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SIGN-CHANGING SOLUTIONS FOR A NONLINEAR DEGENERATE ELLIPTIC SYSTEM

L. THI HONG HANH - D. TRONG

In this article, we study the multiplicity of weak solutions to the nonlinear degenerate elliptic system

$$-\frac{\partial^2 u_j}{\partial x^2} - |x|^2 \frac{\partial^2 u_j}{\partial y^2} + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j, \quad \text{in} \quad \Omega,$$
$$u_j(x,y) = 0, \quad \text{on} \quad \partial\Omega,\ j = 1,\dots,k,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $\lambda_j > 0, k \geq 2, j = 1\dots,k$, β_{ij} are constants satisfying $\beta_{jj} > 0$, $\beta_{ij} = \beta_{ji} \leq 0$ for $1 \leq i < j \leq k$. The existence of sign-changing solutions is proved by the truncation method and the invariant sets of descending flow method.

1. Introduction

In the last decades, the nonlinear Schödinger system

$$-\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j, \quad \text{in} \quad \Omega, \qquad (1)$$
$$u_j(x) = 0, \quad \text{on} \quad \partial\Omega,\ j = 1,\dots,k,$$

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where $\Omega \subset \mathbb{R}^N$ $(N = 2,3)$ is a bounded domain with smooth boundary, and $\lambda_j > 0$, $\beta_{jj} > 0$, $1 \le j \le k$, $\beta_{ij} = \beta_{ji}$, $1 \le i < j \le k$ are constants, has been studied by many authors. Mathematical work has been done extensively in recent years, refer the reader to [1, 4–6, 12, 14, 15, 19, 22, 26] and the references therein, for the existence theory and the studies of qualitative property of solutions to attractive and repulsive systems.

One of the classes of degenerate elliptic equations that has been studied widely in recent years is the class of equations involving an operator of the Grushin type (see [8])

$$G_\alpha := \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha \ge 0.$$

Note that $G_0 \equiv \Delta$ is the Laplacian operator, and G_α, when $\alpha > 0$, is not elliptic in domains intersecting the surface $x = 0$. Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [24, 25] (see also some recent results in [2, 3, 7, 9–11, 16–18, 20, 23] and the references therein).

In this paper, we consider the existence of sign-changing solutions of the nonlinear degenerate elliptic system

$$\begin{gathered} -\frac{\partial^2 u_j}{\partial x^2} - |x|^2 \frac{\partial^2 u_j}{\partial y^2} + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j, \quad \text{in } \Omega, \\ u_j(x,y) = 0, \quad \text{on } \partial\Omega,\ j = 1,\dots,k, \end{gathered}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $\lambda_j > 0, k \ge 2, j = 1\dots,k$, β_{ij} are constants satisfying $\beta_{jj} > 0$, $\beta_{ij} = \beta_{ji} \le 0$ for $1 \le i < j \le k$.

We assume $U = (u_1,\dots,u_k)$, $\varepsilon \in \mathbb{R}$ is a small parameter, $F(U,\varepsilon)$, $\frac{\partial F}{\partial u_j}(U,\varepsilon)$ are continuous functions, and $F(U,\varepsilon) = F(-U,\varepsilon)$. For $\varepsilon = 0$, we understand

$$F(U,0) = 0, \quad \frac{\partial F}{\partial u_j}(U,0) = 0.$$

Then we consider the perturbed problem

$$\begin{gathered} -\frac{\partial^2 u_j}{\partial x^2} - |x|^2 \frac{\partial^2 u_j}{\partial y^2} + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j + \frac{\partial F}{\partial u_j}(U,\varepsilon), \quad \text{in } \Omega, \\ u_j(x,y) = 0, \quad \text{on } \partial\Omega,\ j = 1,\dots,k. \end{gathered} \tag{2}$$

Our main result is given by the following theorem.

Theorem 1.1. *The system* (1) *has infinitely many solutions with each component being sign-changing.*

Theorem 1.2. *Let $l \in \mathbb{N}^+$. Then there exists $\varepsilon_l > 0$ such that for $|\varepsilon| \le \varepsilon_l$, the system*

$$\begin{aligned} -\frac{\partial^2 u_j}{\partial x^2} - |x|^2 \frac{\partial^2 u_j}{\partial y^2} + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j + \frac{\partial F}{\partial u_j}(U,\varepsilon), \quad \text{in } \Omega, \\ u_j(x,y) = 0, \quad \text{on } \partial\Omega,\ j = 1,\dots,k. \end{aligned} \tag{3}$$

has l pairs of sign-changing solutions.

Corollary 1.3. *For each $l \in \mathbb{N}^+$, there exists $\beta_l > 0$ such that for $\beta_{ij} = \beta_{ji} \le \beta_l$ with $1 \le i < j \le k$, system* (1) *has at least l pairs of sign-changing solutions.*

The structure of our note is as follows: In Section 2, we present some definitions and preliminary results. In Section 3, we obtain the sign-changing solutions of the perturbed problem (3), then we obtain the main result.

2. Preliminary results

Definition 2.1. By $S_1^2(\Omega)$ we will denote the set of all functions $u \in L^2(\Omega)$ such that $\frac{\partial u}{\partial x} \in L^2(\Omega), |x|\frac{\partial u}{\partial y} \in L^2(\Omega)$. We define the norm in this space as follows

$$\|u\|_{S_1^2(\Omega)} = \left\{ \int_\Omega \left(|u|^2 + \left|\frac{\partial u}{\partial x}\right|^2 + |x|^2 \left|\frac{\partial u}{\partial y}\right|^2 \right) dxdy \right\}^{\frac{1}{2}}.$$

We can also define the scalar product in $S_1^2(\Omega)$ as follows

$$(u,v)_{S_1^2(\Omega)} = (u,v)_{L^2(\Omega)} + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{L^2(\Omega)} + \left(|x|\frac{\partial u}{\partial y}, |x|\frac{\partial v}{\partial y}\right)_{L^2(\Omega)}.$$

The space $S_{1,0}^2(\Omega)$ is defined as the closure of $C_0^1(\Omega)$ in the space $S_1^2(\Omega)$.

