain that

$$\begin{cases} (AT-TA)x &= f(Tx)x - f(x)Tx \\ (AT-TA)Tx &= f(T^2x)x - f(Tx)Tx. \end{cases}$$

If x and Tx are linearly independent then $AT - TA$ is nilpotent or admits two simple eigenvalues and $x \notin X_{AT-TA}(\{1\}) \subset \mathrm{span}\{x, Tx\}$. Otherwise $X_{AT-TA}(\{1\}) \subset \mathrm{span}\{x\}$, therefore $\dim X_{AT-TA}(\{1\}) \leq 1$.

Conversely, assume that for every $\alpha \in \mathbb{C}$, $A - \alpha I \notin \mathcal{F}_1(X)$. Then there exists a non-zero vector u in X such that $\{u, Au\}$ is linearly independent. Let T be an arbitrary operator in $\mathcal{F}_4(X)$ and consider the following operator $S = AT - TA$, so we discuss three cases.

Case 1. If there exists $x \in X$ such that x, Ax, u and Au are linearly independent. Take an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx = 0, \;\; TAx = -x, \;\; Tu = 0 \text{ and } TAu = -u.$$

It is clear that

$$\begin{cases} Sx &= x \\ Su &= u. \end{cases}$$

Therefore $\mathrm{span}\{x, u\} \subseteq X_S(\{1\})$, which is a contradiction.

Case 2. If u, Au and A^2u are linearly independent. Choose an operator $T \in \mathcal{B}(X)$ satisfying

$$Tu = 0, \;\; TAu = -u \text{ and } TA^2u = -2Au.$$

It follows that

$$\begin{cases} Su &= u \\ SAu &= Au. \end{cases}$$

Therefore $\mathrm{span}\{u,Au\} \subseteq X_S(\{1\})$, a contradiction.
Case 3. If $A^2u \in \mathrm{span}\{u,Au\}$ and for every $x \in X$, $\{x,Ax,u,Au\}$ is linearly dependent. Let M be a supplementary subspace of $\mathrm{span}\{u,Au\}$, this means that $X = \mathrm{span}\{u,Au\} \oplus M$.
If $x \in M$, then $\{x,u,Au\}$ is linearly independent and $Ax \in \mathrm{span}\{x,u,Au\}$. It follows that there exist $\alpha_x, \beta_x, \gamma_x \in \mathbb{C}$ such that $Ax = \alpha_x x + \beta_x u + \gamma_x Au$, which implies that there exist $f,g \in X^*$ and $\alpha \in \mathbb{C}$ such that $Ax = \alpha x + f(x)u + g(x)Au$ for all $x \in M$. Therefore $(A-\alpha I)M \subseteq \mathrm{span}\{u,Au\}$ and we have

$$(A-\alpha I)\mathrm{span}\{u,Au\} \subseteq \mathrm{span}\{u,Au,A^2u\}.$$

Which yields that

$$\mathrm{Ran}(A-\alpha I) \subseteq \mathrm{span}\{u,Au\},$$

and so $\dim \mathrm{Ran}(A-\alpha I) \leq 2$.
If $\dim \mathrm{Ran}(A-\alpha I) = 2$. Since $\mathrm{Ran}(A-\alpha I) \subseteq \mathrm{span}\{u,Au\}$, then there are two linear functionals f and g such that $A - \alpha I = u \otimes f + Au \otimes g$, it follows that $A = \alpha I + u \otimes f + Au \otimes g$. As u and Au are linearly independent and so are f and g. Thus, there exists a vector $y \in X$ such that $f(y) = 1$, $g(y) = 0$ and $\{y,u,Au\}$ is linearly independent. Observe that $Ay = \alpha y + u$. Take an operator $T \in \mathcal{B}(X)$ satisfying

$$Ty = 0, \ \ Tu = -y \text{ and } TAu = -(\alpha y + 2u).$$

We have

$$\begin{cases} Sy &= y \\ Su &= u. \end{cases}$$

Therefore $\mathrm{span}\{u,y\} \subseteq X_S(\{1\})$, which is a contradiction. Consequently, $A - \alpha I$ is a rank-one operator and so $A \in \mathcal{F}_1(X) + \mathbb{C}I$. This complete the proof. $\square$

Now, we can present the main result.