The following embedding inequality was proved in [23, 25]

$$\left(\int_\Omega |u|^p dX\right)^{\frac{1}{p}} \le C(p,\Omega)\, \|u\|_{S_{1,0}^2(\Omega)},$$

where $1 \le p \le 6, C(p,\Omega) > 0$. The embedding $S_{1,0}^2(\Omega) \hookrightarrow L^p(\Omega)$ is compact if $1 \le p < 6$.

Definition 2.2. Let $\mathbb{B}$ be a real Banach space with its dual space $\mathbb{B}^*, \Phi \in C^1(\mathbb{B}, \mathbb{R})$. We say that Φ satisfies the Palais–Smale if for any sequence $\{u_n\}_{n=1}^{n=+\infty} \subset \mathbb{B}$ such that $\Phi(u_n)$ is bounded and

$$\left\|\Phi'(u_n)\right\|_{\mathbb{B}^*} \to 0 \text{ as } n \to \infty,$$

then there exists a subsequence $\{u_{n_k}\}_{k=1}^{k=+\infty}$ that converges strongly in $\mathbb{B}$.

From Theorem A in [21], we have

Theorem 2.3. *Let $\mathbb{B}$ be a Banach space, Φ be an even C^1-functional on $\mathbb{B}$, A be an odd, continuous mapping from $\mathbb{B}$ to $\mathbb{B}$, and P_j, Q_j, $j = 1, \dots, k$ be open convex subsets of $\mathbb{B}$ with $Q_j = -P_j$. Denote $W = \cup_{j=1}^k (P_j \cup Q_j)$, $\Sigma = \cap_{j=1}^k (\partial P_j \cap \partial Q_j)$. Assume*

(A1) Φ satisfies the Palais-Smale condition.

(A2) $c^* = \inf_{x \in \Sigma} \Phi(x) > 0$.

(A3) For each $b_0 > 0$ and $c_0 > 0$, there exists $b = b(b_0, c_0)$, such that if $|\Phi(x)| \leq c_0$, $\|D\Phi(x)\| \geq b_0$, then

$$\langle D\Phi(x), x - Ax\rangle \geq b\|x - Ax\| > 0.$$

(A4) $A(\partial P_j) \subset P_j$, $A(\partial Q_j) \subset Q_j$, $j = 1, \dots, k$.

Put

$$\Gamma_j = \{E \subset \mathbb{B} : E \text{ is compact}, -E = E, \gamma(E \cap \sigma^{-1}(\Sigma)) \geq j \text{ for } \sigma \in \Lambda\},$$
$$\Lambda = \big\{\sigma \in C(\mathbb{B}, \mathbb{B}) : \sigma \text{ is odd}, \sigma(P_j) \subset P_j, \sigma(Q_j) \subset Q_j, j = 1, \dots, k, \\ \sigma(x) = x \text{ if } \Phi(x) < 0\big\}$$

where $\gamma = \gamma(E)$ denotes the genus of a symmetric set E

$$\gamma = \min\{n : \text{there is an odd map } \varphi^{(j)} : E \to \mathbb{R}^n \setminus \{0\}\}.$$

We ssume that

(A5) Γ_j is nonempty for $j = 1, 2, \dots$.

We define

$$c_j = \inf_{E \in \Gamma_j} \sup_{x \in E \setminus W} \Phi(x), \ j = 1, 2, \dots,$$

$$K_c = \{x \in \mathbb{B} : D\Phi(x) = 0, \Phi(x) = c\}, \quad K_c^* = K_c \setminus W.$$

Then

(1) $c_j \geq c_*$, $K_{c_j}^* \neq \emptyset$ for $j = 1, 2, \dots$.

(2) $c_j \to +\infty$, as $j \to \infty$.

(3) If $c_j = c_{j+1} = \dots = c_{j+l-1} = c$, then $\gamma(K_c^*) \geq l$.

3. Proof of the main result

Define the Euler–Lagrange functional associated with the problem (1) as follows

$$\Phi(U)=\frac{1}{2}\int_\Omega\sum_{j=1}^k\left(\left|\frac{\partial u_j}{\partial x}\right|^2+|x|^2\left|\frac{\partial u_j}{\partial y}\right|^2+\lambda_j u_j^2\right)dxdy-\frac{1}{4}\int_\Omega\sum_{i,j=1}^k\beta_{ij}u_i^2u_j^2\,dxdy$$

for $U=(u_1,\ldots,u_k)\in\mathbb{S}=S_{1,0}^2(\Omega)\times\cdots\times S_{1,0}^2(\Omega)$, the k-fold product of $(S_{1,0}^2(\Omega))^k$.

We shall use the equivalent inner products

$$(u_j,v_j)_j=\int_\Omega\left(\frac{\partial u_j}{\partial x}\frac{\partial v_j}{\partial x}+|x|^2\frac{\partial u_j}{\partial y}\frac{\partial v_j}{\partial y}+\lambda_j u_j v_j\right)dxdy,\quad j=1,\ldots,k$$

and the induced norm $\|\cdot\|_j$. The inner product

$$(U,V)=\sum_{j=1}^k(u_j,v_j)_j,\quad U=(u_1,\ldots,u_k),\ V=(v_1,\ldots,v_k),$$

gives rise to a norm $\|\cdot\|$ on $\mathbb{S}$.

Recall that a function $U=(u_1,\ldots,u_k)\in\mathbb{S}$ is called a weak solution of the problem (1) if for all $v_j\in S_{1,0}^2(\Omega)$, $j=1,\ldots,k$, we have

$$\int_\Omega\sum_{j=1}^k\left(\frac{\partial u_j}{\partial x}\frac{\partial v_j}{\partial x}+|x|^2\frac{\partial u_j}{\partial y}\frac{\partial v_j}{\partial y}+\lambda_j u_j v_j\right)dxdy=\int_\Omega\sum_{i,j=1}^k\beta_{ij}u_i^2u_jv_j dxdy.$$

Then the critical points of Φ are weak solutions of the problem (1).

We recall that $U=(u_1,\ldots,u_k)$ is a sign-changing solution of the problem (1) if is a weak solution to the problem (1) and $u_j^\pm\neq 0, j=1,\ldots,k$, where $u_j^+=\max\{0,u_j\}, u_j^-=\min\{0,u_j\}$.

For $M>0$, we define

$$F_M(U,\varepsilon)=F\big(\varphi_M(|U|)\frac{U}{|U|},\varepsilon\big),$$

where φ_M is a monotonic smooth function, satisfying

$$\varphi_M(t):=\begin{cases}t & \text{if } t<M,\\ M+\frac{1}{2} & \text{if } t\geq M.\end{cases}$$

Then we consider the truncated system

$$\begin{gathered}-\frac{\partial^2u_j}{\partial x^2}-|x|^2\frac{\partial^2u_j}{\partial y^2}+\lambda_ju_j=\sum_{i=1}^k\beta_{ij}u_i^2u_j+\frac{\partial F_M}{\partial u_j}(U,\varepsilon),\quad \text{in }\Omega,\\ u_j(x,y)=0,\quad \text{on }\partial\Omega,\ j=1,\ldots,k.\end{gathered}\tag{4}$$

If $U=(u_1,\dots,u_k)$ is a weak solution of the system (4), and there exists $M>0$ such that $|U(x,y)|<M$ for all $(x,y)\in\overline{\Omega}$, then U is also a solution of the perturbed the problem (3). The system (4) has a variational structure given by the functional

$$\begin{aligned}\Phi_M(U) &= \Phi(U) - \int_\Omega F_M(U,\varepsilon)dxdy \\ &= \frac{1}{2}\int_\Omega \sum_{j=1}^{k}\left(\left|\frac{\partial u_j}{\partial x}\right|^2 + |x|^2\left|\frac{\partial u_j}{\partial y}\right|^2 + \lambda_j u_j^2\right)dxdy \\ &- \frac{1}{4}\int_\Omega \sum_{i,j=1}^{k}\beta_{ij}u_i^2u_j^2 dxdy - \int_\Omega F_M(U,\varepsilon)dxdy.\end{aligned} \tag{5}$$

Lemma 3.1. *Φ_M is a C^1-functional on $\mathbb{S}$, and satisfies the Palais-Smale condition.*

Proof. It is easy to verify that Φ_M is a C^1-functional. Also, for $V=(v_1,\dots,v_k)\in\mathbb{S}$, we obtain

$$\begin{aligned}\langle D\Phi_M(U),V\rangle &= \int_\Omega \sum_{j=1}^{k}\left(\frac{\partial u_j}{\partial x}\frac{\partial v_j}{\partial x} + |x|^2\frac{\partial u_j}{\partial y}\frac{\partial v_j}{\partial y} + \lambda_j u_j v_j\right)dxdy \\ &- \int_\Omega \sum_{i,j=1}^{k}\beta_{ij}u_i^2u_jv_j dxdy - \int_\Omega \sum_{j=1}^{k}\frac{\partial F_M}{\partial u_j}(U,\varepsilon)v_j dxdy,\end{aligned} \tag{6}$$

there exists an arbitrary small constant ε_M, such that for $|\varepsilon|\le\varepsilon_M$, we have that

$$\begin{aligned}&\Phi_M(U) - \frac{1}{4}\langle D\Phi_M(U),U\rangle = \frac{1}{4}\int_\Omega \sum_{j=1}^{k}\left(\left|\frac{\partial u_j}{\partial x}\right|^2 + |x|^2\left|\frac{\partial u_j}{\partial y}\right|^2 + \lambda_j u_j^2\right)dxdy \\ &- \int_\Omega\left(F_M(U,\varepsilon) - \frac{1}{4}\sum_{j=1}^{k}\frac{\partial F_M}{\partial u_j}(U,\varepsilon)u_j\right)dxdy \\ &\ge \frac{1}{4}\|U\|^2 - c.\end{aligned} \tag{7}$$

Then any Palais-Smale sequence of Φ_M is bounded in $\mathbb{S}$. Let $U_n=(u_{n,1},\dots,u_{n,k})\in\mathbb{S}$ be a Palais-Smale sequence of the functional Φ_M. Assume that $U_n\rightharpoonup U$ in $\mathbb{S}$. By the imbedding $S^2_{1,0}(\Omega)\hookrightarrow L^4(\Omega)$ is compact hence $U_n\to U$ in $L^4(\Omega)$. Then

$$\int_\Omega \sum_{j=1}^{k}\left(\left|\frac{\partial(u_{n,j}-u_{m,j})}{\partial x}\right|^2 + |x|^2\left|\frac{\partial(u_{n,j}-u_{m,j})}{\partial y}\right|^2 + \lambda_j(u_{n,j}-u_{m,j})^2\right)dxdy$$

$$
\begin{aligned}
&= \langle D\Phi_M(U_n) - D\Phi_M(U_m), U_n - U_m \rangle + \int_\Omega \sum_{i,j=1}^{k} \beta_{ij} u_{n,i}^2 u_{n,j} (u_{n,j} - u_{m,j}) dxdy \\
&\quad - \int\limits_\Omega \sum_{i,j=1}^{k} \beta_{ij} u_{m,i}^2 u_{m,j} \left(u_{n,j} - u_{m,j}\right) dxdy \\
&\quad + \int\limits_\Omega \sum_{j=1}^{k} \left(\frac{\partial F_M}{\partial u_j}(U_n, \varepsilon) - \frac{\partial F_M}{\partial u_j}(U_m, \varepsilon) \right) (u_{n,j} - u_{m,j}) dxdy \\
&\leq o(1) + c\|U_n\|_{L^4(\Omega)}^3 \left(\int\limits_\Omega \sum_{j=1}^{k} (u_{n,j} - u_{m,j})^4 dxdy \right)^{1/4} \\
&\quad + c\|U_m\|_{L^4(\Omega)}^3 \left(\int\limits_\Omega \sum_{j=1}^{k} (u_{n,j} - u_{m,j})^4 \, dx \right)^{1/4} \\
&\quad + \int\limits_\Omega \sum_{j=1}^{k} \left| \frac{\partial F_M}{\partial u_j}(U_n, \varepsilon) - \frac{\partial F_M}{\partial u_j}(U_m, \varepsilon) \right| |u_{n,j} - u_{m,j}| dxdy \\
&\leq o(1) + c\|U_n - U_m\|_{L^4(\Omega)} \to 0, \quad \text{as } n, m \to \infty.
\end{aligned}
$$