Theorem 3.2. Let λ_0 be a fixed scalar in $\mathbb{C}\backslash\{0\}$ and X be a complex Banach space such that $\dim X \geq 4$. Let $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ be a map such that $\mathcal{F}_4(X) \subset \phi(\mathcal{B}(X))$. Then ϕ satisfies

$$X_{\phi(A)\phi(T)-\phi(T)\phi(A)}(\{\lambda_0\}) = X_{AT-TA}(\{\lambda_0\}) \text{ for every } \ A,T \in \mathcal{B}(X),$$

if and only if there exist a scalar $\alpha \in \mathbb{C}$ and a map $\beta : \mathcal{B}(X) \longrightarrow \mathbb{C}$ such that $\alpha^2 = 1$ and

$$\phi(T) = \alpha T + \beta(T)I \text{ for all } T \in \mathcal{B}(X).$$

Proof. Let $\mu_0 \in \mathbb{C}\setminus\{0\}$ such that $\mu_0^2 = \lambda_0$ and let $\psi(A) = \frac{1}{\mu_0}\phi(\mu_0 A)$ for all $A \in \mathcal{B}(X)$. Then

$$X_{\psi(A)\psi(T)-\psi(T)\psi(A)}(\{1\}) = X_{AT-TA}(\{1\}) \text{ for every } A,T \in \mathcal{B}(X),$$

and we reduce the proof for $\lambda_0 = 1$.

The 'if' part is straightforward, leaving us with the task of proving the 'only if' part. Specifically, let ϕ be a map from $\mathcal{B}(X)$ into itself such that, for all $A,T \in \mathcal{B}(X)$, the condition

$$X_{\phi(A)\phi(T)-\phi(T)\phi(A)}(\{1\}) = X_{AT-TA}(\{1\})$$

holds. The proof will be reparted on four claims.

Claim 1. For all $A \in \mathcal{B}(X)$, $\phi(A) \sim 0$ if and only if $A \sim 0$.

Consider $A \in \mathcal{B}(X)$, if $A \sim 0$, then for all $T \in \mathcal{B}(X)$ we have

$$X_{\phi(A)\phi(T)-\phi(T)\phi(A)}(\{1\}) = X_{AT-TA}(\{1\}) = X.$$

Hence for all $T \in \mathcal{B}(X)$

$$X_{\phi(A)\phi(T)-\phi(T)\phi(A)}(\{1\}) = X_{0\phi(T)-\phi(T)0}(\{1\}) = X.$$

Since the range of ϕ contains $\mathcal{F}_4(X)$, then $\phi(A) \sim 0$. The converse is obtained similarly.

Claim 2. For all $A \in \mathcal{B}(X)$, we have $A \in \mathcal{F}_1(X) + \mathbb{C}I$ if and only if $\phi(A) \in \mathcal{F}_1(X) + \mathbb{C}I$.

Let $A \in \mathcal{F}_1(X) + \mathbb{C}I$, it follows from Lemma 3.2 that $\dim X_{AT-TA}(\{1\})) \leq 1$ for all $T \in \mathcal{B}(X)$. Since

$$X_{AT-TA}(\{1\}) = X_{\phi(A)\phi(T)-\phi(T)\phi(A)}(\{1\}),$$

then for every $T \in \mathcal{B}(X)$

$$\dim X_{\phi(A)\phi(T)-\phi(T)\phi(A)}(\{1\}) \leq 1.$$

Since the range of ϕ contains $\mathcal{F}_4(X)$, therefore $\phi(A) \in \mathcal{F}_1(X) + \mathbb{C}I$.
Conversely, just as before, we get if $\phi(A) \in \mathcal{F}_1(X) + \mathbb{C}I$ then $A \in \mathcal{F}_1(X) + \mathbb{C}I$.

Claim 3. There exists a scalar $\alpha \in \mathbb{C}$ such that $\alpha^2 = 1$ and for all $P \in \mathcal{P}_1(X)$, we have $\phi(P) \sim \alpha P$.

Let $x,y \in X$ and $f,g \in X^*$ such that $f(x) = g(y) = 1$. By Claim 2, there exist two vector $x^{'},y^{'} \in X$ and two linear functionals $f^{'},g^{'} \in X^*$ such that $\phi(x\otimes f) \sim x^{'} \otimes f^{'}$ and $\phi(y\otimes g) \sim y^{'} \otimes g^{'}$. Take $R = x\otimes fy\otimes g - y\otimes gx\otimes f$ and $S = x^{'} \otimes f^{'}y^{'} \otimes g^{'} - y^{'} \otimes g^{'}x^{'} \otimes f^{'}$. Suppose that $1-(g(x)f(y))^2 + f(y)g(x) = 0$, then 1 is an eigenvalue of R and $-R$, and we have

$$\text{span}\{x,y\} = X_R\{1\} \oplus X_{-R}\{1\} = X_S\{1\} \oplus X_{-S}\{1\} = \text{span}\{x^{'},y^{'}\}.$$

Therefore $x^{'} \in \operatorname{span}\{x,y\}$ for all $y \in X$ linearly independent with x and so there exists a non-zero scalar $\lambda \in \mathbb{C}$ such that $x^{'} = \lambda x$. Without loss of generality, suppose for all $x \in X$ and $f \in X^*$, there exists $f^{'}_{x,f} \in X^*$ such that $\phi(x \otimes f) \sim x \otimes f^{'}_{x,f}$.
Now, assume that f and $f^{'}_{x,f}$ are linearly independent. Then, there exists $y \in X$ such that $\{x,y\}$ is linearly independent, $f(y) \neq 0$ and $f^{'}(y) = 0$. Pick up $g \in X^*$ such that $g(y) = 1$ and $1 - (g(x)f(y))^2 + f(y)g(x) = 0$.