Therefore, we conclude that up to a subsequence a Palais-Smale sequence U_n is a Cauchy sequence in $\mathbb{S}$, hence a convergent sequence. □

We put

$$
A : U = (u_1, \ldots, u_k) \in \mathbb{S} \mapsto V = (v_1, \ldots, v_k) = AU \in \mathbb{S}
$$

such that

$$
\begin{aligned}
&\int\limits_\Omega \left(\frac{\partial v_j}{\partial x} \frac{\partial \psi_j}{\partial x} + |x|^2 \frac{\partial v_j}{\partial y} \frac{\partial \psi_j}{\partial y} + \lambda_j v_j \psi_j \right) dxdy - \int\limits_\Omega \sum_{i=1, i \neq j}^{k} \beta_{ij} u_i^2 v_j \psi_j dxdy \\
&= \int\limits_\Omega \beta_{jj} u_j^3 \psi_j dxdy + \int\limits_\Omega \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \psi_j dxdy, \forall \Psi = (\psi_1, \ldots, \psi_k) \in \mathbb{S}, j = 1, \ldots, k.
\end{aligned} \tag{8}
$$

Lemma 3.2. *The operator A is well-defined and odd, continuous.*

Proof. It is easy to verify that A is a odd functional. Therefore, $V = AU$ can be obtained by solving the minimization problem

$$
\inf\{G(V) : V \in \mathbb{S}\}
$$

where

$$G(V)=\frac{1}{2}\int_{\Omega}\sum_{j=1}^{k}\left(\left|\frac{\partial v_j}{\partial x}\right|^2+|x|^2\left|\frac{\partial v_j}{\partial y}\right|^2+\lambda_j v_j^2\right)dxdy-\frac{1}{2}\int_{\Omega}\sum_{i,j=1,i\neq j}^{k}\beta_{ij}u_i^2v_j^2\,dxdy$$
$$-\int_{\Omega}\sum_{j=1}^{k}\beta_{jj}u_j^3v_j\,dxdy-\int_{\Omega}\sum_{j=1}^{k}\frac{\partial F_M}{\partial u_j}(U,\varepsilon)v_j\,dxdy.$$

Let $V=AU$, $\bar{V}=A\bar{U}$, $\bar{V}=(\bar{v}_1,\dots,\bar{v}_k)$, $\bar{U}=(\bar{u}_1,\dots,\bar{u}_k)$. From (8), we have

$$\begin{aligned}
&\|V-\bar{V}\|^2\\
&=\int_{\Omega}\sum_{j=1}^{k}\left(\left|\frac{\partial(v_j-\bar{v}_j)}{\partial x}\right|^2+|x|^2\left|\frac{\partial(v_j-\bar{v}_j)}{\partial y}\right|^2+\lambda_j(v_j-\bar{v}_j)^2\right)dxdy\\
&=\int_{\Omega}\sum_{i,j=1,i\neq j}^{k}\beta_{ij}(u_i^2v_j-\bar{u}_i^2\bar{v}_j)(v_j-\bar{v}_j)dxdy+\int_{\Omega}\sum_{j=1}^{k}\beta_{jj}(u_j^3-\bar{u}_j^3)(v_j-\bar{v}_j)\,dxdy\\
&\quad+\int_{\Omega}\sum_{j=1}^{k}\left(\frac{\partial F_M}{\partial u_j}(U,\varepsilon)-\frac{\partial F_M}{\partial u_j}(\bar{U},\varepsilon)\right)(v_j-\bar{v}_j)\,dxdy\\
&\le c\int_{\Omega}\sum_{i,j=1,i\neq j}^{k}|u_i^2-\bar{u}_i^2|\,|v_j|\,|v_j-\bar{v}_j|dxdy+c\int_{\Omega}\sum_{j=1}^{k}|u_j^3-\bar{u}_j^3|\,|v_j-\bar{v}_j|dxdy\\
&\quad+\int_{\Omega}\sum_{j=1}^{k}\left|\frac{\partial F_M}{\partial u_j}(U,\varepsilon)-\frac{\partial F_M}{\partial u_j}(\bar{U},\varepsilon)\right|\,|v_j-\bar{v}_j|dxdy\\
&\le c(\|U-\bar{U}\|\,\|V-\bar{V}\|+\left\|\frac{\partial F_M}{\partial u_j}(U,\varepsilon)-\frac{\partial F_M}{\partial u_j}(\bar{U},\varepsilon)\right\|\,\|V-\bar{V}\|),
\end{aligned}$$

hence $AU-A\bar{U}=V-\bar{V}\to 0$ as $U\to\bar{U}$ in $\mathbb{S}$. □

Lemma 3.3. *For each $b_0,c_0>0$, then the following property holds: if $|\Phi_M(U)|\le c_0$ and $\|\Phi_M(U)\|\ge b_0$, then there exists $b=b(b_0,c_0)$ such that*

$$\langle D\Phi_M(U),U-AU\rangle\ge b\|U-AU\|>0.$$

Proof. We have

$$
\begin{aligned}
&\langle D\Phi_M(U), \Psi\rangle \\
&= \int_\Omega \sum_{j=1}^{k} \left(\frac{\partial(u_j - v_j)}{\partial x} \frac{\partial \psi_j}{\partial x} + |x|^2 \frac{\partial(u_j - v_j)}{\partial y} \frac{\partial \psi_j}{\partial y} + \lambda_j (u_j - v_j)\psi_j \right) dxdy \\
&- \int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 (u_j - v_j)\psi_j \, dxdy \\
&= \langle U - V, \Psi\rangle - \int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 (u_j - v_j)\psi_j \, dxdy,
\end{aligned}
\tag{9}
$$

for $\Psi = (\psi_1, \ldots, \psi_k) \in \mathbb{S}$. By using $\Psi = U - V$ in (9), we obtain

$$
\langle D\Phi_M(U), U - V\rangle = \|U - V\|^2 - \int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 (u_j - v_j)^2 \, dxdy.
$$

By $\beta_{ij} = \beta_{ji} \leq 0$ for $1 \leq i < j \leq k$, hence

$$
\langle D\Phi_M(U), U - V\rangle \geq \|U - V\|^2 \tag{10}
$$

and

$$
\langle D\Phi_M(U), U - V\rangle \geq -\int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 (u_j - v_j)^2 \, dxdy. \tag{11}
$$

It follows from (9) and (11) that

$$
\begin{aligned}
&|\langle D\Phi_M(U), \Psi\rangle| = \left| \langle U - V, \Psi\rangle - \int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 (u_j - v_j)\psi_j \, dxdy \right| \\
&\leq \|U - V\| \, \|\Psi\| + \left(-\int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 (u_j - v_j)^2 \, dxdy \right)^{1/2} \left(-\int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 \varphi_j^2 \, dxdy \right)^{1/2} \\
&\leq \|U - V\| \, \|\Psi\| + c\|U\|_{L^4(\Omega)} \|\Psi\|_{L^4(\Omega)} \langle D\Phi_M(U), U - V\rangle^{1/2}
\end{aligned}
$$

which implies that

$$
\|D\Phi_M(U)\| \leq \|U - V\| + c\|U\| \langle D\Phi_M(U), U - V\rangle^{1/2}. \tag{12}
$$

There exists a small constant ε_M, so that for $|\varepsilon| \leq \varepsilon_M$, by (5) and (9), we have

$$\begin{aligned}
&\Phi_M(U) - \frac{1}{4}\langle U-V, U\rangle \\
&= \Phi_M(U) - \frac{1}{4}\langle D\Phi_M(U), U\rangle - \frac{1}{4}\int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 u_j (u_j - v_j) dxdy \\
&= \frac{1}{4}\|U\|^2 + \int_\Omega \left(\frac{1}{4}\sum_{j=1}^{k} \frac{\partial F_M}{\partial u_j}(U,\varepsilon) u_j - F_M(U,\varepsilon) \right) dxdy \\
&\quad - \frac{1}{4}\int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 u_j (u_j - v_j)\, dxdy \\
&\geq \frac{1}{4}\|U\|^2 - \frac{1}{4}\int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 u_j (u_j - v_j)\, dxdy - c.
\end{aligned} \tag{13}$$

So by (13), we obtain

$$\begin{aligned}
&\|U\|^2 \\
&\leq c\,(1 + |\Phi_M(U)|) + c|\langle U-V, U\rangle| + c\left| \int_\Omega \sum_{i,j=1, i\neq j}^{k} \beta_{ij} u_i^2 u_j (u_j - v_j)\, dx \right| \\
&\leq c\,(1 + |\Phi_M(U)|) + c\|U-V\|^2 + \frac{1}{4}\|U\|^2 + c\|U\|^2_{L^4(\Omega)} \langle D\Phi_M(U), U-V\rangle^{1/2}.
\end{aligned} \tag{14}$$

Given a positive constant $\overline{C}$, if

$$\langle D\Phi_M(U), U-V\rangle \geq \overline{C}^2,$$

then by (10) we can easily obtain

$$\langle D\Phi_M(U), U-V\rangle \geq \overline{C}\|U-V\| > 0.$$

The conclusion holds; if not, let

$$\langle D\Phi_M(U), U-V\rangle \leq \overline{C}^2, \tag{15}$$

by (14) and (15), we have

$$\|U\|^2 \leq c\left(1 + |\Phi_M(U)| + \|U-V\|^2\right) + c_0\overline{C}\|U\|^2. \tag{16}$$

Hence, taking $\overline{C}$ such that $c_0\overline{C} \leq 1/2$, then we have

$$\|U\|^2 \leq c\left(1 + |\Phi_M(U)| + \|U-V\|^2\right). \tag{17}$$

Substituting (17) into (12), we obtain

$$\begin{aligned}
&\|D\Phi_M(U)\| \\
&\le \|U-V\| + c\left(1+|\Phi_M(U)|+\|U-V\|^2\right)^{1/2} \langle D\Phi_M(U), U-V\rangle^{1/2} \\
&\le \|U-V\| + \frac{1}{2}\|D\Phi_M(U)\| + c\left(1+|\Phi_M(U)|+\|U-V\|^2\right)\|U-V\|.
\end{aligned} \tag{18}$$

Therefore

$$\|D\Phi_M(U)\| \le c(1+|\Phi_M(U)|+\|U-V\|^2)\|U-V\|.$$

If $|\Phi_M(U)| \le c_0$ and $\|D\Phi_M(U)\| \ge b_0 > 0$, we deduce that there exists $b = b(b_0, c_0)$ such that $\|U-V\| > b$. So it follows from (10) that

$$\langle D\Phi_M(U), U-AU\rangle \ge b\|U-AU\| > 0. \qquad \square$$

Let P_j, Q_j for $j = 1, \dots, k$ be open convex subsets of $\mathbb{S}$, defined by

$$\begin{aligned}
P_j = P_j(\delta) &= \{U = (u_1, \dots, u_k) \in \mathbb{S} : \|u_j^-\|_{L^4(\Omega)} < \delta\}, \\
Q_j = Q_j(\delta) &= \{U = (u_1, \dots, u_k) \in \mathbb{S} : \|u_j^+\|_{L^4(\Omega)} < \delta\}.
\end{aligned}$$

Lemma 3.4. *There exist $\delta > 0$ and $\varepsilon_M > 0$ such that for $|\varepsilon| \le \varepsilon_M$, it holds that*

$$A(\partial P_j) \subset P_j, \quad A(\partial Q_j) \subset Q_j, \quad \text{for } j = 1, \dots, k.$$