It follows that $X_R(\{1\}) \neq \{0\}$. On the other hand, we have

$$X_{(x\otimes f^{\prime}_{x,f})(y\otimes g^{\prime}_{y,g})-(y\otimes g^{\prime}_{y,g})(x\otimes f^{\prime}_{x,f})}\{1\} = \{0\}$$

which is a contradiction. Hence f and $f^{'}_{x,f}$ are linearly dependent.
Now, let $x,y \in X$ such that $\{x,y\}$ is linearly independent or $x = y$ and $f,g \in X^*$ such that $f(x) = g(y) = 1$. We can find $z \in X$ such that $f(z) = g(z) \neq 0$ and $h \in X^*$ such that $h(z) = 1, h(x) = h(y)$ and $1 - (h(x)f(z))^2 + f(z)h(x) = 0$. Then

$$X_{x\otimes fz\otimes h - z\otimes hx\otimes f}(\{1\}) \neq \{0\}.$$

On the other hand, we have $\phi(x \otimes f) \sim \alpha_{x,f} x \otimes f$ and $\phi(z \otimes h) \sim \alpha_{z,h} z \otimes h$. Hence

$$\begin{aligned}
\{0\} \neq X_{x\otimes fz\otimes h - z\otimes hx\otimes f}(\{1\}) &= X_{\phi(x\otimes f)\phi(z\otimes h)-\phi(z\otimes h)\phi(x\otimes f)}(\{1\}) \\
&= X_{\alpha_{x,f}\alpha_{z,h}x\otimes fz\otimes h-\alpha_{x,f}\alpha_{z,h}z\otimes hx\otimes f}(\{1\}) \\
&= X_{\alpha_{x,f}\alpha_{z,h}(x\otimes fz\otimes h-z\otimes hx\otimes f)}(\{1\}).
\end{aligned}$$

This proves that $\alpha_{x,f}\alpha_{z,h} = 1$, and similarly we can find $\alpha_{y,g}\alpha_{z,h} = 1$. We conclude that there exists a scalar $\alpha \in \mathbb{C}$ such that $\alpha^2 = 1$ and $\phi(P) \sim \alpha P$ for all $P \in \mathcal{P}_1(X)$.

Claim 4. $\phi(T) \sim \alpha T$ for all $T \in \mathcal{B}(X)$.
Let $P \in \mathcal{P}_1(X)$ and $T \in \mathcal{B}(X)$, we obtain that

$$\begin{aligned}
X_{TP-PT}(\{1\}) &= X_{\phi(T)\phi(P)-\phi(P)\phi(T)}(\{1\}) \\
&= X_{\phi(T)\alpha P-\alpha P\phi(T)}(\{1\}).
\end{aligned}$$

By Lemma 3.1, $\alpha\phi(T) \sim T$. Since $\alpha^2 = 1$, then $\phi(T) \sim \alpha T$. Therefore there exists a map $\beta : \mathcal{B}(X) \longrightarrow \mathbb{C}$ such that $\phi(T) = \alpha T + \beta(T)I$ for all $T \in \mathcal{B}(X)$. $\square$

As a consequence of Theorem 3.2, the result of [13] can be extended as follows.

Corollary 3.3. Let X be a complex Banach space with $\dim X \geq 4$. Let $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ be a map such that $\mathcal{F}_4(X) \subset \phi(\mathcal{B}(X))$. Then ϕ satisfies

$$F(\phi(A)\phi(T) - \phi(T)\phi(A)) = F(AT - TA) \text{ for every } A, T \in \mathcal{B}(X),$$

if and only if there exist a scalar $\alpha \in \mathbb{C}$ and a map $\beta : \mathcal{B}(X) \longrightarrow \mathbb{C}$ such that $\alpha^2 = 1$ and

$$\phi(T) = \alpha T + \beta(T)I \text{ for all } T \in \mathcal{B}(X).$$

Proof. Since Lemma 2.3 of [13] holds for every $T \in \mathcal{F}_4(X)$ ("$(ii) \Longrightarrow (i)$") and $F(AT - TA) \subset X_{AT-TA}(\{1\})$, the desired result follows by the same arguments as that used in [13]. □

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S. SABER
Department of Mathematics,
Ecole Normale Supérieure
Mohammed V University,
B. P. 5118, 10105, Rabat, Morocco
e-mail: saber.somayaa@gmail.com

M. ELHODAIBI
Department of Mathematics,
Faculty of Sciences, Mohammed First University,
Labo LIABM Oujda, Morocco
e-mail: m.elhodaibi@ump.ac.ma

S. ELOUAZZANI
Department of Mathematics,
Faculty of Sciences, Mohammed First University,
Labo LIABM Oujda, Morocco
e-mail: elouazzani.soufiane@ump.ac.ma