Proof. Choose $\Psi = V^+ = (v_1^+, \dots, v_k^+)$ as test function in (8), we have

$$\begin{aligned}
&\int_\Omega \left(\left|\frac{\partial v_j^+}{\partial x}\right|^2 + |x|^2 \left|\frac{\partial v_j^+}{\partial y}\right|^2 + \lambda_j (v_j^+)^2 \right) dxdy - \int_\Omega \sum_{i=1, i\neq j}^{k} \beta_{ij} u_i^2 (v_j^+)^2 dxdy \\
&= \int_\Omega \beta_{jj} u_j^3 v_j^+ \, dxdy + \int_\Omega \frac{\partial F_M}{\partial u_j}(U,\varepsilon) v_j^+ \, dxdy \\
&\le c\left(\int_\Omega (u_j^+)^3 v_j^+ \, dxdy + \int_\Omega \left|\frac{\partial F_M}{\partial u_j}(U,\varepsilon)\right| v_j^+ \, dxdy \right).
\end{aligned}$$

Then

$$\|v_j^+\|_{L^4(\Omega)}^2 \le c_1 \|u_j^+\|_{L^4(\Omega)}^3 \|v_j^+\|_{L^4(\Omega)} + c_2 \left\| \frac{\partial F_M}{\partial u_j}(U,\varepsilon) \right\|_{L^\infty(\Omega)} \|v_j^+\|_{L^4(\Omega)}. \tag{19}$$

Take $\delta > 0$ such that $c_1 \delta^2 \le 1/4$ and choose $\varepsilon_M > 0$, such that

$$|\varepsilon| \le \varepsilon_M, \; c_2 \left\| \frac{\partial F_M}{\partial u_j}(U,\varepsilon) \right\|_{L^\infty(\Omega)} \le \frac{\delta}{4}.$$

Then for $U \in \partial Q_j$, $\|u_j^+\|_{L^4(\Omega)} = \delta$, we have

$$\|v_j^+\|^2_{L^4(\Omega)} \le \frac{1}{4}\delta\|v_j^+\|_{L^4(\Omega)} + \frac{1}{4}\delta\|v_j^+\|_{L^4(\Omega)},$$

hence

$$\|v_j^+\|_{L^4(\Omega)} \le \frac{1}{2}\delta.$$

That is, for $U \in \partial Q_j$, we have $V = AU \in Q_j$ and $A(\partial Q_j) \subset Q_j$, $j = 1, \dots, k$. Similarly, $A(\partial P_j) \subset P_j$, $j = 1, \dots, k$. □

Lemma 3.5. *There exist $\delta > 0$ and $c^* > 0$, such that if $U \in \Sigma$ and $|\varepsilon| \le \varepsilon_M$, then $\Phi_M(U) \ge c^*$.*

Proof. We have that

$$\begin{aligned}\Phi_M(U) &= \frac{1}{2}\int_\Omega \sum_{j=1}^k \left(\left|\frac{\partial u_j}{\partial x}\right|^2 + |x|^2 \left|\frac{\partial u_j}{\partial y}\right|^2 + \lambda_j u_j^2 \right) dxdy - \frac{1}{4}\int_\Omega \sum_{i,j=1}^k \beta_{ij} u_i^2 u_j^2\, dxdy \\ &\quad - \int_\Omega F_M(U,\varepsilon)\, dxdy \ge \frac{1}{2}\|U\|^2 - \frac{1}{4}\int_\Omega \sum_{j=1}^k \beta_{jj} u_j^4\, dxdy - \int_\Omega F_M(U,\varepsilon)\, dxdy \\ &\quad \ge c_1\|U\|^2_{L^4(\Omega)} - c_2\|U\|^4_{L^4(\Omega)} - \|F_M(U,\varepsilon)\|_{L^\infty(\Omega)}.\end{aligned}$$

For $U \in \Sigma = \cap_{j=1}^k (\partial P_j \cap \partial Q_j)$, we have

$$\|U\|^4_{L^4(\Omega)} = \int_\Omega \sum_{j=1}^k \left((u_j^+)^4 + (u_j^-)^4 \right) dxdy = 2k\|u_j^+\|^4_{L^4(\Omega)} = 2k\delta^4.$$

By Lemma 3.4, taking $\delta > 0$ such that $c_2\delta^2 \le \frac{1}{4}c_1$, and choosing ε_M such that for $|\varepsilon| \le \varepsilon_M$, we have $\|F_M(U,\varepsilon)\|_{L^\infty(\Omega)} \le \frac{1}{4}c_1\delta^2$. Therefore,

$$\Phi_M(U) \ge c_1\delta^2 - c_2\delta^4 - \frac{1}{4}c_1\delta^2 \ge \frac{1}{2}c_1\delta^2 := c^* > 0.$$

□

Let

$$\begin{aligned}\Gamma_j &= \left\{ E \subset X : E \text{ is compact}, -E = E, \gamma(E \cap \sigma^{-1}(\Sigma)) \ge j \text{ for } \sigma \in \Lambda \right\}, \\ \Lambda &= \Big\{ \sigma \in C(X,X) : \sigma \text{ odd}, \sigma(P_j) \subset P_j,\ \sigma(Q_j) \subset Q_j,\ j = 1, \dots, k, \\ &\qquad \sigma(U) = U \text{ if } I_M(U) < 0 \Big\},\end{aligned}$$

and $\gamma = \gamma(E)$ is the genus of E,

$$\gamma = \min\{n : \text{there is an odd map } \varphi^{(j)} : E \to \mathbb{R}^n \setminus \{0\}\}.$$

Now we define a sequence of critical values of the truncated functional Φ_M,

$$c_j(M,\varepsilon) = \inf_{E\in\Gamma_j} \sup_{U\in E\setminus W} \Phi_M(U), \quad j = 1,2,\ldots$$

where $W = \cup_{j=1}^{k}(P_j \cup Q_j)$.

Lemma 3.6. *The set Γ_j is nonempty, and there exist $d_j > 0$ independent of M, ε and $\varepsilon_M^{(j)} > 0$, such that if $|\varepsilon| \le \varepsilon_M^{(j)}$, then $c_j(M,\varepsilon) \le d_j$.*

Proof. Let B^{nk} be the unit closed ball of $\mathbb{R}^{nk}$. Assume $n = j+k$. Denote $t \in \mathbb{R}^{nk}$ by $t = (t_1,\ldots,t_k)$ and $t_m = (t_{1m}, t_{2m}, \ldots, t_{nm}) \in \mathbb{R}^n$ for $m = 1,\ldots,k$. Let $v_{im} \in C_0^\infty(\Omega)$, $i = 1,\ldots,n$, $m = 1,\ldots,k$ be nk functions in $\mathbb{S}$ with disjoint supports. Define

$$\begin{aligned} \varphi^{(j)} &: B^{nk} \longrightarrow \mathbb{S} \\ &t \longmapsto \varphi^{(j)}(t) = R\Big(\sum_{i=1}^{n} t_{i1}v_{i1}, \ldots, \sum_{i=1}^{n} t_{ik}v_{ik}\Big) \in \mathbb{S}, \end{aligned}$$

where R is large enough such that $\Phi(\varphi^{(j)}(t)) < -10$ for $t \in \partial B^{nk}$. Then there exists $\varepsilon_M > 0$, so that if $|\varepsilon| \le \varepsilon_M$, then we have

$$\Phi_M(\varphi^{(j)}(t)) \le \Phi(\varphi^{(j)}(t)) + 1 < 0$$

for $t \in \partial B^{nk}$. By [13, Lemma 5.6], we have $E_j := \varphi^{(j)}(B^{nk}) \in \Gamma_j$. Then Γ_j is nonempty.

Next we estimate $c_j(M,\varepsilon)$ for $|\varepsilon| \le \varepsilon_M$. We have

$$c_j(M,\varepsilon) = \inf_{E\in\Gamma_j} \sup_{U\in E\setminus W} \Phi_M(U) \le \sup_{U\in E_j} \Phi_M(U) \le \sup_{U\in E_j} (\Phi(U)+1) := d_j\,.$$

□

We complete the proof of Theorem 1.1 and Theorem 1.2. For fixed $M > 0$ and $\varepsilon = 0$, we will obtain the critical point U of Φ.

Lemma 3.7. *Assume $D\Phi_M(U) = 0$, $\Phi_M(U) \le L$. Then there exist $\varepsilon_M > 0$ and $K = K(L)$ independent of M, ε, such that for $|\varepsilon| \le \varepsilon_M$,*

$$\|U(x)\|_{L^\infty(\Omega)} \le K.$$

Proof. Denote $U = (u_1, \ldots, u_k)$. By (7), for $|\varepsilon| \leq \varepsilon_M$, we have

$$
\begin{aligned}
L &\geq \Phi_M(U) - \frac{1}{4}\langle D\Phi_M(U), U\rangle \\
&= \frac{1}{4}\int_\Omega \sum_{j=1}^{k}\left(\left|\frac{\partial u_j}{\partial x}\right|^2 + |x|^2\left|\frac{\partial u_j}{\partial y}\right|^2 + \lambda_j u_j^2\right) dxdy \\
&- \int_\Omega \left(F_M(U,\varepsilon) - \frac{1}{4}\sum_{j=1}^{k}\frac{\partial F_M}{\partial u_j}(U,\varepsilon)u_j\right) dxdy \geq \frac{1}{4}\|U\|^2 - c.
\end{aligned}
\tag{20}
$$

We know that there exists $C(L) > 0$, such that $\|U\| \leq C(L)$. Choose $\psi_j = u_{jT}|u_{jT}|^{2r-2}, \Psi = (\psi_1, \ldots, \psi_k)$ as the test function in $\langle D\Phi_M(u_j), \Psi\rangle = 0$, where $r \geq 1$, $T > 1$, and

$$
u_{jT}(x) := \begin{cases} T & \text{if } \; u_j(x) \geq T, \\ -T & \text{if } \; -u_j(x) \geq T, \\ u_j(x) & \text{if } \; |u_j(x)| < T. \end{cases}
$$

We have

$$
\begin{aligned}
&\int_\Omega \left(\frac{\partial u_j}{\partial x}\frac{\partial \psi_j}{\partial x} + |x|^2\frac{\partial u_j}{\partial y}\frac{\partial \psi_j}{\partial y} + \lambda_j u_j \psi_j\right) dxdy \\
&= \int_\Omega \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j \psi_j \, dxdy + \int_\Omega \frac{\partial F_M}{\partial u_j}(U,\varepsilon)\psi_j \, dxdy.
\end{aligned}
\tag{21}
$$

By (21), it is easy to obtain the inequality

$$
\begin{aligned}
&\int_\Omega \left(\frac{\partial u_j}{\partial x}\frac{\partial \psi_j}{\partial x} + |x|^2\frac{\partial u_j}{\partial y}\frac{\partial \psi_j}{\partial y}\right) dxdy \\
&\leq \int_\Omega \beta_{jj} u_j^3 \psi_j \, dxdy + \int_\Omega \left|\frac{\partial F_M}{\partial u_j}(U,\varepsilon)\psi_j\right| dxdy.
\end{aligned}
\tag{22}
$$

Firstly, we estimate the left-hand side of (22). By $S^2_{1,0}(\Omega) \hookrightarrow L^6(\Omega)$ we have that

$$\begin{aligned}
&\int_\Omega \left(\frac{\partial u_j}{\partial x}\frac{\partial \psi_j}{\partial x}+|x|^2\frac{\partial u_j}{\partial y}\frac{\partial \psi_j}{\partial y}\right)dxdy \\
&\geq (2r-1)\int_\Omega \left(\left|\frac{\partial u_{jT}}{\partial x}\right|^2+|x|^2\left|\frac{\partial u_{jT}}{\partial y}\right|^2\right)|u_{jT}|^{2r-2}\,dxdy \\
&\geq \frac{2r-1}{r^2}\int_\Omega \left(\left|\frac{\partial |u_{jT}|^r}{\partial x}\right|^2+|x|^2\left|\frac{\partial |u_{jT}|^r}{\partial y}\right|^2\right)dxdy \\
&\geq \frac{c(2r-1)}{r^2}\left(\int_\Omega |u_{jT}|^{6r}\,dxdy\right)^{1/3}.
\end{aligned}\tag{23}$$

Let $M>0$, there exists ε_M such that for $|\varepsilon|\leq\varepsilon_M$, we have

$$\left\|\frac{\partial F_M}{\partial u_j}(U,\varepsilon)\right\|_{L^\infty(\Omega)}<1.$$

Then the right-hand side of (22) satisfies

$$\begin{aligned}
&\int_\Omega \beta_{jj}u_j^3\psi_j\,dxdy+\int_\Omega \left|\frac{\partial F_M}{\partial u_j}(U,\varepsilon)\psi_j\right|dxdy \\
&\leq c\left(\int_\Omega |u_j|^3|u_{jT}|^{2r-1}\,dxdy+\int_\Omega 1\cdot|u_{jT}|^{2r-1}\,dxdy\right) \\
&\leq c\left(\int_\Omega \left(1+|u_j|^3\right)|u_j|^{2r-1}\,dxdy\right) \\
&\leq c\left(1+\int_\Omega |u_j|^3|u_j|^{2r-1}\,dxdy\right) \\
&\leq c\left(1+\left(\int_\Omega |u_j|^6dxdy\right)^{\frac{1}{3}}\left(\int_\Omega |u_j|^{3r}\,dxdy\right)^{\frac{2}{3}}\right) \\
&\leq c\left(1+\left(\int_\Omega |u_j|^{3r}\,dxdy\right)^{\frac{2}{3}}\right) \\
&\leq c\max\left\{1,\left(\int_\Omega |u_j|^{3r}\,dxdy\right)^{\frac{2}{3}}\right\}.
\end{aligned}\tag{24}$$

Let $T\to\infty$ such that $u_{jT}(x,y)\to u_j(x,y)$. By (23) and (24), we obtain

$$\left(\int_\Omega |u_{jT}|^{6r}dxdy\right)^{\frac{1}{3}} \le \frac{cr^2}{2r-1}\max\left\{1,\left(\int_\Omega |u_j|^{3r}dxdy\right)^{\frac{2}{3}}\right\}. \tag{25}$$

Using iteration, we have that

$$\left(\int_\Omega |u_j|^{12}dxdy\right)^{\frac{1}{12}} \le \left(\frac{4c}{3}\right)^{\frac{1}{4}}\max\left\{1,\left(\int_\Omega |u_j|^{6}dxdy\right)^{\frac{1}{6}}\right\}. \tag{26}$$

Therefore, for any $m=1,2,\ldots$ and by (26), we obtain

$$\begin{aligned}\left(\int_\Omega |u_j|^{3.2^{m+2}}dxdy\right)^{\frac{1}{3.2^{m+2}}} &\le \left(\frac{2^{2m+2}c}{2^{m+2}-1}\right)^{\frac{1}{2^{m+2}}}\max\left\{1,\left(\int_\Omega |u_j|^{3.2^{m+1}}dxdy\right)^{\frac{1}{3.2^{m+1}}}\right\}\\ &\le \prod_{i=0}^{m}\left(\frac{2^{i+1}c}{2^{i+2}-1}\right)^{\frac{1}{2^{i+2}}}\max\left\{1,\left(\int_\Omega |u_j|^{6}dxdy\right)^{\frac{1}{6}}\right\},\end{aligned}$$

We put $C_0=\prod_{i=0}^{m}\left(\frac{2^{i+1}c}{2^{i+2}-1}\right)^{\frac{1}{2^{i+2}}}$, then

$$\|u_j\|_{L^{3.2^{m+2}}(\Omega)} \le C_0(1+\|u_j\|_{L^6(\Omega)}). \tag{27}$$

Let $m\to\infty$ in (27), by (20), we have

$$\|u_j\|_{L^\infty(\Omega)} \le C_0(1+\|u_j\|_{L^6(\Omega)}) \le c=c(L), \forall j=1,\ldots,k.$$

Hence

$$\|U(x)\|_{L^\infty(\Omega)} \le K.$$

□

Proof of Theorem 1.2. By Lemmas 3.1, 3.3-3.6, for a sufficiently small parameter ε, the functional Φ_M satisfies the conditions (A3), (A4)–(A7) of the Theorem 2.3. Then, $c_j(M,\varepsilon)$ is a critical value of the functional Φ_M, and each component of the corresponding critical point $U_j(M,\varepsilon)$ is sign-changing. That is, $U_j(M,\varepsilon)$ is a sign-changing solution of the truncated the system (4). Moreover, given $l\in\mathbb{N}^+$, $L^*>0$, by Lemma 3.6, there exists $\varepsilon_M^*>0$ such that for $|\varepsilon|\le\varepsilon_M^*=\min\{\varepsilon_M^{(1)},\ldots,\varepsilon_M^{(l)}\}$,

$$c_j(M,\varepsilon)\le L^*=\max\{d_1,\ldots,d_l\},\quad j=1,\ldots,l.$$

By Lemma 3.7, there exist the constant K^* independent of M, ε, and $\varepsilon_M > 0$, such that for $|\varepsilon| \leq \varepsilon_M$, we obtain

$$\|U_j(M,\varepsilon)\|_{L^\infty(\Omega)} \leq K^*, \quad j = 1,\dots,l.$$

Now take $M \geq K^* + 1$, then for $|\varepsilon| \leq \varepsilon_l$, $U_j(\varepsilon) := U_j(M,\varepsilon)$, $j = 1,\dots,l$ are sign-changing solutions of the perturbed the system (3). $\square$

Taking $\varepsilon = 0$, we have

$$F(U,0) = 0, \ \frac{\partial F}{\partial u_j}(U,0) = 0,$$

then the solutions to the perturbed system (3) are also solutions to the original system (1).

We have obtained the sign-changing critical points of the truncated functional Φ_M. Therefore, by Theorem 1.2, we know that system (3) has l pairs of sign-changing solutions. Then, for $\varepsilon = 0$, the system (1) has infinitely many sign-changing solutions, and we have thus proved the main result.

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L. THI HONG HANH
Department of Mathematics, Hoa Lu University, Ninh Binh, Vietnam.
e-mail: lthhanh@hluv.edu.vn

D. TRONG
Department of Mathematics, Hoa Lu University, Ninh Binh, Vietnam.
e-mail: dtluyen.dnb@moet.edu.vn, dtluyen@hluv.edu.vn